Advanced Real Analysis

refered ~~to~~ Real Analysis by folland ; Real Analysis by stein

Measure Theory and fine properties of functions by Evans.

from 24. 2. 2015

~~to~~ 2

Chapter 1 Measure ; for this part it has been introduced in RA note just review ...

~~1.1 σ-Algebra~~

some preliminary for a sequence of sets $\{E_n\} \subseteq P(X)$

$\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} E_j$ $\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_j$

$\downarrow$ $\downarrow$

$x \in E_n$ for infinitely many n $x \in E_n$ ~~for~~ for all but finitely many n

De morgan $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap E_\alpha^c$ $\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$

map $f^{-1}: P(X) \to P(X)$ preserve the union, intersection and complement of sets

i.e. $f^{-1}(\bigcup_\alpha E_\alpha) = \bigcup_\alpha f^{-1}(E_\alpha)$. $f^{-1}(\bigcap_\alpha E_\alpha) = \bigcap_\alpha f^{-1}(E_\alpha)$ $f^{-1}(E^c) = (f^{-1}(E))^c$

A form of Axiom of Choice

if $\{X_\alpha\}$ is non-empty $\Rightarrow \prod_\alpha X_\alpha$ is non empty

In folland's book, We have seen that if we want to construct a map

$\mu : \boxed{P(X)} \to [0,+\infty]$ satisfying

(1) $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ if $\{E_n\}$ is a disjoint sequence

(2) $\mu(E_n) = \mu(F)$ if if E can transformed into F by "translation", "rotation" & "reflection"

(3) $\mu(Q) = 1$ Q denote unit cube in $\mathbb{R}^n$

using the construction process of ~~the~~ Vitali set N

$\Rightarrow \mu$ should have a smaller domain

An Algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements.

A σ-Algebra is an algebra that is closed under countable unions.

$\boxed{\text{notation}}$ suppose $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{A}$ set $F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$

$\Rightarrow \bigcup_1^k F_j = \bigcup_1^k E_j \Rightarrow \bigcup_1^\infty F_j = \bigcup_1^\infty E_j$

So $\mathcal{A}$ is a σ-Algebra provided that it's closed under countable disjoint ~~sets~~ unions
an algebra

Intersection of any family of σ-algebra on X is again a σ-algebra

$\mathcal{M}(\mathcal{E}) :=$ the smallest σ-Algebra containing $\mathcal{E}$
called the σ-algebra generated by $\mathcal{E}$

$\boxed{\text{Lem}}$ $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F}) \Rightarrow \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$

elementary family (Semi-algebra) is a collection $\mathcal{E}$ of subsets of X
s.t.
- $\phi \in \mathcal{E}$
- $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$
- $E \in \mathcal{E} \Rightarrow E^c$ is a finite disjoint union of members of $\mathcal{E}$.

$\boxed{\text{prop}}$ If $\mathcal{E}$ is an "elementary family", the collection $\mathcal{A}$ of finite disjoint union of members of $\mathcal{E}$ is an algebra.

proof: if $A, B \in \mathcal{E}$, $B^c = \bigcup_1^J C_j$

$A \setminus B = A \cap (B^c) = A \cap \left(\bigcup_1^J C_j\right) = \bigcup_1^J (A \cap C_j)$ is finite union of members of $\mathcal{E}$

$A \cup B = (A \setminus B) \cup B = \left[\bigcup_1^J (A \cap C_j)\right] \cup B \in \mathcal{A}$

then $\bigcup_1^n A_j = A_n \cup \left(\bigcup_1^{n-1} A_j \setminus A_n\right) \in \mathcal{A}$ (assume $A_1 \cdots A_{n-1}$ are disjoint)

suppose $A_m^c = \bigcup_1^{J_m} B_m^j$

$\left(\bigcup_1^n A_j\right)^c = \bigcap_1^n A_j^c = \bigcap_1^n \left(\bigcup_1^{J_m} B_m^j\right) = \bigcup \left(B_1^{j_1} \cap \cdots \cap B_n^{j_n}\right)$ $\square$

Measure is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfies
- $\mu(\phi) = 0$
- $\mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu(E_j)$ with $\{E_j\}$ are disjoint

like Lebesgue measure, a measure has the properties below

$\boxed{\text{thm}}$
- Monotonicity $E \subseteq F$. $E, F \in \mathcal{M} \Rightarrow \mu(E) \leq \mu(F)$
- Subadditivity $\{E_j\} \subseteq \mathcal{M} \Rightarrow \mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu(E_j)$
- Continuity from below $\{E_j\} \subseteq \mathcal{M}$ $E_1 \subseteq \cdots \subseteq E_n \subseteq \cdots$
$\Rightarrow \lim_{j \to \infty} \mu(E_j) = \mu\left(\bigcup_1^\infty E_j\right)$
- Continuity from above $\{E_j\} \subseteq \mathcal{M}$ $E_1 \supseteq \cdots \supseteq E_n \supseteq \cdots$ & $\mu(E_{n_0}) < \infty$ for ~~some~~
$\Rightarrow \lim_{j \to \infty} \mu(E_j) = \mu\left(\bigcap_1^\infty E_j\right)$

$\boxed{\text{EX}}$ 1.2.1 $\mathcal{R} \subseteq \mathcal{P}(X)$ is a ring if it's closed under ~~finite~~ finite and differences.

A ring is closed under countable unions is called a σ-ring.

a. "rings are closed under ~~finite~~ intersection"

it's trivial to see finite intersections of rings ~~are~~ is still a ring.

b. "A ring becomes an algebra iff $X \in \mathcal{R}$"

$(\Leftarrow)$ $E \in R$ $E^c = X \setminus E \in \mathcal{R}$

$(\Rightarrow)$ $\mathcal{R}$ is an algebra $E \setminus E = \phi \in R$. $\phi^c = X \in \mathcal{R}$ "

c. "S ~~denote~~ $:=$ $\{E \subseteq X : E \in \mathcal{R}$ or $E^c \in \mathcal{R}\}$ with $\mathcal{R}$ a σ-ring. then S is ~~a~~ σ-algebra"

$S \ni X$ since $\phi \in \mathcal{R}$

S is closed in complement trivial

S is closed in countable unions : consider $A: \{S_n\}_1^\infty \subseteq S$.

WLOG. divide it into two ~~sets~~ class $S_1 = \{s \in A \mid s \in R\}$

$S_2 = \{S \in A \mid S^c \in R\}$

rewrite their notations as $S_1, \cdots S_n \cdots$

$\tilde{S}_1^c, \cdots \tilde{S}_n^c \cdots$ (with $\tilde{S}_j \in R$)

then $\left(\bigcup_1^\infty S_j\right) \cup \left(\bigcup_1^\infty \tilde{S}_j^c\right) = \cancel{\left(\bigcup_1^\infty S_j\right) \cup \left(\bigcap_1^\infty \tilde{S}_j\right)^c}$

$\underbrace{\bigcup_1^\infty S_j}_{\in R} =: B$

$\cancel{B \cup \bigcap_1^\infty \tilde{S}_j}$

$\cancel{B \cup \left(\bigcup_1^\infty \tilde{S}_j^c\right) = \bigcup_1^\infty (B \cup \tilde{S}_j^c)}$

$\cancel{= \left(\bigcap_1^\infty (B \cup \tilde{S}_j^c)^c\right)^c}$

$\cancel{= \left(\bigcap_1^\infty (B^c \cap \tilde{S}_j)\right)^c}$

Consider $C = \bigcap_1^\infty \tilde{S}_j = \left(\bigcup \tilde{S}_j^c\right)^c \in R$

ring is closed under finite intersection

$$E_1 \setminus (E_1 \setminus E_2) = E_1 \cap (E_1 \cap E_2^c)^c = E_1 \cap (E_1^c \cup E_2) = E_1 \cap E_2$$

σ-ring

$$\bigcap_1^\infty E_k = E \setminus \left(E \setminus \left(\bigcap_1^\infty E_k\right)\right) = E \cap \left(E \cap \left(\bigcap_1^\infty E_k\right)^c\right)^c$$

$$= E \setminus \left(E \cap \left(\bigcup_1^\infty E_k^c\right)\right) = E \setminus \left(\bigcup_1^\infty (E \setminus E_k)\right)$$

$B \cup C^c$ $(B \cup C^c)^c = C \cap B^c = C \setminus B \in R$

$\Rightarrow B \cup C^c \in S$.

d. If $E \cap F \in \mathcal{R}$ for $\forall F \in R$

$E^c \cap F = F \setminus E = F \setminus (F \cap E) \in R \Rightarrow E^c \in S$.

$\left(\bigcup_1^\infty E_j\right) \cap F = \bigcup_1^\infty \underbrace{(E_j \cap F)}_{\in R} \in R \Rightarrow \bigcup_1^\infty E_j \in S$. $\square$

$\boxed{\text{Def}}$. A measure whose domain includes all subsets of null sets is called complete.

We can always make a measure complete by enlarging its domain as following.

$\boxed{\text{Thm}}$ suppose $(X, \mathcal{M}, \mu)$ is a measure space. ~~At~~ $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subseteq N \text{ for some } N \in \mathcal{N}\}$. then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension ~~of~~ $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

proof. since $\mathcal{M}$ and $\mathcal{N}$ ~~is C~~ are both closed under countable unions so is $\overline{\mathcal{M}}$. if $E \cup F \in \overline{\mathcal{M}}$ where $E \in \mathcal{M}$, $F \in \mathcal{N}$

F may not belong to $\mathcal{M}$

$F \subseteq N$

$\cancel{(E\cup F)^c = E^c \cap F^c \subseteq E^c \in \mathcal{M}}$

~~NOTE that $E \cup F = (E \setminus N) \cup (E \cup N) \cup \cap (N^c \cap F)$~~

WLOG. We can assume $E \cap N = \phi$. otherwise replace F & N by $\cancel{E\setminus N}$ $F\setminus E$ & $N \setminus E$

so $E \cup (F \setminus E) = E \cup F$ $\quad E \cap (N \setminus E) = \phi$

then $\quad E \cup F = (E \cup N) \cap (N^c \cup F)$

$(E \cup F)^c = (E^c \cap N^c) \cup (N \cap F) \in \overline{\mathcal{M}}$ so $\overline{\mathcal{M}}$ is a σ-algebra.

Set $\bar{\mu}(E \cup F) = \mu(E)$ if $E_1 \subseteq E_1 \cup F_1 = E_2 \cup F_2 \subseteq E_2 \cup N_2$

$\Rightarrow \mu_0(E_1) \leq \mu_0(E_2)$ and likewise $\mu(E_2) \leq \mu(E_1)$

$\Rightarrow \bar{\mu}$ is well-defined.

It is easy to show $\bar{\mu}$ is a measure, especially a complete one.

If $\hat{\mu}$ is another measure ~~to~~ extended from μ on $\overline{\mathcal{M}}$

$\forall\, E \cup F \in \overline{\mathcal{M}} \quad F \subseteq N$

$\boxed{\text{EX}}$ 1.3.6 $\Rightarrow \hat{\mu}(E \cup F) \leq \hat{\mu}(E \cup N) = \mu(E \cup N) = \mu(E) = \hat{\mu}(E)$

$\Rightarrow \cancel{\hat{\mu}(F) = 0}$ $\boxed{\hat{\mu}(E \cup F) = \mu(E)}$. $\square$

$\boxed{\text{EX}}$ 1.3.8 $\cancel{\mu(\bigcup_j \bigcap_{k} E_j)}$ $\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j\right) = \mu\left(\lim_{k\to\infty} \bigcap_{j=k}^{\infty} E_j\right)$

$\bigcap_{j=k}^{\infty} E_j \nearrow \liminf E_j$

$= \lim_{k\to\infty} \mu\left(\bigcap_{j=k}^{\infty} E_j\right)$

$\leq \lim_{k\to\infty} \inf_{j\geq k} \mu(E_j) = \liminf_{k\to\infty} \mu(E_j)$

$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j\right) = \mu\left(\lim_{k\to\infty} \bigcup_{j=k}^{\infty} E_j\right) \underline{\underline{\text{condition}}} \lim_{k\to\infty} \mu\left(\bigcup_{j=k}^{\infty} E_j\right) \geq \lim_{k\to\infty} \sup_{j\geq k} \mu(E_j)$ $\square$

$= \limsup_{k\to\infty} \mu(E_j)$

· EX 1.3.10 $\mu_E(\bigcup_1^\infty A_j) = \mu((\bigcup_1^\infty A_j)\cap E) = \mu(\bigcup_1^\infty(A_j\cap E)) = \sum_1^\infty \mu(A_j\cap E)$

with $\{A_j\}$ a sequence of disjoint sets. $= \sum_1^\infty \mu_E A_j$ □

Just like what we did in undergraduate real analysis, we introduce outer measure.

Def An outer measure on a nonempty set X is a function $\mu^*: \mathcal{P}(X)\to[0,\infty]$ that satisfies

- $\mu^*(\phi)=0$
- $\mu^*(A)\le\mu^*(B)$ if $A\subseteq B$
- $\mu^*(\bigcup_1^\infty A_j)\le\sum_1^\infty\mu^*(A_j)$

We previously achieve a premeasure to a outer measure, now we have

a generalization

Prop. $\mathcal{E}\subseteq\mathcal{P}(X)$ is a semi-algebra, and $\rho:\mathcal{E}\to[0,\infty]$ be such that $\phi\in\mathcal{E}$. $X\in\mathcal{E}$ and $\rho(\phi)=0$. For any $A\subseteq X$ define

$$\mu^*(A)=\inf\{\sum_1^\infty\rho(E_j): E_j\in\mathcal{E} \text{ and } A\subseteq\bigcup_1^\infty E_j\}$$

then μ^* is an outer-measure

proof: $\forall\varepsilon, \exists \bigcup_1^\infty E_j^i\supseteq A_i$ $\quad \mu^*(A_i)+2^{-i}\varepsilon\geqslant\sum\rho(E_j^i)$

$\Rightarrow \bigcup_1^\infty\bigcup_1^\infty E_j^i\supseteq\bigcup_1^\infty A_i$

$\sum_1^\infty\mu^*(A_i)+\varepsilon\geqslant\sum_1^\infty\sum_1^\infty\rho(E_j^i)\geqslant\mu^*(\bigcup_1^\infty A_i)$ □

Def A set $A\subseteq X$ is called μ^*-measure if $\mu^*(E)=\mu^*(E\cap A)+\mu^*(E\cap A^c)$ $\forall E\subseteq X$.

Rmk We only need to have the inequality $\boxed{\mu^*(E)\geqslant\mu^*(E\cap A)+\mu^*(E\cap A^c)}$

Thm (Carathéodory) If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measure sets is σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure. $\mu^*|_{\mathcal{M}}:=\mu$

proof: See real analysis note. □

Using caratheodory thm, we can extend measure from algebra to σ-algebra.

If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra. a function $\mu_0 : \mathcal{A} \to [0,\infty]$ is called **premeasure** if

- $\mu(\emptyset) = 0$
- if $\{A_j\}$ is a sequence of disjoint sets in $\mathcal{A}$, s.t. $\bigcup_1^\infty A_j \in \mathcal{A}$

then $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$

using last $\boxed{\text{prop}}$, we have the result below.

$\boxed{\text{prop}}$ If μ_0 is a premeasure on $\mathcal{A}$ and $\mu^*(E) := \inf\{\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_1^\infty A_j\}$ (*)

then • $\mu^*|_{\mathcal{A}} = \mu_0$

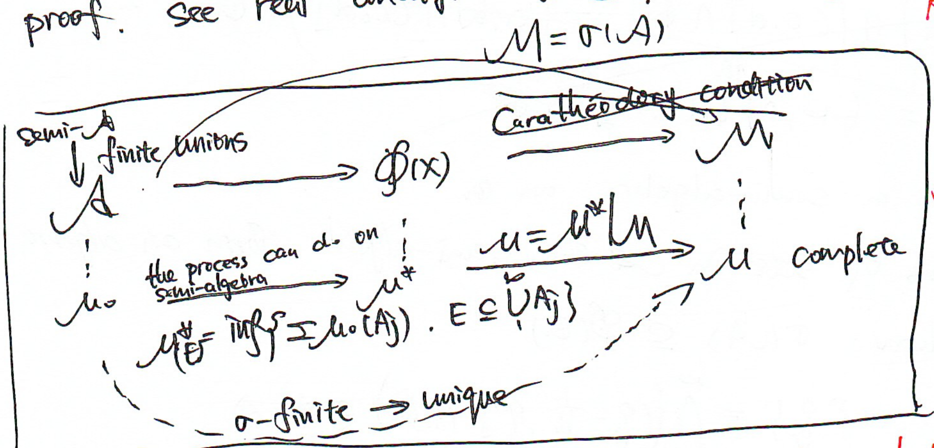
• every sets in $\mathcal{A}$ is μ^*-measure.

proof: see real analysis note. □

The thm we will introduce below concludes what we did previously and notes the uniqueness of our work.

$\boxed{\text{Thm}}$ Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$. $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 ($\mu|_{\mathcal{A}} = \mu_0$), ~~sim~~ likewise, $\mu = \mu^*|_{\mathcal{M}}$ (μ^* is defined by (*)). If ν is another measure defined by μ_0 (i.e. $\nu|_{\mathcal{A}} = \mu_0$), then $\nu(E) \leq \mu(E)$ for $\forall E \in \mathcal{M}$ with equality when $\mu(E) < \infty$. In particular, $\nu = \mu$ if μ_0 is σ-finite ($X = \bigcup_1^\infty A_j$, $\mu_0(A_j) < \infty$).

proof. see real analysis note. □



A mistake: in Folland's book, $\mathcal{M} := \sigma(\mathcal{A})$, But in my real analysis note $\mathcal{M} := \{\mu^*\text{-measurable sets}\}$. And we should use $\mathcal{M}^*$ to denote the latter one.

Fortunately, we needn't polish my proof a lot. (and the) Only thing requiring it's short "$\mathcal{M}^* \supseteq \mathcal{A}$, & $\mathcal{M}^*$ is a σ-algebra $\Rightarrow$ $\sigma(\mathcal{A}) \subseteq \mathcal{M}^*$".

The second part is similar.

~~If $E \in \mathcal{M}$, $E \subseteq \bigcup_1^\infty A_j$, $A_j \in \mathcal{A}$, $\nu(E) \leq \sum_1^\infty \nu(A_j) = \sum_1^\infty \mu_0(A_j)$ take inf~~

~~$\Rightarrow \nu(E) \leq \mu(E)$~~

$\boxed{\text{EX}}$ 1.4.17 $\boxed{\{A_j\} \text{ disjoint } \mu^*\text{-measure}}$

$$\sum_1^\infty \mu^*(E\cap A_j) \geq \mu^*\left(E\cap\left(\bigcup_1^\infty A_j\right)\right)$$

$$\mu^*(E) = \mu^*(E\cap A_j) + \mu^*(E\cap A_j^c)$$

since All the μ^*-measure sets become a σ-algebra $\mathcal{M}$

$$\Rightarrow \bigcup_1^\infty A_j \in \mathcal{M}$$

$$\Rightarrow \mu^*(E) = \mu^*(E\cap(\cup A_j)) + \mu^*\left(E\cap\left(\bigcup_1^\infty A_j\right)^c\right)$$

$$\Rightarrow \mu^*(E\cap(\cup A_j)) = \mu^*(E) - \mu^*\left(E\setminus\left(\bigcup_1^\infty A_j\right)\right)$$

$$\geq \mu^*(E) - \mu^*\left(E\setminus\left(\bigcup_1^n A_j\right)\right)$$

since ~~$\mu^*(E) = \mu^*\left(E\cap\left(\bigcup_1^n A_j\right)\right) + \mu^*\left(E\setminus\left(\bigcup_1^n A_j\right)\right)$~~

~~$= \mu^*\left(E\cap\left(\bigcup_1^{n-1} A_j\right)\cap\right.$~~

$$\mu^*(E) = \mu^*(E\cap A_1) + \mu^*(E\setminus A_1)$$

$$= \mu^*(E\cap A_1\cap A_2) + \mu^*((E\cap A_1)\setminus A_2) + \mu^*((E_1\setminus A_1)\cap A_2) + \mu^*((E_1\setminus A_1)\setminus A_2)$$

$$= \mu^*(E\cap(A_1\cup A_2)) + \mu^*(E\cap A_1) + \mu^*(E\cap A_2) + \mu^*(E\setminus(A_1\cup A_2))$$

$$\cdots \geq \sum_1^n \mu^*(E\cap A_j) + \mu^*\left(E\setminus\left(\bigcup_1^n A_j\right)\right)$$

in fact $\mu^*\left(E\cap\left(\bigcup_1^n A_j\right)\right) = \sum_1^n \mu^*(E\cap A_j)$

$$\Rightarrow \mu^*\left(E\cap\left(\bigcup_1^\infty A_j\right)\right) \geq \sum_1^n \mu^*(E\cap A_j)$$

$\square$

$\boxed{\text{EX}}$ 1.4.23. (a) ~~$((a,b]\cap\mathbb{Q})\cap((c,d]\cap\mathbb{Q}) = ((a,b]\cap(c,d])\cap\mathbb{Q} \in \cdots$~~

~~$((a,b]\cap\mathbb{Q})^c_{\mathbb{Q}} = (-\infty,a]\cap\mathbb{Q}\cup(b,+\infty]\cap\mathbb{Q}\cup$~~

trivial. since it's a semi-algebra on $\mathbb{Q}$

then finite unions of sets in the semi-algebra form an algebra

(b) it's easy to show $\sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$

On the other hand. $\{q\} = \bigcap_{k=1}^\infty \left((q-\frac{1}{k}, q]\cap\mathbb{Q}\right) \in \sigma(\mathcal{A})$

Any subsets in $\mathbb{Q}$ has countable elements. i.e. $\mathcal{P}(\mathbb{Q}) \subseteq \sigma(\mathcal{A})$

$$\Rightarrow \sigma(\mathcal{A}) = \mathcal{P}(\mathbb{Q}).$$

(c). counting measure & $\mu(A) = \begin{cases} 0 & A=\emptyset \\ \infty & A\neq\emptyset \end{cases}$

$\square$

~~We have shown that how to get an outer measure by μ_0 and then a~~

What we have shown:

$$\begin{tikzcd} \mu_0 \arrow[r] & \mu^* \arrow[r] & \mu \\ \mathcal{A} \arrow[rr, bend right] & \phi(X) & \mathcal{M} \end{tikzcd}$$

$\mathcal{M} = \sigma(\mathcal{A})$

A natural ~~of~~ question is that "$\mu^*|_{\mathcal{M}}$ is a measure $\mu^*|_{\mathcal{M}^*}$ is also a measure which is even complete, then what's the relation between them?"

NOTE that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is the smallest complete measure space containing $(X, \mathcal{M}, \mu)$

(So it's called ~~$(X, \mathcal{M}, \mu)$'s completion~~ completion space of $(X, \mathcal{M}, \mu)$)

so what's the relation between $\overline{\mu}$ & $\mu^*|_{\mathcal{M}^*}$?

Preciously, we should begin from a measure μ and space $(X, \mathcal{M}, \mu)$.

μ^* is the outer measure induced by μ with "$\forall E \subseteq X$, $\mu^*(E) = \inf\{\sum_1^\infty \mu(E_j), E_j \in \mathcal{M}, E \subseteq \bigcup_1^\infty E_j$

$\mathcal{M}$ is obviously an algebra $\xRightarrow{\text{by Prop}}$ μ^* is an outer measure.

Thm $\forall E \subseteq X$, $\exists C \in \mathcal{M}$, $C \supseteq E$, s.t. $\mu^*(E) = \mu(C)$

proof. $\boxed{1.}$ $\mu^*(C) = \mu(C)$ $\forall C \in \mathcal{M}$

- It's obvious $\mu^*(C) \leq \mu(C)$
- $\forall \varepsilon > 0$, $\exists \bigcup_1^\infty E_j \supseteq C$, s.t. $\mu^*(C) + \varepsilon \geq \sum_1^\infty \mu(E_j) \geq \mu(C)$

$\Rightarrow \mu^*(C) \geq \mu(C)$

$\boxed{2.}$ (Idea: equi-measure hull)

$\forall n \in \mathbb{N}$, $\exists \underbrace{\bigcup_1^\infty E_j^{(n)}}_{\in \mathcal{M}} \supseteq E$. s.t. $\mu^*(E) + \frac{1}{n} > \sum_1^\infty \mu(E_j^{(n)})$

let $C = \bigcap_{n=1}^\infty \bigcup_{j=1}^\infty E_j^{(n)} \in \mathcal{M}$, we claim $\mu(C) = \mu^*(E)$

On the one hand $E \subseteq C \in \mathcal{M} \Rightarrow \mu^*(E) \leq \mu(C)$

on the other hand, $\mu(C) = \mu(\bigcap_{n=1}^\infty \bigcup_{j=1}^\infty E_j^{(n)}) \leq \mu(\bigcup_{j=1}^\infty E_j^{(n)}) < \mu^*(E) + \frac{1}{n}$

the two sides of inequality above is independent of n

$\Rightarrow \mu(C) \leq \mu^*(E)$ $\square$

~~Thm~~ **Rmk** Consider the example of Borel measure and Lebesgue measure.

A lebesgue measurable set is a G_δ set $\setminus$ a null set.

here if $\mu^*(E) < \infty$, then $\mu^*(E) = \underbrace{\mu^*(C \cap E)}_{\mu^*(E)} + \underbrace{\mu^*(E \setminus C)}_{0}$

9. $\boxed{\text{Thm}}$ Let $(X, \mathcal{M}, \mu)$ a σ-finite measure space. Then $\overline{\mu} = \mu^*$

~~proof. One side is easy. By caratheodory theorem, $\mu^*|_{\mathcal{M}^*}$ is complete $\Rightarrow \overline{\mathcal{M}} \subset \mathcal{M}^*$.~~

~~For the other hand. If $\mu(X) < \infty$~~

~~$\forall E \in \mathcal{M}^*$ $\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \cap E^c)$ It suffices to show~~

~~E is the union of a measure set and subset set of a μ-null set.~~

~~By last thm, $\exists C \supseteq E$. s.t $\mu^*(E) = \mu(C) = \mu^*(C)$~~

~~$N := C \setminus E$ $\mu^*(N) = \mu^*(\tilde{N})$ with $\tilde{N} \subseteq N$, $\tilde{N} \in \mathcal{M}$~~

~~$E = (C \setminus \tilde{N}) \cup (E \cap \tilde{N}) \Rightarrow E \in \overline{\mathcal{M}}$~~

lemma. if μ is σ-finite, then $\forall E \subseteq X$. $\exists B \supseteq E$. $\mu^*(B \setminus E) = 0$

Suppose $X = \bigcup_1^\infty X_j$ $\mu(X_j) < \infty$, set $E_j = E \cap E_j$

$\forall \varepsilon > 0$. $\exists C_j \in \mathcal{A}_\sigma$ s.t $\mu^*(C_j) \leq \varepsilon \cdot \frac{1}{2^j} + \mu^*(E_j)$ with $E_j \subseteq C_j$

$\mu^*(C_j) = \mu^*(E_j \cap C_j) + \mu^*(E_j^c \cap C_j) = \mu^*(E_j) + \mu^*(C_j \setminus E_j)$

$\Rightarrow \mu^*(C_j \setminus E_j) \leq \varepsilon \cdot \frac{1}{2^j}$ set $B_\varepsilon = \bigcup_1^\infty C_j \in \mathcal{A}_\sigma$

$\mu^*(B_\varepsilon \setminus E) = \mu^*\left(\bigcup_{j=1}^\infty (C_j \cap E^c)\right) \leq \sum_1^\infty \mu^*(C_j \setminus E_j) \leq \varepsilon$

$\varepsilon = \frac{1}{k}$ $B = \bigcap_{k=1}^\infty B_k \in \mathcal{A}_{\sigma\delta}$ $\Rightarrow \mu^*(B \setminus E) = 0$ $\square$

In fact, we can show $\mu^*|_{\mathcal{M}^*} = \overline{\mu}$

If $E \in \mathcal{M}^* \Rightarrow E^c \in \mathcal{M}^*$ $\exists B^c \in \mathcal{M}$. $E^c \subseteq B^c$ with $\mu^*(B^c \setminus E^c) = 0$

i.e $\mu^*(E \setminus B) = 0$ Now. $E = (E \setminus B) \cup B$ $B \in \mathcal{M}$

$E \setminus B \subseteq A_n$ $\mu(A_n) \leq \mu^*(E \setminus B) + \frac{1}{n}$ $A := \bigcap_{n=1}^\infty A_n \in \mathcal{M}$

$\Rightarrow \mu(A) = \mu(E \setminus B) = 0$ $\Rightarrow E \subseteq A \cup B \Rightarrow E \in \overline{\mathcal{M}} \Rightarrow \mathcal{M}^* \subseteq \overline{\mathcal{M}}$

$E \cup F \in \overline{\mathcal{M}}$ $\boxed{F \subseteq N, \mu(N) = 0}$ $E, N \in \mathcal{M} \subseteq \mathcal{M}^*$

~~$\overline{\mu}(F) = \mu(N) = 0$~~ $\mu^*(N) = 0 \Rightarrow \mu^*(F) = 0$

$[\mu^*(S) \leq \mu^*(S \cap F) + \mu^*(S \cup F) = \mu^*(S \cup F) \leq \mu^*(S) + \mu^*(F) = \mu^*(S)]$

$\Rightarrow F \in \mathcal{M}^* \Rightarrow \overline{\mathcal{M}} \subseteq \mathcal{M}^*$ $\square$

~~By Caratheodory theorem~~ Rmk $A \in \overline{\mathcal{M}} \Rightarrow A = B \cup C$

$\overline{\mu}(A) = \mu^*(B) \leq \underset{\mu^*(A)}{\mu^*(B \cup C)} \leq \mu^*(B) + \underset{0}{\mu^*(C)} = \mu^*(B) = \overline{\mu}(B) = \overline{\mu}(A)$

Before introducing Borel measure, we first discuss something about "$\otimes$ Product".

Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection ~~component~~ of nonempty sets, $X=\prod_{\alpha\in A} X_\alpha$, $\pi_\alpha : X \to X_\alpha$ be the canonical projection. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the product σ-algebra is generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}, \alpha\in A\} =: \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$$

For the moment that A is countable, $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ can be represented by a more natural method shown below.

$\boxed{\text{prop}}$ If A is countable, then $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is the σ-algebra generated by

$\{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$ (temporarily we denote it by $\widetilde{\mathcal{M}}$)

proof. $\boxed{\prod_\alpha E_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(E_\alpha)} \Rightarrow \{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{M}_\alpha\} \in \bigotimes_\alpha \mathcal{M}_\alpha \Rightarrow \widetilde{\mathcal{M}} \subseteq \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$

On the other hand, $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta\in A} E_\beta$ ~~which~~ whose $E_\beta = X_\beta$ except $\beta\neq\alpha$ $\square$

Like what we met in Topology, we will "shrink" the origin of the algebra.

$\boxed{\text{prop}}$ Suppose $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$, $\alpha\in A$. Then $\bigotimes_\alpha \mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha\in A\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all $\alpha\in A$.

$\underbrace{\bigotimes_\alpha \mathcal{M}_\alpha \{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}}_{\text{is generated by } \mathcal{F}_2 =}$

proof. (1) Obviously $\mathcal{M}(\mathcal{F}_1) \subseteq \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$. On the other hand, for each $\alpha\in A$.

$\{$ ~~$\pi_\alpha^{-1}(E) : E\in\mathcal{E}_\alpha$~~ $E\subseteq X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\}$ is a σ-algebra, which contains $\mathcal{E}_\alpha$

$\Rightarrow \{E\subseteq X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\} = \mathcal{M}_\alpha$

(2) ~~$\mathcal{M}(\mathcal{F}_2) \subseteq \bigotimes_\alpha \mathcal{M}_\alpha$. On the other hand, $\{E\subseteq X_\alpha$~~

~~$\prod_\alpha E_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(E_\alpha)$~~ $\mathcal{M}(\mathcal{F}_2) \subseteq \bigotimes_{\alpha\in A} \mathcal{M}_\alpha$

$\bigotimes_\alpha \mathcal{M}_\alpha = \mathcal{M}(\mathcal{F}_1) = \mathcal{M}(\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\})$

$\subseteq \mathcal{M}(\mathcal{F}_2)$. $\square$

~~$\{E_\alpha : \exists \prod_\beta E_\beta \in \mathcal{M}(\mathcal{F}_2)\}$~~

~~$\bigcap_\beta \pi_\beta^{-1}(E_\beta^\alpha)$~~

1.

$\boxed{\text{prop}}$ Let $X_1, \cdots X_n$ be metric space. $X = \prod_1^n X_j$ ~~...~~ equipped with the $\boxed{\text{product metric}}$

(Here product metric take the maximum of each tuple). $\Rightarrow \bigotimes_1^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$

If X_j is separable for each $j \Rightarrow \bigotimes_1^n \mathcal{B}_n = \mathcal{B}_X$.

Why do we require "separable"

~~...~~ Consider $X_1 = (\mathbb{R}, d_{dis})$

every point is open

$X_2 = (\mathbb{R}, d_{usual})$

$\mathcal{B}_{X_1} = \mathcal{P}(\mathbb{R})$

$\mathcal{B}_{X_2} = \mathcal{B}_{\mathbb{R}}$

$\mathcal{B}_X = \mathcal{P}(\mathbb{R}^2)$

proof: $\bigotimes_1^n \mathcal{B}_{X_j}$ is generated by $\underbrace{\boxed{\pi_j^{-1}(U_j)}}_{\text{open in } X}$, $U_j \in \mathcal{T}_{X_j}$

$\Rightarrow \bigotimes_1^n \mathcal{B}_j \subseteq \mathcal{B}_X$

If X_j is separable for each j. ~~$\exists\, \mathcal{E}_j \subseteq X_j$~~

~~$\mathcal{E}_j$ is countable dense set.~~ $A_2 \overset{\text{metric}}{\Longleftrightarrow}$ separable

$\Rightarrow \mathcal{E}_j \subseteq \mathcal{P}(X_j)$ ~~...~~ and $\mathcal{E}_j$ is a countable collection of some balls.

$\Rightarrow \mathcal{B}_{X_j}$ is generated by $\mathcal{E}_j$ $\mathcal{B}_X$ is generated by $\{\prod_1^n E_j : E_j \in \mathcal{E}_j\}$ $\square$.

$\boxed{\text{Cor}}$ $\bigotimes_1^n \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$

Now, We can discuss Borel measure, and ~~...~~ we will obtain Lebesgue-Stieltjes measure.

Let $\mathcal{C} =$ ~~$\{\prod (a_i, b_i] : a_i \le b_i\}$~~ $\{\prod_{i=1}^n (a_i, b_i] : a_i \le b_i,\ a_i, b_i \in \mathbb{R}\}$

$\mu_0(x) = \prod_{i=1}^n (b_i - a_i)$ $\Rightarrow$ $\mathcal{C}$ is a semi-ring. μ_0 is a ~~...~~ function with σ-additive on $\mathcal{C}$.

Enlarge our basic set a little, we ~~...~~ actually possess a semi-algebra ~~and~~ and a premeasure on it.

$\boxed{\text{prop}}$ $F: \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$ $(j = 1, \cdots n)$ are disjoint h-intervals, let $\mu_0(\bigcup_1^n (a_j, b_j]) = \sum_1^n (F(b_j) - F(a_j))$

and let $\mu_0(\emptyset) = 0$. Then μ_0 is a premeasure on the $\mathcal{A}$ (finite union of h-intervals)

proof. (1) μ_0 is well-defined. If $\{(a_j, b_j]\}_{j=1}^n$ are disjoint and $\bigcup_1^n (a_j, b_j] = (a, b]$

$a = a_1 < b_1 = a_2 < \cdots < b_n = b$. So $\sum_1^n F(b_j) - F(a_j) = F(b) - F(a)$

Generally, $\{I_i\}_{i=1}^n$ & $\{J_j\}_{j=1}^m$ with $\bigcup_1^n I_i = \bigcup_1^m J_j$

Then we have $\sum_1^n \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_j \mu_0(J_j)$

It remains to show that if $\{I_j\}$ is a sequence of disjoint h-intervals with $\bigcup_1^\infty I_j \in \mathcal{A}$

Since $\bigcup_1^\infty I_j \in \mathcal{A}$ (finite unions of h-intervals), WLOG, we assume $\bigcup_1^\infty I_j = (a, b] = I$

$\mu_0(I) = \mu_0(\bigcup_1^n I_j) + \mu_0(I \setminus \bigcup_1^n I_j) \ge \sum_1^n \mu_0(I_j) \Rightarrow \mu_0(I) \ge \sum_1^\infty \mu_0(I_j)$

For the reverse inequality.

$\forall \varepsilon > 0$. $\exists \delta > 0$, s.t $F(a+\delta) - F(a) < \varepsilon$, then we can consider $[a+\delta, b]$.

For $I_j = (a_j, b_j]$. $\exists \delta_j > 0$, s.t $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$, $\hat{I}_j = (a_j, b_j + \delta_j)$

then we possess a new sequence of intervals covering ~~$I = (a,b]$~~ $[a+\delta, b]$

~~μ_0~~ using compactness $\bigcup_1^n \hat{I}_j \supseteq [a+\delta, b]$ (After realabelling)

$\boxed{\mu_0(I)} = F(b) - F(a) < F(b) - F(a+\delta) + \varepsilon$

$\leq F(b_n + \delta_n) - F(a_1) + \varepsilon = F(b_n + \delta_n) - F(a_n) + \sum_1^{n-1} (F(a_{j+1}) - F(a_j)) + \varepsilon$

$(b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1}))$

$\leq F(b+\delta_n) - F(a_n) + \sum_1^{n-1} (F(b_{j+1} + \delta_{j+1}) - F(a_j)) + \varepsilon$

$\leq \sum_1^N (F(b_j) + 2^{-j}\varepsilon - F(a_j)) + \varepsilon$

$< \sum_1^\infty \mu_0(I_j) + 2\varepsilon$

$\varepsilon \to 0$ If $a = -\infty$ or $b = \infty$. consider $M > 0$ $(M, b]$ or $(a, M]$ $\begin{cases} \varepsilon \to 0 \\ M \to \infty. \end{cases}$ $\square$

The technique above is worth noticing.

$\boxed{\text{Thm}}$ If $F: \mathbb{R} \to \mathbb{R}$ is any increasing and right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) := \mu(b) - \mu(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets. and we define

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0. \end{cases}$$

then F is increasing and right continuous and $\mu = \mu_F$

proof: Each F induced a premeasure on $\mathcal{A}$ by last prop.

$\begin{cases} F - G \text{ is constant} \Rightarrow \mu_F = \mu_G \\ \mu_F = \mu_G \quad \mu_F((0,x]) = F(x) - F(0) = G(x) - G(0) = \mu_G((0,x]) \end{cases}$

$\Rightarrow \boxed{F(x) - G(x) = F(0) - G(0)}$.

since $\mathbb{R} = \bigcup_{-\infty}^{\infty} (j, j+1] \Rightarrow \mu_0$ is σ-finite $\Rightarrow \mu$ is unique!

The monotone of $\mu \Rightarrow F$ is increasing

The continuity of $\mu \Rightarrow F$ is right continuous. $\begin{cases} x > 0 \\ x < 0 \end{cases}$ so $\mu = \mu_F$ on $\mathcal{A}$

$\Rightarrow \mu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$ by the uniqueness $\square$

Rmk. 1. Consider $[a, b)$ is the same.

2. If μ is finite Borel measure on $\mathbb{R} \Rightarrow \mu = \mu_F$ where $F = \mu((-\infty, x])$ is the cumulative distribution function of μ.

3. One can show $\bar{\mu}_F$'s domain is always strictly larger than $\mathcal{B}_{\mathbb{R}}$

And the complete measure is called "Lebesgue ~~Stie~~ Stieltjes measure associated to F

3. Now we fix a ~~complete~~ measure μ on $\mathbb{R}$ associated to increasing, right continuous
Lebesgue stieltjes
function F. We ~~den~~ denote by $\mathcal{M}_\mu$ the domain of μ. Thus, for any

$$E \in \mathcal{M}_\mu, \quad \mu(E) = \inf\left\{\sum_1^\infty F(b_j) - F(a_j) : E \subseteq \bigcup_1^\infty (a_j, b_j]\right\}$$
$$= \inf\left\{\sum_1^\infty \mu((a_j, b_j]) : E \subseteq \bigcup_1^\infty (a_j, b_j]\right\}$$

Now, to better deal with inner and outer regularity of Borel measure, we shall shift the half-interval in the definition to open sets. And it's a preparation for Radon measure actually.

$\boxed{\text{Lem}}$ for $\forall E \in \mathcal{M}_\mu$, $\mu(E) = \inf\left\{\sum_1^\infty \mu(a_j, b_j) : E \subseteq \bigcup_1^\infty (a_j, b_j)\right\}$. $(=: \nu(E))$

proof. ~~$\forall \varepsilon > 0.\ \exists \bigcup_1^\infty (a_j, b_j] \supseteq E$~~ ~~$\leq \mu(E) + \varepsilon$~~

NOTE: In the premeasure we don't directly understand (a_j, b_j)'s measure from itself

Each (a_j, b_j) is a countable disjoint unions of h-interval I_j^k

for $I_j^k = (c_j^k, c_j^{k+1}]$ $\quad c_j^1 = a_j \quad c_j^k \nearrow b_k$

$$\Rightarrow E \subseteq \bigcup_{j,k=1}^\infty I_j^k \quad \sum_1^\infty \mu((a_j, b_j)) \overset{\text{cts}}{=} \sum_{j,k=1}^\infty \mu(I_j^k) \geq \mu(E)$$

$\Rightarrow \nu(E) \geq \mu(E)$

On the other hand $\forall \varepsilon > 0$. $\exists \bigcup_1^\infty (a_j, b_j] \supseteq E$. with $\sum_1^\infty \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$

$$F(b_j + \delta_j) - F(b_j) < \varepsilon \cdot 2^{-j} \Rightarrow E \subseteq \bigcup(a_j, b_j + \delta_j)$$

$$\nu(E) \leq \sum_1^\infty \mu(a_j, b_j + \delta_j) \leq \sum_1^\infty \mu((a_j, b_j + \delta_j]) \leq \sum_1^\infty F(b_j) - F(a_j) + \varepsilon \leq \mu(E) + 2\varepsilon$$

$\Rightarrow \nu(E) \leq \mu(E)$ $\square$

For latter discussion, we shall always consider $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{\text{stand}})$ as an example.

$\boxed{\text{Thm}}$ If $E \in \mathcal{M}_\mu$, then $\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ is open}\}$
$= \sup\{\mu(K) : K \subseteq E \text{ is compact}\}$

proof By lemma, $\forall \varepsilon > 0$ $\exists \underbrace{\bigcup_1^\infty (a_j, b_j)}_{= U} \supseteq E$ & $\underbrace{\sum_1^\infty \mu((a_j, b_j))}_{\mu(U) \leq} \leq \mu(E) + \varepsilon$

(1) $\Rightarrow \mu(E) \leq \mu(U) \leq \mu(E) + \varepsilon$

(2) If E is ~~closed~~ Bounded
- E is closed $\Rightarrow E$ is compact ✓
- E is not ~~open~~ compact $\Rightarrow \mu(\bar{E} \setminus E) + \varepsilon \geq \mu(U)$ $U \supseteq \bar{E} \setminus E$

$K = \bar{E} \setminus U = (E \cup (\bar{E} \setminus E)) \setminus U$
$= (E \setminus U) \cup (\bar{E} \setminus E \setminus U)$
$= E \setminus U)$

let $K = \bar{E} \setminus U$ is compact $K \subseteq E$

$\mu(E) \geq \mu(K) = \mu(E) - \mu(E \cap U)$
$= \mu(E) - (\mu(U) - \mu(U \setminus E))$
$\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E)$
$\geq \mu(E) - \varepsilon$

$\Rightarrow \mu(E) = \cdots$

If E is unbounded, let $E_j = E \cap [j, j+1]$.
$\mu(K_j) \geq \mu(E_j) - \varepsilon \frac{1}{3 \cdot 2^{-j}}$ $\quad H_n = \bigcup_{-n}^{n} K_j$ $\quad \mu(H_n) \geq \mu\left(\bigcup_{-n}^{n} E_j\right) - \varepsilon$ $\square$

$\boxed{\text{Thm}}$ If $E \subseteq \mathbb{R}$, TFAE

- $E \in \mathcal{M}_\mu$
- $E = V \setminus N_1$, V is a G_δ set and $\mu(N_1)=0$
- $E = H \cup N_2$, H is an F_σ set and $\mu(N_2)=0$

proof: If $E \in \mathcal{M}_\mu$ and $\mu(E)<\infty$. $\forall j \in \mathbb{N}$

$\exists U_j, K_j$ s.t. $K_j \subseteq E \subseteq U_j$

$\mu(U_j) - 2^{-j} \le \mu(E) \le \mu(K_j) + 2^{-j}$

let $V = \bigcap\limits_1^\infty U_j$ $\quad H = \bigcup\limits_1^\infty K_j \Rightarrow H \subseteq E \subseteq V$

$\mu(V) = \cancel{\lim \mu(U_j)}\ \mu(E) = \mu(H) < \infty$

$\Rightarrow \mu(V \setminus E) = \mu(E \setminus H) = 0$

If $\mu(E)=\infty$ $\quad E_j = \cancel{[j,j]\cap E}\ E \cap [j, j+1)$ $\quad \mu(E_j) \le 1 < \infty$

$\cancel{\Rightarrow \mu(E_j)<\infty.\ E_j = U_j \setminus N_j}$ $\exists K_n^j \subseteq E_j$ $\quad K_n^j$ is an F_σ set

$\cancel{U(V_j \cap N_j)\ \text{let}}$. $\mu(E_n \setminus K_n^j) \le \frac{1}{2^{|n|+k}}$

let $K_j = \bigcup\limits_{-\infty}^{\infty} K_n^j$ $\quad E \setminus K_j = \left(\bigcup\limits_{n=-\infty}^{\infty} E_n\right) \cap \left(\bigcup\limits_{n=-\infty}^{\infty} K_n^j\right)^c$

$= \bigcup\limits_{n=-\infty}^{\infty} (E_n \setminus K_n^j)$

$\mu(E \setminus K_j) \le \frac{3}{2^k}$ $\quad K = \bigcup K_j \Rightarrow \mu(E \setminus K) = 0$

$\mu(E^c \setminus K) = 0 \longrightarrow E^c \cap K^c = \underbrace{K^c}_{G_\delta} \setminus E$ $\quad \mu(K^c \setminus E) = 0.$ $\square$

$\boxed{\text{prop}}$. If $E \in \mathcal{M}_\mu$, $\mu(E)<\infty$. ~~then~~ for $\forall \varepsilon>0$. ~~there~~ is a set A ~~that~~ is a finite union of open intervals such ~~that~~ $\mu(E \triangle A) < \varepsilon$

proof. $\forall \varepsilon>0$. $\exists U \supseteq E$ $\quad \mu(E) + \varepsilon/3 \ge \mu(U)$

$U \subseteq \mathbb{R}$ is consisted ~~of countable~~ disjoint open intervals. suppose $U = \bigcup\limits_{n=1}^{\infty} I_n$

Since $\mu(E)<\infty \Rightarrow \mu(U)<\infty$ i.e. $\sum\limits_{n=1}^{\infty} \mu(I_n) < \infty$

$\exists N>0$. s.t. $\sum\limits_{n=N+1}^{\infty} \mu(I_n) < \varepsilon/3$. $\quad$ let $A = \bigcup\limits_{n=1}^{N} I_n$

$E \triangle A = (E \setminus A) \cup (A \setminus E)$

$\mu(E \setminus A) \le \mu\left(\bigcup\limits_{N+1}^{\infty} I_n\right) < \varepsilon/3$

$\mu(A \setminus E) \le \mu(U \setminus E) \le \varepsilon/3$

$\Rightarrow \mu(E \triangle A) < \varepsilon$ $\quad \square$

5. Now we take ~~one~~ further step from Borel measure on $\mathbb{R}$.

In Evans' book, ~~the~~ concept "Borel regular" is based on ~~topological~~ space $X=\mathbb{R}^n$

An outer measure μ^* on X is said ~~to~~ be Borel regular if all Borel sets are μ^*-measure and ~~if~~ for each $A \subseteq X$. ~~there~~ exists a Borel set B s.t. $A \subseteq B$ and $\mu(A) = \mu(B)$

~~$\boxed{\text{Def}}$ A Borel regular measure μ on X is "open σ-finite" if $X = \bigcup_j V_j$ where V_j is open in X and $\mu(V_j) < \infty$ for each $j = 1, \cdots$~~

~~$\boxed{\text{Thm}}$~~

Here I decided not to introduce too much about Radon measure, which is a measure defined on a LCH space (A space with ~~properties~~ good enough to do some abstract analysis) with ~~great~~ regularity.