Comparing to the definition of measurable functions in Real Analysis note, we have the generalized definition.

$\boxed{\text{Def}}$ If $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are measurable spaces, a map $f: X \to Y$ is called $((\mathcal{M}, \mathcal{N})-)$ measurable if $f^{-1}(E) \in \mathcal{M}$ for $\forall E \in \mathcal{N}$

It is obvious ~~that~~ the composition of measurable maps is measurable, more precisely, if $f: (X, \mathcal{M}) \to (Y, \mathcal{N})$, $g: (Y, \mathcal{N}) \to (Z, \mathcal{L})$

$g \circ f$ is measurable.

$\boxed{\text{prop}}$ if $\mathcal{N} = \sigma(\mathcal{E})$, then $f: X \to Y$ is measurable iff $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{E}$

proof: $\{f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra that contains $\mathcal{E}$

$\Rightarrow \{f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{N}$ $\square$

$\boxed{\text{Cor}}$ If X, Y are topological spaces, then every continuous function $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.

Now, $\boxed{\mathcal{B}_{\mathbb{R}} \text{ or } \mathcal{B}_{\mathbb{C}}}$ is always understood as the σ-algebra on the range ~~algebra~~ space unless otherwise specified.

So, the composition of two Lebesgue measurable functions may not be measurable. $f: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, $g: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Like induced ~~product~~ topology, given a set X, $\{(Y_\alpha, \mathcal{N}_\alpha)\}$ is a family of measurable spaces, $f_\alpha: X \to Y_\alpha$. there is a unique smallest σ-algebra on X ~~that~~ wrt which the f_α are all measurable, namely, the σ-algebra generated by $\{f_\alpha^{-1}(E): E \in \mathcal{N}_\alpha, \alpha \in A\}$. And as for product measure, we have $f_\alpha = \pi_\alpha$

$\boxed{\text{prop}}$ $(X, \mathcal{M})$, $(Y_\alpha, \mathcal{N}_\alpha)$ $(\alpha \in A)$ are measurable spaces. $Y = \prod_{\alpha \in A} Y_\alpha$

$\mathcal{N} = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha$. $\pi_\alpha: Y \to Y_\alpha$. then $f: X \to Y$ is measurable iff $\pi_\alpha \circ f = f_\alpha$ is measurable

proof ($\Rightarrow$) trivial

($\Leftarrow$) $f^{-1}(\underline{\pi_\alpha^{-1}(E_\alpha)}) = f_\alpha^{-1}(E_\alpha)$ $\in \mathcal{M}$ is measurable for $\forall E_\alpha \in \mathcal{N}_\alpha$

they generate $\bigotimes_{\alpha \in A} \mathcal{N}_\alpha$. $\square$

7. The following propositions are familiar enough.

[Cor] $f: X \to \mathbb{C}$ is measurable $\Leftrightarrow$ $\mathrm{Re} f$, $\mathrm{Im} f$ are measurable

[Prop] $f, g: X \to \mathbb{C}$ are $\mathcal{M}$-measurable, so are $f+g$ & $f \cdot g$.

[Prop] If $\{f_j\}$ is a sequence of $\bar{\mathbb{R}}$ valued measurable functions on $(X, \mathcal{M})$

then $g_1(x) = \sup_j f_j(x)$ $\quad g_3(x) = \limsup_{j\to\infty} f_j(x)$ are measurable.

$g_2(x) = \inf_j f_j(x)$ $\quad g_4(x) = \liminf_{j\to\infty} f_j(x)$

I forgot it...

[Def] ~~for f: X →~~ $E \in \mathcal{M}$ $\quad f$ is measurable if $f^{-1}(B) \cap E$ $(B \in \mathcal{B})$

i.e. $f|_E$ is $\mathcal{M}_E$-measurable where $\mathcal{M}_E = \{F \cap E : F \in \mathcal{M}\}$.

[Cor] $f, g: X \to \bar{\mathbb{R}}$ are measurable then $\max(f, g)$ & $\min(f, g)$ are measurable

[Cor] --- $f^+(x) := \max(f(x), 0)$ $\quad f^-(x) = \max(-f(x), 0)$

$\Rightarrow f = f^+ - f^-$

If $f: X \to \mathbb{C}$, we have its polar decomposition

$$f = (\mathrm{sgn}\, f) \cdot |f| \qquad \mathrm{sgn}\, z = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$$

sgn is cts except $\{0\}$

$\mathrm{sgn}^{-1}(U)$ = open sets or $V \cup \{0\}$ $\Rightarrow$ sgn is Borel-measurable.

$\Rightarrow$ $|\cdot| \circ f$, $\mathrm{sgn} \cdot f$ are measurable.

[Def] A simple function $f: X \to \mathbb{C}$ is a finite linearity combination of characteristic functions of measurable sets with complex coefficients (Not allow to assume ∞)

$$f = \sum_1^n z_j \chi_{E_j} \qquad E_j = f^{-1}(z_j) \qquad \mathrm{range}(f) = \{z_1, \dots, z_n\}$$

f's stand representation

[Thm] $(X, \mathcal{M})$,

(1) $f: X \to [0, \infty]$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions s.t. $0 \le \phi_1 \le \cdots \le f$, $\phi_n \nearrow f$ pointwise and $\phi_n \nearrow f$ uniformly on any sets ~~on the~~ where f is bounded

(2) $f: X \to \mathbb{C}$. $0 \le |\phi_1| \le \cdots \le |f|$. $\phi_n \nearrow f$ pointwise. $\phi_n \rightrightarrows f$ on any sets where $|f| < \infty$.

Why do we consider completion of a measure? The discussion following will show an answer to it.

$\boxed{\text{prop}}$ The followings are valid iff μ is complete

- If f is measurable and $f = g$ μ-a.e, then g is measurable
- If f_n is measurable for $n \in \mathbb{N}$, and $f_n \to f$ μ-a.e. then f is measurable

proof: (1) ~~If μ is complete. $E = F_0 \cup N \in \mathcal{N}$~~

~~$f^{-1}(F_0 \cup N) = f^{-1}(F_0) \cup f^{-1}(N)$ is measurable~~

~~$g^{-1}(E)$ differs $f^{-1}(E)$ from a subset of null-set~~ $\Rightarrow g$ is measurable.

$A = \{f \neq g\}$ is a μ-null set.

then for $\forall E \in \mathcal{N}$. $g^{-1}(E)$ differs $f^{-1}(E)$ from a subset of A

i.e. a μ-null set $\Rightarrow g^{-1}(E)$ is measurable $\forall E \in \mathcal{N}$

Let $f = \chi_F \Rightarrow g = \chi_N$ $\quad N \subseteq F \in \mu\text{-null}$ $\Rightarrow N$ is measurable $\Rightarrow \mu$ is complete

for that F, N are arbitrary.

(2) ~~convergent point set: $\bigcap_{l} \bigcup_{j} \bigcap_{k \geq j} \{|f_k - f| < \frac{1}{l}\}$~~

~~$\bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{|f_k - f| \geq \frac{1}{l}\} = A$ is μ-null set~~

E denote the convergent point set. $f_n \to f \Rightarrow f$ is measurable on E

& E^c is μ-null set $\Rightarrow f$ is measurable

~~$f^{-1}(S) = f^{-1}(S \cap E)$~~

N denote a μ-null set $F \subseteq N$ $\quad f_n = 0 \quad f = \chi_F$

$f_n \to f$ on $N^c \Rightarrow f$ is measurable $\Rightarrow F$ is measurable

---

Rmk. How to understand μ-a.e.

In wikipedia on Folland's book. a property P holds μ-a.e. if there exists a μ-null set N s.t. all $x \in X \setminus N$ have the property P.

So μ-a.e. doesn't mean the $\{x \in X, \neg P(x)\}$ has measure zero.

9. $\boxed{\text{prop}}$ Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \bar{\mathcal{M}}, \bar{\mu})$ be its completion. If f is a $\bar{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g$ $\bar{\mu}$-a.e.

proof. For this type of questions, we first ~~se~~ start from Characteristic func.

If $f = \chi_E$ where E is $\bar{\mu}$-measurable, we can write E as $E = E_1 \cup N_1$ where N_1 is a subset of μ-null set $\underline{\underline{N}}$. let $g = \chi_E$

$\Rightarrow f = g$ ~~a~~ in N^c. For general case, if f is $\bar{\mathcal{M}}$-measurable

Hence f is a $\bar{\mathcal{M}}$-measurable simple function

there exists a sequence of $\bar{\mu}$-measurable simple functions $\{\phi_n\}$ with $\phi_n \to f$ pointwise. For each n, let ψ_n be an $\mathcal{M}$-measurable simple function with $\phi_n = \psi_n$ except $E_n \in \bar{\mathcal{M}}$ with $\bar{\mu}(E_n) = 0$.

Choose $N \in \mathcal{M}$ such that $\mu(N) = 0$ and $N \supseteq \bigcup_1^\infty E_n$

And set $g = \lim \chi_{X \setminus N} \psi_n$. Then g is $\mathcal{M}$-measurable and $g = f$ on $X \setminus N$. $\square$

Now, we start to define our integrals on $(X, \mathcal{M}, \mu)$ which is a fixed measure space. Firstly, clarify some notations.

L^+ = the space of all measurable functions from X to $[0, \infty]$

If ~~ϕ~~ a simple function in L^+ with $\boxed{\text{standard representation}}$ $\phi = \sum_1^n a_j \chi_{E_j}$

$$\int \phi \, d\mu := \sum_1^n a_j \mu(E_j) \quad \text{(may equal } \infty\text{)}$$

With convention. $0 \cdot \infty = 0$

$$\int_A \phi \, d\mu = \int_A \phi = \int_A \phi(x) \, d\mu(x) = \int \phi \chi_A \, d\mu \quad (A \in \mathcal{M}) \quad \int = \int_X$$

$\boxed{\text{prop}}$ Let ϕ and ψ be simple functions in L^+; $\phi = \sum_1^n a_j \chi_{E_j}$, $\psi = \sum_1^m b_k \chi_{F_k}$

(a) If $c \geq 0$ $\int c\phi = c \int \phi$

(b) $\int (\phi + \psi) = \int \phi + \int \psi$

(c) If $\phi \leq \psi$, then $\int \phi \leq \int \psi$

(d) The map $A \mapsto \int_A \phi \, d\mu$ is a measure on $\mathcal{M}$

proof. The first three properties require us to use the intersections to divide set into disjoint parts. For example $\int \phi = \sum_{j,k} a_j \mu(E_j \cap F_k) \leq \sum_{j,k} b_k \mu(E_j \cap F_k) = \int \psi$

The last one is interesting to discuss.

All terms are positive

Let $A = \underset{\text{disjoint}}{\bigcup A_k}$: $\int_A \phi \overset{\text{def}}{=} \sum_j a_j \mu(A \cap E_j) = \sum_j a_j \sum_k \mu(A_k \cap E_j) \overset{\Downarrow}{=} \sum_k \int_{A_k} \phi \, d\mu$ $\square$

We now extend the integral to functions in L^+ by defining

$$\int f d\mu = \sup\left\{ \int \phi \, d\mu : 0 \le \phi \le f, \phi \text{ is simple} \right\}$$

If f is simple. the two definitions of f agree.

By the definition, it's not hard to see:

(a) $\int f d\mu \le \int g d\mu$ if $f \le g$

(b) $c \int f d\mu = \int cf d\mu$ for $\forall c \in [0, \infty)$

Then, we aim to establish some ~~converge~~ convergence Theorem.

[Rmk] The following results ~~we~~ that we will introduce immediately have been shown in my real analysis note. But here I would prove them one more ~~time~~ as reviewing.

[Thm] MCT If $\{f_n\}_{n=1}^{\infty} \subseteq L^+$, $f_j \le f_{j+1}$ $j \in \mathbb{N}$, $f = \lim_{n\to\infty} f_n (= \sup f_n)$ ~~i.e. pointw~~

then $\int f = \lim_{n\to\infty} \int f_n$

proof: It's obvious that $\int f \ge \int f_n$. It suffices to show the reverse.

Fix $\alpha \in (0,1)$, let ϕ be a simple function ~~that~~ with $0 \le \phi \le f$.

let $E_n = \{ f_n \ge \alpha\phi \}$. Then we shall set $\{E_n\}$ is an increasing sequence of measurable sets whose limit is the whole space X.

$$\Rightarrow \int f_n d\mu \ge \int_{E_n} f_n d\mu \ge \int_{E_n} \alpha\phi \, d\mu = \alpha \int_{E_n} \phi \, d\mu$$

By the continuity of measure, we have

$$\lim_{n\to\infty} \int f_n d\mu \ge \alpha \int \phi \, d\mu \text{ for } \forall \alpha \in (0,1) \text{ \& } 0 \le \phi \le f$$

$$\Rightarrow \lim_{n\to\infty} \int f_n \ge \int f \qquad \square$$

[Cor] $\{f_n\}$ is a finite ~~or~~ or infinite sequence in L^+ and $f = \sum_n f_n$ then $\int f = \sum_n \int f_n$

[prop] If $f \in L^+$, then $\int f = 0$ iff $f = 0$ μ-a.e.

proof: If $f = 0$ a.e. $\Rightarrow \forall\, 0 \le \phi \le f$, $\phi = 0$ a.e. $\int f \overset{\sup}{=} \int \phi = 0$

on the other hand $\{ f(x) > 0 \} = \bigcup_n \underbrace{\{ f > \tfrac{1}{n} \}}_{:E_n}$ If $f = 0$ a.e. ~~$\mu(E_n)$~~ is false,

$\exists n$ $\mu(E_n) > 0$. $\Rightarrow \int f \ge \int_{E_n} f > \frac{1}{n}\mu(E_n) > 0$ ↯ $\square$

[Cor] $\{f_n\} \subseteq L^+$, $f \in L^+$, $f_n \nearrow f$ for a.e. x, then $\int f = \lim_{n\to\infty} \int f_n$

proof. $f_n \nearrow f$ on E where $\mu(E^c) = 0$ $\Rightarrow f - f\chi_E = 0$ a.e. $\overset{MCT}{\Rightarrow}$ $\square$

$f_n - f_n\chi_E = 0$ a.e.

21.

Note that, there are many cases that the limit of integral not equals the integral of the limit, but in that case, there is an inequality that remains valid.

$\boxed{\text{Thm}}$ (Fatou's lemma) If $\{f_n\}$ is any sequence ~~of~~ in L^+, then

$$\int \liminf f_n \le \liminf \int f_n$$

proof. For each $k \ge 1$ $\inf_{n\ge k} f_n \le f_j$ for $j \ge k$

hence $\int \inf_{n\ge k} f_n \le \int f_j$ for $j \ge k$

hence $\int \inf_{n\ge k} f_n \le \inf_{j\ge k} \int f_j$. Let $k \to \infty$, by MCT, we have

$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$ □

$\boxed{\text{Cor}}$ $\{f_n\} \subseteq L^+$, $f \in L^+$, $f_n \to f$ a.e. $\Rightarrow \int f \le \liminf_{n\to\infty} \int f_n$

$\boxed{\text{prop}}$ If $f \in L^+$, $\int f < \infty$, then $\{f(x) = \infty\}$ is μ-null set and $\{f(x) > 0\}$ is σ-finite.

proof. If $\{f = \infty\}$ is not a null set. $\mu(\{f=\infty\}) = a > 0$

$\Rightarrow \mu(\{f > N\}) \ge \mu(\{f = \infty\}) = a > 0$

$\Rightarrow \int f \ge \int_{\{f>N\}} > Na$ let $N \to \infty \Rightarrow \int f = \infty$ ↯

~~$E_n = \{n-1 < f \le n\} \Rightarrow X = \bigcup_1^\infty E_n \cup \{f = \infty\}$~~

~~Since $\int f < \infty$, $\int f \ge \int_{E_n} f \ge (n-1)\mu(E_n) \Rightarrow \mu$~~

$E_n = \{f \ge \frac{1}{n}\}$ $\{f(x) > 0\} = \bigcup_1^\infty E_n$

$\infty > \int f \ge \int_{E_n} f \ge \frac{1}{n}\mu(E_n) \Rightarrow \mu(E_n) < \infty$ □

$\boxed{\text{EX}}$ 2.1.2.

lem. $f: X \to \overline{\mathbb{R}}$. $Y = f^{-1}(\mathbb{R})$, then f is measurable iff $\begin{cases} f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M} \\ f \text{ is measurable on } Y \end{cases}$

proof of lemma. ($\Rightarrow$) trivial

($\Leftarrow$) WLOG. assume $E = A \cup \{+\infty\}$ $A \in \mathcal{B}_{\mathbb{R}} \Rightarrow f^{-1}(E) = f^{-1}(A) \cup f^{-1}(\{+\infty\})$ ...

Back to the original problem. $(fg)^{-1}(\{+\infty\}) = \Big(\{f>0\} \cap \{g = +\infty\}\Big) \cup \Big(\{f<0\} \cap \{g = -\infty\}\Big)$ $\cup \cdots$ $\in \mathcal{M}$

Similarly, $(fg)^{-1}(\{-\infty\}) \in \mathcal{M}$.

... The proof that "$f+g$" & "fg" are measurable has been forgotten...

$F(x) = (f(x), g(x))$ $\phi(z,w) = z+w$. $\psi(z,w) = zw$ are all measurable

$\Rightarrow f+g = \phi \circ F$. $fg = \psi \circ F$ ... □

$\boxed{\text{EX}}$ 2.15 $(\Rightarrow)$ trivial

$(\Leftarrow)$ $\forall E \in \mathcal{B}$ ~~Note $X = f^{-1}$~~

$$f^{-1}(E) = (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) \in \mathcal{M}$$

$\Rightarrow f$ is measurable $\square$

We have defined integral in simple functions, L^+, ~~or~~ and we can extend it to $\boxed{\text{real-valued}}$ measurable in an obvious way; namely. If f^+ & f^- are the positive $\mathbb{R}$ and negative parts of f and at least one of them is finite, (to avoid $\infty - \infty$), we define $\int f = \int f^+ - \int f^-$.

We shall be mainly concerned with the case with $\int f^+$ & $\int f^-$ are both finite

We then say that f is integrable. Since $|f| = f^+ + f^-$, it's clear that f is integrable iff $\int |f| < \infty$

$\boxed{\text{prop}}$ The set of integrable real-valued functions on X is a real vector space and the integral is a linear funtional on it.

proof. The only thing needs to notice is that

$$h = f + g \Rightarrow h^+ + f^- + g^- = h^- + f^+ + g^+$$
$$\Rightarrow \int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+$$
$$\Rightarrow \int h = \int f + \int g. \quad \square$$

Generally. measurable functions $f: X \to \mathbb{C}$ are called integrable if $\int |f| < \infty$. on $E \in \mathcal{M}$ $\int_E$

Since $|f| \le |\mathrm{Re} f| + |\mathrm{Im} f| \le 2|f| \Rightarrow$ " f is integrable $\Leftrightarrow$ $\mathrm{Re} f$, $\mathrm{Im} f$ are both integra[ble]"

$$\int f := \int \mathrm{Re} f + i \int \mathrm{Im} f.$$

~~$\boxed{\text{prop}}$~~ $\boxed{\text{prop}}$ If f is integrable $\Rightarrow \left|\int f\right| \le \int |f|$.

$\boxed{\text{prop}}$ (1) If f is integrable, $\{f(x) \neq 0\}$ is σ-finite

(2) If f, g are integrable, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f-g| = 0$ iff $f = g$ a.e.

proof. (1) Similar as before

(2) "$\int |f-g| = 0 \Leftrightarrow f = g$ a.e." is from $\int f = 0 \Leftrightarrow f = 0$ a.e., $f \in L^+$

$$\left|\int_E f - \int_E g\right| \le \int \chi_E |f-g| \le \int |f-g| = 0. \text{ If } \int |f-g| = 0$$

If "$\int_E f = \int_E g$ $\forall E \in \mathcal{M}$. $\not\Rightarrow$ $f = g$ a.e." ~~#~~ then, WLOG, $\exists$ a positive set $E \{\mathrm{Re}(f-g)^+ > 0\} \Rightarrow \int_E f - g > 0$ on E ↯ $\square$

$\boxed{\text{Rmk}}$ Integration makes no difference by altering functions on a null set.

So we can treat $\overline{\mathbb{R}}$-valued functions that are a.e. finite as real-valued functions for the purposes of integration.

$L^1(\mu) = \{ f : X \to \mathbb{R} \text{ (or } \mathbb{C}) : f \text{ is integrable} \} / \sim$ is a vector space.

$f \sim g$ iff $f = g$ a.e.

And although $L^1(\mu)$ is a set of equivalent classes of a.e. defined integrable functions on X, we shall still employ ~~the~~ notation " $f \in L^1(\mu)$ ".

Advantages: ① $L^1(\mu)$ is naturally ~~equivalent~~ corresponsed to $L^1(\bar{\mu})$

(since f is $\bar{\mu}$-measurable. $\exists\, g = f$ a.e. and g is μ-measurable)

② $L^1(\mu)$ becomes a metric space.

$\boxed{\text{Thm}}$ DCT. $\{f_n\} \subseteq L^1$ with (1) $f_n \to f$ a.e. (2) $\exists\, g \in L^1$ s.t. $|f_n| \le g$ a.e.

Then $f \in L^1$ and $\int f = \lim\limits_{n\to\infty} \int f_n$.

proof. f is measurable (maybe after redefinition on a null set)

Since $|f| \le g$ a.e $\Rightarrow f \in L^1$

WLOG assume f_n & f are real-valued, $\begin{cases} g + f_n \ge 0 & \text{a.e.} \\ g - f_n \ge 0 & \text{a.e.} \end{cases}$

$\Rightarrow$ By Fatou's lemma ~~$\int g + f_n$~~

$$\int f + \int g \le \liminf \int (f_n + g) = \int g + \liminf \int f_n$$

$$\int g - \int f \le \liminf \int (g - f_n) = \int g - \limsup \int f_n$$

$$\Rightarrow \liminf \int f_n \ge \int f \ge \limsup \int f_n$$ □

$\boxed{\text{Thm}}$ $\{f_j\} \subseteq L^1$ s.t. $\sum_1^\infty \int |f_j| < \infty$. then $\sum_1^\infty f_j$ converge a.e. to a function in L^1 and $\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$

proof By MCT. $\int \sum_1^\infty |f_j| = \sum_1^\infty \int |f_j| \quad g = \sum_1^\infty |f_j| \in L^1$

$\Rightarrow g$ is finite a.e. $\Rightarrow \sum_1^\infty f_j$ converges a.e.

By DCT. $\sum_1^\infty f_j \in L^1$ □

$\boxed{\text{Thm}}$ If $f \in L^1(\mu)$ and $\varepsilon > 0$. there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ s.t $\int |f - \phi| d\mu < \varepsilon$ (integrable simple functions are dense in L^1 space in L^1 metric)

If μ is a Lebesgue - Stieltjes measure on $\mathbb{R}$, the sets E_j in the definition of ϕ can be ~~taken~~ to be finite unions of open intervals ; moreover, there is a continuous function g that vanish outside a bound interval s.t $\int |f - g| d\mu < \varepsilon$

proof. let $\phi_n \to f$ pointwise By DCT. $\int |\phi_n - f| \to 0$ as $n \to \infty$

since $|\phi_n - f| \le 2|f|$ $\Rightarrow \exists\ \phi$ s.t $\int |\phi - f| d\mu < \varepsilon$

If $\phi_n = \sum a_j \chi_{E_j}$ $\mu(E_j) = |a_j|^{-1} \int_{E_j} |\phi_n| \le |a_j^{-1}| \int |f| < \infty$

$\mu(E \Delta F) = \int |\chi_E - \chi_F| d\mu$

For $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$, any measurable set of finite measure can be approximated arbitrarily closely by finite unions of open intervals.

All facts listed above lead to the desired result. $\square$

$\boxed{\text{Thm}}$ suppose that $f: X \times [a,b] \to \mathbb{C}$ $-\infty < a < b < \infty$, $f(\cdot, t)$ is integrable for each $t \in [a,b]$, ~~then lim~~ Let $F(t) = \int_X f(x,t)\, d\mu(x)$

a. suppose $\exists\ g \in L^1(\mu)$ s.t $|f(x,t)| \le g(x)$ for all x, t.

If $\lim\limits_{t \to t_0} f(x,t) = f(x,t_0)$ for every x, then $\lim\limits_{t \to t_0} F(t) = F(t_0)$

(In particular, if $f(x, t\cdot)$ is cts for each $x \Rightarrow F(t)$ is cts)

b. suppose ~~that~~ $\frac{\partial f}{\partial t}$ $\exists$, $g \in L^1(\mu)$ s.t $\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$ for all x, t.

$\Rightarrow F$ is differentiable & $F'(t) = \int \frac{\partial f}{\partial t}(x,t)\, d\mu(x)$.

proof a. (By Heine Thm) $\forall \{t_n\} \to t_0$ $\int f(x,t_n)\, d\mu \overset{DCT}{\Rightarrow} \int f(x,t_0)$

b. $\dfrac{f(x,t_n) - f(x,t)}{t_n - t} = h_n(x)$ $|h_n(x)| \overset{\text{mean value}}{\le} g(x)$

$h_n(x) \to \frac{\partial f}{\partial t}(x,t)$ (is measurable)

$F'(t_0) = \lim\limits_{n \to \infty} \int h_n(x) \overset{DCT}{=} \int \frac{\partial f}{\partial t}\, d\mu(x)$. $\square$

$\boxed{\text{Rmk}}$ a. the continuity of integral

b. the differentiability of integral.

25.

$\boxed{\text{Thm}}$ (Generalized DCT) If f_n, g_n $f, g \in L^1$ $f_n \to f$, & $g_n \to g$ a.e.,
$|f_n| \le g_n$ and $\int g_n \to \int g$, then $\int f_n \to \int f$

proof. we have $\begin{cases} g_n + f_n \ge 0 \\ g_n - f_n \ge 0 \end{cases}$. By Fatou's lemma,

$$\left.\begin{aligned} \int g + \liminf \int f_n = \liminf \int g_n + f_n \ge \int f + g \\ g - \limsup \int f_n \liminf \int g_n - f_n \ge \int g - f \end{aligned}\right\} \Rightarrow \liminf \int f_n \ge \int f \ge \limsup \int f_n$$

□

$\boxed{\text{EX}}$ 2.3.21 If. $\int |f_n| \to \int |f|$ $\Rightarrow \int |f_n| + |f| \to \int 2|f|$

$|f_n - f| \to 0$ a.e. $|f_n - f| \le |f_n| + |f|$

$\Rightarrow \int |f_n - f| \to 0.$

If $\int |f_n - f| \to 0$. $\left| \int |f_n| - \int |f| \right| \le \int |f_n - f| \to 0.$

□

Please remember ~~the~~ following four examples.

1. $f_n = \frac{1}{n} \chi_{(0,n)} \to 0$ uni
2. $f_n = \chi_{(n,n+1)} \to 0$ pw
3. $f_n = n \chi_{[0,\frac{1}{n}]} \to$ a.e.

$\mathbb{R}$ with Lebesgue measure. not L^1

4. $f_n = \chi_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}$ $n = 2^k + j$, $0 \le j < 2^k$ not a.e. but L^1

$\boxed{\text{Def}}$ A sequence $\{f_n\}$ of measurable complex-valued functions on $(X, \mathcal{M}, \mu)$ is Cauchy in measure if for every $\varepsilon > 0$

$$\mu(\{x : |f_n(x) - f_m(x)| \ge \varepsilon\}) \to 0 \quad \text{as } n, m \to \infty$$

Converge in measure to f if for $\forall \varepsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) \to 0 \quad \text{as } n \to \infty.$$

$\boxed{\text{prop}}$ If $f_n \xrightarrow{L^1} f$, then $f_n \to f$ in measure ($f_n \xrightarrow{\mu} f$).

proof. $\forall \varepsilon > 0$. $E_{n,\varepsilon} = \{|f_n - f| \ge \varepsilon\}$ measurable

$$\int |f_n - f| \ge \int_{E_{n,\varepsilon}} |f_n - f| \ge \varepsilon \mu(E_{n,\varepsilon}) \Rightarrow \mu(E_{n,\varepsilon}) \le \frac{1}{\varepsilon} \int |f_n - f|$$

let $n \to \infty$. $\Rightarrow \mu(E_{n,\varepsilon}) \to 0$ □

$\boxed{\text{Rmk}}$ $f_n = \frac{1}{n} \chi_{(0,n)} \xrightarrow{\mu} 0$ but $f_n \not\xrightarrow{L^1} f = 0.$

$\boxed{\text{Thm}}$ suppose $\{f_n\}$ is Cauchy in measure. Then there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence $\{f_{n_j}\}$ that converges to f a.e. Moreover, if also $f_n \xrightarrow{\mu} g$ then $g = f$ a.e.

proof. To show the result more preciously, we introduce the following lem.

lemma. $f_n \xrightarrow{\mu} f$ iff for $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $\mu(\{|f_n - f| \geq \varepsilon\}) < \varepsilon$ for $n > N$

proof of lemma. $(\Rightarrow)$ since ~~for~~ an arbitrary fixed $\varepsilon > 0$. $\mu(\{|f_n - f| \geq \varepsilon\}) \to 0$ as $n \to \infty$ we can lead to right immediately.

$(\Leftarrow)$ $\mu(\{|f_n - f| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$ $\quad n > N(k)$

$\quad \vee \!\! \mid$

$\mu(\{|f_n - f| \geq \varepsilon\})$ $\quad \varepsilon > \frac{1}{2^k}$ $\quad$ let $k \to \infty$.

$\Rightarrow \mu(\{|f_n - f| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

choose a subsequence $\{g_j\} = \{f_{n_j}\}$ s.t. $E_j = \{|g_j - g_{j+1}| \geq 2^{-j}\}$, $\mu(E_j) \leq 2^{-j}$

If $F_k = \bigcup_{j=k}^{\infty} E_j$. then $\mu(F_k) \leq 2^{1-k}$. And if $x \notin F_k$, for $i \geq j \geq k$. we have

$$|g_j(x) - g_i(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq 2^{1-j}$$

$\Rightarrow \{g_j\}$ is pointwise cauchy on F_k^c. Let $F = \bigcap_1^{\infty} F_k = \limsup E_j$. It's not hard to see $\mu(F) = 0$. Set $f(x) = \begin{cases} \lim g_i(x) & x \in F^c \\ 0 & x \in F \end{cases}$ is measurable

$g_j \xrightarrow{a.e.} f$. Moreover, in $|g_j(x) - g_i(x)| \leq 2^{1-j}$ $\quad i \geq j \geq k$ on $F_k^c \subseteq F^c$

$\overset{i \to \infty}{\Rightarrow} |g_j - f| \leq 2^{1-j}$ in F_k^c since $\mu(F_k) \to 0$ as $k \to \infty$

$\Rightarrow g_j \xrightarrow{\mu} f$.

By $\{|f_n - g| \geq \varepsilon\} \subseteq \{|f - f_n| \geq \varepsilon/2\} \cup \{|f_n - g| \geq \varepsilon/2\}$

Idea. Like Cauchy convergence principle in sequence, we hope to construct a subsequence of functions and prove that it is exactly the limit we want.

proof. like lemma $\forall k > 0$. $\exists N(k) > 0$. $\mu(\{|f_m - f_n| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$ when $m, n > N(k)$.

So our strategy is that we always require $n > m$, $n > N(k+1)$, and $m > N(k)$, that is to say, we've already taken a subsequence $\{g_j\} \subseteq \{f_n\}$ that

$\mu(\{|g_{j+1} - g_j| \geq \frac{1}{2^j}\}) < \frac{1}{2^j}$. Let $E_j = \{|g_{j+1} - g_j| \geq \frac{1}{2^j}\}$, $\mu(E_j) < \frac{1}{2^j}$.

We shall have such intuition, that E_j's are the bad and small error sets whose measure tends to zero.

7. Let $F_j = \bigcup_{k=j}^{\infty} E_k \Rightarrow \mu(F_j) < 2^{1-j}$. Then, as what we have expected initially,

What happens on the outside of F_k? $|g_j - g_{j+1}| = \frac{1}{2^j}$ $j \geq k$. What's more,

$$|g_i - g_j| \leq |g_i - g_{i-1}| + \cdots + |g_{j+1} - g_j| \leq 2^{1-j} \qquad i \geq j \geq k. \quad (*)$$

It's a Cauchy sequence! $\Rightarrow g_j \to f$ on F_k^c $j \geq k$

We want to obtain more convergent set as we can, on the other hand, take the union of them. $\boxed{\bigcup_{k=1} F_k^c = (\bigcap_{k=1} F_k)^c}$ So, Let $F = \bigcap_{k=1}^{\infty} F_k = \limsup_j E_j$.

It's obvious F has measure zero.

let $f(x) = \begin{cases} \lim_j g_j(x) & x \in F^c \\ 0 & x \in F \end{cases}$. so, $\boxed{g_j \xrightarrow{a.e.} f}$ and f is measurable

$\cancel{|g_j - f| \geq \frac{1}{2^k}}$

Using $(*)$, letting $i \to \infty$. $\Rightarrow |f - g_j| \leq 2^{1-j}$ $x \in F_k^c$ $j \geq k$.

$\Rightarrow g_j \xrightarrow{\mu} f$. (Note that $g_j = f_{n_j}$)

By $\{|f_n - f| \geq \varepsilon\} \subseteq \underbrace{\{|f_n - f_{n_j}| \geq \varepsilon/2\}}_{\text{Cauchy}} \cup \underbrace{\{|f_{n_j} - f| \geq \varepsilon/2\}}_{\text{Converge.}}$

$\Rightarrow \boxed{f_n \xrightarrow{\mu} f.}$

If $f_n \xrightarrow{\mu} g$.

$$\{|f - g| \geq \varepsilon\} \subseteq \underbrace{\{|f - f_n| \geq \varepsilon/2\}}_{\to 0} \cup \underbrace{\{|f_n - g| \geq \varepsilon/2\}}_{\to 0}$$

$\Rightarrow \mu\{|f - g| \geq \varepsilon\} = 0$ $\forall \varepsilon > 0$. $\Rightarrow f = g$ a.e. $\square$

Cor $f_n \xrightarrow{L^1} f \Rightarrow \exists \{f_{n_j}\} \subseteq \{f_n\}$ s.t $f_{n_j} \xrightarrow{a.e.} f$.

Thm (Egoroff) Suppose $\mu(X) < \infty$, $f_1 \cdots f_n \cdots, f$ are measurable complex-valued functions on X s.t $f_n \xrightarrow{a.e.} f$.

Then, $\forall \varepsilon > 0$. $\exists E \subseteq X$. $\mu(E) < \varepsilon$ & $f_n \rightrightarrows f$ on E^c (~~subspace topology~~)

Proof. For convenience, suppose $f_n \to f$ pointwise. NOTE that $\bigcap_{\ell=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} \{|f_k - f| < \frac{1}{\ell}\} = X$

$\boxed{f_n \xrightarrow{un.} f : \bigcap_{\ell=1}^{\infty} \bigcap_{n=N} \{|f_n - f| < \frac{1}{\ell}\}}$ $\Rightarrow \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} \{|f_k - f| < \frac{1}{\ell}\} = X \Rightarrow \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{|f_k - f| \geq \frac{1}{\ell}\} = \emptyset$

Since $\mu(X) < \infty$. By continuity of measure, ~~let $E_j(\ell) = \bigcup_{k=j}^{\infty} \{|f_k - f| \geq \frac{1}{\ell}\}$~~

let $E_j(\ell) = \bigcup_{k=j}^{\infty} \{|f_k - f| \geq \frac{1}{\ell}\}$ $\exists n_\ell$ s.t $E_{n_\ell}(\ell) \leq \varepsilon \cdot 2^{-\ell}$. let $E = \bigcup_{\ell=1}^{\infty} E_{n_\ell}(\ell)$. $\mu(E) < \varepsilon$

Then. for $\forall x \in E^c = \bigcap_{l=1}^{\infty} \bigcap_{k=n_l}^{\infty} \{|f_k - f| \le \frac{1}{l}\}$

We have $|f_n(x) - f(x)| \le \frac{1}{l}$ for $n > n_l$. Thus $f_n \rightrightarrows f$ on E^c □

$\boxed{\text{Rmk}}$ This type of convergence mode is called almost uniform convergence.

$\boxed{\text{EX}}$ 2.4.39. $f_n \xrightarrow{a.un.} f \Rightarrow f_n \xrightarrow{\mu} f$ & $f_n \xrightarrow{a.e.} f$

proof. (1) "$f_n \xrightarrow{\mu} f$: $\mu(\{|f_n - f| \ge \varepsilon\}) \cancel{\to 0 \text{ as } n \to \infty}$ $< \varepsilon$ for $n > N(\varepsilon)$"

Since $f_n \xrightarrow{a.un.} f$ $\forall \varepsilon > 0$. $\exists E_\varepsilon$ s.t. $\mu(E_\varepsilon) < \varepsilon$. $f_n \rightrightarrows f$ on E_ε^c

$\Rightarrow$ $\cancel{\{|f_n - f| \ge \varepsilon\} \subseteq \{|f_n - f| \ge \varepsilon\}}$

$|f_n - f| < \varepsilon$ on E_ε^c when $n > N(\varepsilon)$

$\Rightarrow \{|f_n - f| \ge \varepsilon\}_{n > N(\varepsilon)} \subseteq E_\varepsilon$ ...

(2) $\forall \varepsilon > 0$. $f_n \rightrightarrows f$ on E_ε^c $\mu(E_\varepsilon) < \varepsilon$

So $f_n \not\to f$ on $\bigcap_{\varepsilon > 0} E_\varepsilon \Rightarrow \{f_n \not\to f\} \subseteq \bigcap E_\varepsilon$ is a null set. □

$\boxed{\text{EX}}$ 2.4.44. (Lusin) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measure and $\varepsilon > 0$, there is a compact set $E \subseteq [a,b]$ such that $\mu(E^c) < \varepsilon$ and $f|_E$ is cts.

proof. Let $\{\phi_n\}$ is a sequence of simple functions such that $|\phi_1| \le \cdots \le |f|$

$\phi_n \to f$. for each ϕ_n, we can change its value on a small ~~set~~ open E_n with $\mu(E_n) = \frac{\varepsilon}{4^n}$. and get $\tilde{\phi}_n$ on $[a,b] \setminus E_n$. We have $\tilde{\phi}_n = \phi_n$

So $\tilde{\phi}_n \to f$ on $[a,b] \setminus (\bigcup_{n=1}^{\infty} E_n)$. Using Egoroff Thm, $\exists E \subseteq [a,b] \setminus (\bigcup_{n=1}^{\infty} E_n)$

(E can be chosen as ~~open~~ closed set) $\tilde{\phi}_n \xrightarrow{a.un.} f$ on E $\Rightarrow$ f is cts on E □

$\cancel{\mu(E) \le} \mu([a,b] \setminus (\bigcup E_n) \setminus E) < \varepsilon/2 \Rightarrow \mu(E^c) < \varepsilon$

$\boxed{\text{Rmk}}$ In egoroff. we can ~~choose the error set to be an open set (outer regularity of measurable set)~~

We don't assume too much of the background measure space.

But for Lusin's Thm, we shall consider $f : [a,b]_{\mathcal{L}} \to \mathbb{C}_{\mathcal{B}_{\mathbb{C}}}$, and E should be a compact set (it can be extend to closed set if $[a,b] = \mathbb{R}^n$ or $\mathbb{C}$) ...

29. Now, we start to discuss about product measure. Recall that, product σ-product is generated by $\{\pi_\alpha^{-1}(E_\alpha)\ \forall E_\alpha \in \mathcal{M}_\alpha,\ \alpha \in A\}$, that in a certain sense, is a canonical construction.

Let $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ be two measure space.

Since this part has been written in Real Analysis note. Just rewrite the result below.

$\boxed{\text{Def}}$ $E_x = \{y \in Y : (x,y) \in E\}$. $E^y = \{x \in X, (x,y) \in E\}$

$f_x(y) = f^y(x) = f(x,y)$

$\boxed{\text{Rmk}}$ $(A \times B)_x = \begin{cases} B & x \in A \\ \phi & x \notin A \end{cases}$ $\quad (\bigcup E_j)_x = \bigcup (E_j)_x \quad (\bigcap E_j)_x = \bigcap (E_j)_x$

$(E^c)_x = (E_x)^c$

$\boxed{\text{Prop}}$ $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E_x \in \mathcal{N}$. $E^y \in \mathcal{M}$

f is $\mathcal{M} \otimes \mathcal{N}$-measurable $\Rightarrow f_x$ is $\mathcal{N}$-measurable, f^y is $\mathcal{M}$-measurable.

$\boxed{\text{Def}}$ $\mathcal{C} \subseteq \mathcal{P}(X)$ is called monotone class if $\mathcal{C}$ is closed under countable increasing unions and countable decreasing intersections.

$\boxed{\text{Thm}}$ (Monotone class lemma) An algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ then $\mathcal{C} = \mathcal{C}(\mathcal{A})$ and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ coincide.

$\boxed{\text{Thm}}$ suppose $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are $\boxed{\sigma\text{-finite}}$ measures spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $x \mapsto \nu(E_x)$ & $y \mapsto \mu(E^y)$ are measurable on X and Y. and

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

$\boxed{\text{Thm}}$ (Fubini & Tonelli Thm) $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

(a) If $f \in L^+(X \times Y) \Rightarrow g(x) = \int f_x\, d\nu$, $h(y) = \int f^y\, d\mu$ in $L^+(X)$ & $L^+(Y)$

$$\int f\, d(\mu \times \nu) = \int \left[\int f(x,y)\, d\nu(y) \right] d\mu(x)$$
$$= \int \left[\int f(x,y)\, d\mu(x) \right] d\nu(y) \quad (*)$$

(b) If $f \in L^1(X \times Y) \Rightarrow$ $f_x \in L^1(\nu)$ for a.e. x & $g(x) \in L^1(\mu)$

$f_y \in L^1(\mu)$ for a.e. y $\quad h(y) \in L^1(\nu)$

$(*)$ holds.

$\boxed{\text{Rmk}}$ the condition of σ-finite can't be removed;

even if μ, ν are complete, $\mu \times \nu$ can be usually not complete.

If we hope to obtain "complete version" of Fubini – Tonelli Theorem, we can consider $\mathcal{L} := \overline{\mathcal{M} \otimes \mathcal{N}}$, $\lambda := \overline{\mu \times \nu}$.

$\boxed{\text{Thm}}$ (Fubini – Tonelli Theorem) Let $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ be complete, σ-finite measure space, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable, and either belongs to $L^+(\lambda)$ or $L^1(\lambda)$, then f_x & (f^y) is $\mathcal{N}$- ($\mathcal{M}$-) measurable for a.e. x (y). Moreover, $x \mapsto \int f_x \, d\nu$ ($y \mapsto \int f^y \, d\mu$) is measurable. (and in case $L^1(\lambda)$, also integrable.). Moreover, following result holds both

$$\int f \, d\lambda = \iint f(x,y) \, d\mu(x) \, d\nu(y) = \iint f(x,y) \, d\nu(y) \, d\mu(x)$$

(omit ~~blank~~ brackets)

proof $\boxed{\text{lem 1}}$ If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x, y

proof of lem 1. $0 = \mu \times \nu(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)$

$\boxed{\text{lem 2}}$ If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e. then f_x & f^y are ~~measurable~~ integrable for a.e. x, y, & $\int f_x \, d\nu = \int f_y \, d\mu = 0$ for a.e. x & y

proof of lem 2. ~~$f = 0$ λ-a.e. $\Rightarrow \exists g$ is $\mathcal{M} \otimes \mathcal{N}$-measurable $f = g$ λ-a.e.~~

~~$\Rightarrow \iint g_x \, d\nu \, d\mu = \iint g^y \, d\mu \, d\nu = 0$~~

~~$\Rightarrow \int g_x \, d\nu = \int g^y \, d\mu = 0$ for a.e. x, y.~~

~~$E_0 = \{f(x,y) \neq 0\}$. there exists a set E that contains E_0 and lies in $\mathcal{M} \otimes \mathcal{N}$. with E $\mu \times \nu(E) = 0$~~

~~$\Rightarrow \int |f_x| \, d\nu(y) = \int \chi_{E_x} |f_x| \, d\nu(y) = 0$ $(0 \cdot \infty = 0)$~~

By the definition of completion, $E = F \cup Z$ with $F \in \mathcal{M} \otimes \mathcal{N}$, $Z \subseteq N \in \mathcal{M} \otimes \mathcal{N}$, $\mu \times \nu(N) = 0$

$E_x = F_x \cup Z_x \subseteq F_x \cup N_x$ since $\mu(N_x) = 0$ for a.e. $x \Rightarrow \mu(Z_x) = 0$ for a.e. x

$(E = \{f \neq 0\}) \Rightarrow \mu(E_x) = \mu(E^y) = 0$ for a.e. x, y

$\Rightarrow f_x = 0$ on $E_x^c \Rightarrow f_x$ is $\mathcal{N}$-measurable (non-zero on a null set) by completeness

$\Rightarrow \int f_x \, d\nu = 0$ for a.e. x. ...

31. NOW, back ~~to~~ the original problem.

$f \in \mathcal{L} \Rightarrow \exists\, g \in \mathcal{M} \otimes \mathcal{N}$ ~~with~~ that $f = g$ λ-a.e.

$\Rightarrow f - g = 0$ λ-a.e.

$\Rightarrow f_x - g_x = 0$ λ-a.e. $\quad f^y - g^y = 0$ λ-a.e.

If f, g is $L^1(\lambda)$, so are $f_x - g_x$, $f^y - g^y$.

$\cancel{f_x - g_x}$

$\Rightarrow f_x$ is $\mathcal{N}$-measurable $\quad f^y \ldots$

$\int f_x = \int g_x$ for a.e. x. $\quad f^y \ldots$

$$\Rightarrow \int\int f_x \, d\nu \, d\mu = \int\int g_x \, d\nu \, d\mu = \int g \, d\lambda = \int f \, d\lambda. \qquad \square$$

Now, we shall introduce two small topics and end up the chapter.

1°. n-dim Lebesgue integral. ($\mathbb{R}^n$)

$\mathcal{L}^n$ is the completion of $\mathcal{L} \otimes \cdots \otimes \mathcal{L}$.

and it's not surprise that it shares similar, even ~~the~~ same properties of Borel Stieltjes measure, ~~that~~ in other words, outer regularity, inner-regularity and if $m(E) < \infty$. E can be approximate by finite cubes.

So, with the techniques above, $f \in L^1(m)$ can be L^1 approximated by continuous functions with compact support.

And the most important theorem ~~one~~ is the following one.

$\boxed{\text{Thm}}$ Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set. $G: \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(x)\,dx = \int_\Omega f \circ G(x) \left| \det D_x G \right| dx.$$

2° Integration in polar coordinates.

The construction process is kind of boring ~~though~~ conforming geometric intuition. And ~~the~~ result is great ~~to~~ and worth remembering.

$\boxed{\text{Thm}}$ There is a $\boxed{\text{unique}}$ Borel measure $\sigma=\sigma_{n-1}$ on S^{n-1} such that $m_* = \rho\times\sigma$. If f is Borel measurable on $\mathbb{R}^n$ and $f\geq 0$ or $f\in L^1(m)$, then

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{S^{n-1}} f(rx')\,r^{n-1}\,d\sigma(x')\,dr$$

where $x'\in S^{n-1}$, $\rho=\rho_n$ on $(0,\infty)$ by $\rho(E)=\int_E r^{n-1}dr$

$\boxed{\text{Cor}}$ If f is a measurable function on $\mathbb{R}^n$, $f\geq 0$ or $f\in L^1$ s.t. $f(x)=g(|x|)$ for some g on $(0,\infty)$, ~~then~~

$$\int_{\mathbb{R}^n} f(x)\,dx = \sigma(S^{n-1})\int_0^\infty g(r)\,r^{n-1}\,dr.$$

$\boxed{\text{Example}}$ 1° $a>0$. $\int_{\mathbb{R}^n} \exp(-a|x|^2)\,dx = I_n$

By Tonelli's thm. $I_n=(I_1)^n$ $\quad I_1 = I_2^{\frac{1}{2}}$ $\quad I_n=\left(\frac{\pi}{a}\right)^{n/2}$

$$I_2 = 2\pi\int_0^\infty r e^{-ar^2}\,dr = \frac{\pi}{a}$$

2°. $\sigma(S^{n-1}) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$

$$\pi^{n/2} = \int_{\mathbb{R}^n}\exp(-|x|^2)\,dx = \sigma(S^{n-1})\int_0^\infty r^{n-1}e^{-r^2}\,dr$$

$$= \frac{\sigma(S^{n-1})}{2}\int_0^\infty s^{n/2-1}\cdot e^{-s}\,ds = \frac{\sigma(S^{n-1})}{2}\cdot\Gamma(n/2).$$

$\square$

A graph of different convergence modes