Chapter 3. Signed measure and differentiation.

Many results are already shown in Real Analysis note, here we shall simplify most of the discussion.

$\boxed{\text{Def}}$ A signed measure on a measurable space $(X,\mathcal{M})$ is a function $\nu: \mathcal{M} \to [-\infty, \infty]$ such that

- $\nu(\phi) = 0$
- ν assumes at most one of the values $\pm\infty$.
- If $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ where the latter sum converges absolutely if $\nu(\bigcup_1^\infty E_j)$ is finite.

There are two examples, what's more, only two examples of signed measures.

$1°$. μ_1, μ_2 are measures on $(X,\mathcal{M})$ where at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure

(For the third condition, WLOG, we can suppose μ_2 is of finite. Then

$$\nu(\bigcup_1^\infty E_j) = \mu_1(\bigcup_1^\infty E_j) - \mu_2(\bigcup_1^\infty E_j) < \infty \Rightarrow \mu_1(\bigcup_1^\infty E_j) < \infty$$

$$\Rightarrow \mu_1(\bigcup_1^\infty E_j) + \mu_2(\bigcup_1^\infty E_j) < \infty \overset{\text{Cauchy}}{\Rightarrow} \sum_1^\infty |\mu_1(E_j) - \mu_2(E_j)| < \infty \text{)}$$

$2°$. $f: X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite. $\nu(E) = \int_E f\, d\mu$

$$\left(\nu(\bigcup_1^\infty E_j) = \int_{\bigcup_1^\infty E_j} f\, d\mu = \cancel{\sum_1^\infty \int_{E_j} f\, d\mu} = \int_{\bigcup_1^\infty E_j} f^+ - \underbrace{\boxed{\int_{\bigcup_1^\infty E_j} f^-}}_{\text{definitely}} < \infty \right.$$

$$\left. \Rightarrow \int_{\bigcup_1^\infty E_j} f^+ d\mu \xrightarrow{\text{Similarly}} \right)$$

$\boxed{\text{Thm}}$ (continuity) $E_j \nearrow \quad \nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$

$E_j \searrow \quad \nu(E_n) < \infty$ for some $n \Rightarrow \nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

$\boxed{\text{Def}}$ A set $E \in \mathcal{M}$ is called positive (negative or null) for ν, if $\nu(F) \geq 0$ (≤ 0, $= 0$) for $\forall F \in \mathcal{M}$ s.t. $F \subseteq E$.

$\boxed{\text{Lem}}$ Any ~~positive set~~ measurable subset of a positive set is positive

The union of any countable positive sets is positive.

$\boxed{\text{Thm}}$ (Hahn decomposition) If ν is a signed measure on $(X, \mathcal{M})$, there exists a positive set P and a negative set N for ν such that $X = P \cup N$, and $P \cap N = \phi$. If P', N' is another such pair, then $P \Delta P'$ $(N \Delta N')$ is null for ν.

proof. WLOG, ν doesn't assume $+\infty$. Now we aim to find the "largest" positive set and claim its complement is negative.

Let $m = \sup\{\nu(E) : E \text{ is positive}\} \Rightarrow \exists P_j \quad \nu(P_j) \to m$

$P = \bigcup_1^\infty P_j$ is positive. $\boxed{\nu(P) = m} < \infty$

If $X \setminus P$ is not negative $\Rightarrow E \subseteq N$. $\nu(E) > 0$

E can't be positive $\Rightarrow E_1 \subseteq E \quad \nu(E_1) < 0$

$\Rightarrow \nu(E \setminus E_1) = \nu(E) - \nu(E_1) > \nu(E)$

Such a process can continue! We need to precisely deal with the increasing speed of signed measure

$\{A_j\} \searrow$ n_j is the smallest ~~number~~ integer s.t $\exists A_j \subseteq A_{j-1}$

$\nu(A_j) > \nu(A_{j-1}) + \frac{1}{n_j}$

Let $A = \bigcap_{j=1}^\infty A_j$, $\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) > \sum_1^\infty \frac{1}{n_j} \Rightarrow n_j \to \infty$ as $j \to \infty$

But for A, we can still find a subset B s.t $\nu(B) > \nu(A) + \frac{1}{n}$

for some integer n. since $n_j \to \infty$ as $j \to \infty$. We have $n_j > n$.

$B \subseteq A \subseteq A_j \Rightarrow \nu$

Now we shall use a special method ~~to~~ to lead contradiction.

n_1 is the smallest ~~number~~ integer s.t there exists $B \subseteq N$, $\nu(B) > \frac{1}{n_1}$, and A_1 is such a set

Inductively, n_j is the smallest integer for which there exists a set $B \subseteq A_{j-1}$ with $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$. $A = \bigcap_{j=1}^\infty A_j$ $\infty > \nu(A) = \lim \nu(A_j) = \sum_{j=1}^\infty \frac{1}{n_j} \Rightarrow n_j \to \infty$ as $j \to \infty$

$\exists B \subseteq A \quad \nu(B) > \nu(A) + \frac{1}{n} \quad n < n_j$

$\Rightarrow \nu(B) > \nu(A_j) + \frac{1}{n} > \nu(A_j) + \frac{1}{n_j}$ ↯

$X = P \cup N = P' \cup N$. $\quad P \setminus P' \subseteq P \Rightarrow P \setminus P'$ is null

$P \cap P'^c = P \cap N' \subseteq N' \Rightarrow P \Delta P'$ is null $\square$

$\boxed{\text{Def}}$ We say two signed measures μ and ν are mutually singular, if there exists $E, F \in \mathcal{M}$, $E \cup F = X$, $E \cap F = \phi$. E is null for μ. F is null for ν.

$\boxed{\mu \perp \nu}$ "live on disjoint sets"

$\boxed{\text{Thm}}$ (Jordan decomposition) If ν is a signed measure, there exists unique positive measures ν^+ & ν^- s.t. $\nu = \nu^+ - \nu^-$, $\nu^+ \perp \nu^-$.

proof. $X = P \cup N$. $\nu^+(E) = \nu(E \cap P)$ $\nu^- = -\nu(E \cap N)$

$\Rightarrow \nu^+(E) - \nu^-(E) = \nu(E)$

ν^+ is null on N. ν^- is null on P.

It's unique! If $\nu = \mu^+ - \mu^-$. $\mu^+ \perp \mu^-$. $E \cup F = X$. $E \cap F = \phi$

$\mu^+(F) = \mu^-(E) = 0$ $\Rightarrow E, F$ is another Hahn decomposition

$\Rightarrow E \Delta P$ & $F \Delta N$ is null for ν.

$\mu^+(S) = \mu^+(S \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(S)$ $\square$

ν^+, ν^- are called positive and negative variations of ν. and $\nu = \nu^+ - \nu^-$ is called ~~the~~ Jordan decomposition.

Total variation of ν : $|\nu| = \nu^+ + \nu^-$.

$\boxed{\text{Rmk}}$ Hahn decomposition $\rightsquigarrow$ Jordan decomposition

~~Set~~ $\longmapsto$ Measure

(may) Not unique $\longrightarrow$ unique.

$\boxed{\text{EX}}$ 3.1.2. ~~E is ν-null $\Rightarrow \forall \hat{E} \subseteq E, \hat{E} \in \mathcal{M}$. $\nu(\hat{E}) = \nu^+(\hat{E}) - \nu^-(\hat{E}) = 0$~~

Not hard ~~$\Rightarrow \nu^+(\hat{E}) = \nu^-(\hat{E})$ for $\forall \hat{E} \subseteq E$. But $\nu^+ \perp \nu^-$. $\exists X = A \sqcup B$~~

But for rigorous process. ~~$\nu^+(\hat{E}) = \nu^+(\hat{E} \cap A)$~~

~~$\forall \hat{E} \subseteq A$. $\nu^+(\hat{E}) = \nu^-(\hat{E}) = 0$ $\Rightarrow \nu^+(\hat{E} \cup \hat{E}') = 0$~~

E is a ν-null set $\Rightarrow \forall \hat{E} \subseteq E$, $\hat{E} \in \mathcal{M}$. $\nu(\hat{E}) = \nu^+(\hat{E}) - \nu^-(\hat{E}) = 0$

Suppose $X = P \cup N$ is a Hahn decomposition.

$\nu^+(\hat{E}) = \nu^+((\hat{E} \cap P) \cup (\hat{E} \cap N)) = \nu^+(\hat{E} \cap P) = \nu^-(\hat{E} \cap P) = 0$

similarly $\nu^-(\hat{E}) = 0$ $\Rightarrow |\nu| = 0$

$|\nu|(E) = 0 \Rightarrow \nu^+(\hat{E}) + \nu^-(\hat{E}) = 0 \Rightarrow \nu^+(\hat{E}) = \nu^-(\hat{E}) = 0 \Rightarrow E$ is ν-null.

$\square$

$\boxed{EX}$ 3.1.2. $\nu \perp \mu \Rightarrow X = E \cup F$. F is ν-null, E is μ-null

$\Rightarrow F$ is $|\nu|$ null $\Rightarrow |\nu| \perp \mu$

$\Rightarrow F$ is ν^+, ν^--null

$\Rightarrow \nu^+ \perp \mu$, $\nu^- \perp \mu$ The process is natural!

$\Rightarrow F$ is ν-null $\square$

~~EX 3.1.3.~~

By $X = P \cup N$, $\nu = \nu^+ - \nu^-$, We can consider the integral wrt ν.

See what we do before $\mu(E) = \int_E \chi \, d\mu$

$$\Rightarrow \nu(E) = \nu^+(E) - \nu^-(E) = \int_E \chi_P \, d\nu^+ - \int_E \chi_N \, d\nu^-$$

$$d|\nu| = d\nu^+ + d\nu^- \qquad = \int_E \chi_P (d\nu^+ + d\nu^-) + \int_E \chi_N (d\nu^+ + d\nu^-)$$

$$= \int_E (\chi_P - \chi_N) \, d|\nu|$$

$$f = \chi_P - \chi_N \qquad = \int_E f \, d|\nu|$$

It's not hard too see that ν is "finite" $\Longleftrightarrow |\nu|$ is finite

So we can define finite (r.t σ-finite) for ν by $|\nu|$.

Integrals with respect to ν is defined by an obvious way below:

$$\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$$

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-).$$

$\boxed{EX}$ 3.13 if $f \in L^1(\nu) \Rightarrow \int f \, d\nu^+, \int f \, d\nu^-$ are finite

$\Rightarrow \int f (d\nu^+ + d\nu^-)$ is finite

$\quad \overset{\shortparallel}{\int f \, d|\nu|} \Rightarrow f \in L^1(|\nu|)$

~~$f \in L^1(|\nu|)$ $\int f \, d\nu^+ + \int f \, d\nu$ is finite.~~

$f \in L^1(|\nu|) \Rightarrow \int |f| \, d\nu^+ + \int |f| \, d\nu^- < \infty$

$\Rightarrow \int f \, d\nu^+, \int f \, d\nu^- < \infty$

$\Rightarrow \int f \, d\nu < \infty$

$\boxed{Rmk}$ The definition of L^1 assumes the integrals of positive and negative part are both finite.

$$\left|\int f \, d\nu\right| = \left|\int f \, d\nu^+ - \int f \, d\nu^-\right| \le \left|\int f \, d\nu^+\right| + \left|\int f \, d\nu^-\right|$$

$$\le \int |f| \, d\nu^+ + \int |f| \, d\nu^- = \int |f| \, d|\nu|.$$

$$\left|\int_E f \, d\nu\right| \le \int_E |f| \, d|\nu| \le |\nu|(E) \Rightarrow \sup \cdots \le |\nu|(E)$$

$$\sup \cdots \ge \left|\int_E \chi_P - \chi_N \, d\nu\right| = |\nu|(E) \qquad \square$$

3).

$\boxed{\text{Def}}$ (absolutely continuous) suppose ν is a signed measure, μ is a positive measure. We say ~~that~~ ν is absolutely continuous with respect to μ if $\nu(E)=0$ for $\forall E\in\mathcal{M}$. $\mu(E)=0$. Write $\nu \ll \mu$.

$\boxed{\text{EX}}$ 3.2.8. $\nu \ll \mu \Rightarrow \forall E\in\mathcal{M}$. with $\mu(E)=0$. we see that

E is ν-null by μ's monotone

$\Rightarrow E$ is $|\nu|$-null $\Rightarrow |\nu| \ll \mu$.

$\Rightarrow \nu^+ \ll \mu$, $\nu^- \ll \mu$

$\Rightarrow \nu^+ - \nu^- \ll \mu$. $\square$

key is "ν-null $\Leftrightarrow |\nu|$-null"

$\boxed{\text{EX}}$ 3.2.9. $\nu_j \perp \mu \Rightarrow X = E_j \cup F_j$ where $\nu_j(F_j)=0$ ~~$\mu(E_j)=0$~~ E_j is μ-null.

$\Rightarrow E = \bigcup_{j=1}^{\infty} E_j \Rightarrow E$ is μ-null

$F = X \setminus \left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcap_{j=1}^{\infty} E_j^c = \bigcap_{j=1}^{\infty} F \Rightarrow \sum_{j=1}^{\infty} \nu_j(F) = 0$

$\nu_j \ll \mu \Rightarrow \sum_{j=1}^{\infty} \nu_j \ll \mu$ is trivial $\square$

In some sense, absolutely continuity is the antithesis of of mutual singularity. if $\nu \perp \mu$ & $\nu \ll \mu$. $X = E\cup F$

$0=\mu(E) = \nu(E)$. ~~$\nu(E)$~~ F is ν-null $\Rightarrow \nu = 0$

$\Downarrow$

E is ν-null

$\boxed{\text{Rmk}}$ The term "absolutely continuous" is derived from real-variable theory.

$\boxed{\text{Thm}}$ ν is a finite signed measure, μ is a positive measure on $(X,\mathcal{M})$ Then $\nu \ll \mu$ iff for every $\varepsilon>0$. $\exists\, \delta>0$. s.t. $|\nu(E)|<\varepsilon$ whenever $\mu(E)<\delta$

proof. Since $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu$. $\nu \le |\nu|$. WLOG, we can assume ν is positive.

$\varepsilon-\delta$: $\mu(E)=0 \Rightarrow \nu(E)=0$

Conversely, If $\varepsilon-\delta$ condition doesn't satisfy, then there exists a $\varepsilon>0$

$\forall \delta>0$. $\exists\, E_\delta$. $\nu(E_\delta) \ge \varepsilon$. but $\mu(E_\delta)<\delta$

$\{E_n\}$ with $\nu(E_n) \ge \varepsilon$. $\mu(E_n) < \frac{1}{2^n}$.

Let $E = \limsup_n E_n = \boxed{\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j}$ $\quad \mu(E) < 2^{1-n} \to_{n\to\infty}$

$\mu(E)=0$

$\nu(\bigcup_{j=n}^{\infty} E_j) \ge \varepsilon \Rightarrow \nu(E) = \lim_j (\bigcup_{j=n} E_j) \ge \varepsilon$. $\square$

$\boxed{\text{Cor}}$ If $f \in L^1(\mu)$. for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|\int_E f\right| < \varepsilon$ whenever $\mu(E) < \delta$.

Notation: $d\nu = f d\mu$ for $\int_E f d\mu = \nu(E)$.

$\boxed{\text{Lem}}$ ν, μ are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exists $\varepsilon > 0$ & $E \in \mathcal{M}$ s.t. $\mu(E) > 0$. and $\nu \geq \varepsilon \mu$

proof. $X = P_n \cup N_n$ for $\nu - \frac{1}{n}\mu$

$P = \bigcup_{n=1}^{\infty} P_n$ $\quad N = P^c$. $\quad N$ is negative set for $\forall n$, $\nu - \frac{1}{n}\mu$.

$0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$. let $n \to \infty$ $\nu(N) = 0$

If $\mu(P) = 0 \Rightarrow \mu \perp \nu$.

If $\mu(P) > 0 \Rightarrow \mu(P_n) > 0$. for some n. $\square$

$\boxed{\text{Thm}}$ let ν be a $\boxed{\sigma\text{-finite}}$ signed measure and μ a $\boxed{\sigma\text{-finite}}$ positive measure on $(X, \mathcal{M})$. There exists $\boxed{\text{unique}}$ $\boxed{\sigma\text{-finite}}$ signed measure λ, ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \mu, \ \rho \ll \mu \quad \nu = \lambda + \rho.$$

Moreover, there exists an extended μ-integrable (at least one of $\int f^+ d\mu$, $\int f^- d\mu$ is finite) function $f: X \to \mathbb{R}$. such that $d\rho = f d\mu$

and any of two such functions are equal μ-a.e.

proof. The proof is on Real Analysis note. Here I shall state the idea I've understood.

STEP 1. of ~~finite~~ positive measure first.

$$\mathcal{F} = \{ f: X \to [0, \infty] : \int_E f d\mu \leq \nu(E). \ \forall E \in \mathcal{M} \}$$

using sequence and MCT to attain the "highest" f

claim: $\underset{\text{positive}}{\underline{d\lambda}} = d\nu - f d\mu$ is singular wre μ.

lem $\lambda - \varepsilon\mu > 0$ on E with $\mu(E) > 0$

$$\varepsilon \chi_E d\mu \leq d\lambda = d\nu - f d\mu$$

$$\Rightarrow (f + \varepsilon\chi_E) d\mu \leq d\nu \qquad f + \varepsilon\chi_E \geq f \ \text{↯}$$

Uniqueness: $d\nu = d\lambda + f d\mu = d\lambda' + f' d\mu$

$$\underline{d\lambda - d\lambda'} = \underline{(f - f')d\mu} \ll \mu$$

STEP 2. paste ... $\quad$ STEP 3. positive $\to$ signed.

λ : $E_1 \cup F_1$

λ' : $E_2 \cup F_2$

$(E_1 \cup E_2) \cup (F_1 \cap F_2)$

$\mu = 0 \qquad \lambda_1 - \lambda_2 = 0$

Another view is that of $\mu \perp \nu$

$\mu \cdot \nu = 0 \Rightarrow \mu(E) \neq 0 \Rightarrow \nu_j(E) = 0$

$\Rightarrow \cdots \Rightarrow \sum \nu_j(E) = 0$

$\sum \nu_j(E) \neq 0 \Rightarrow \nu_j(E) \neq 0$ ~

39.

f in last thm is called Radon-Nikodym derivative. of ν w.r.t μ.

$$d\nu = f d\nu := \frac{d\mu}{d\nu} d\nu.$$

linear operations hold true naturally.

What about chain rule?

$\boxed{\text{prop}}$ ν is σ-finite signed measure. μ, λ is σ-finite measure

$\nu \ll \mu$. $\mu \ll \lambda$

(1) If $g \in L^1(\nu)$ then $g \frac{d\nu}{d\mu} \in L^1(\mu)$

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

proof. assume $\nu \geq 0$. let $g = \chi_E$ $\int \chi_E\, d\nu = \nu(E)$

$$\int \chi_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu} d\mu \overset{\text{by def}}{=} \nu(E)$$

By steps of ~~def~~ constructing integrals, ...

$\nu \ll \lambda$ is trivial

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_E \boxed{\left(\frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}\right)} d\lambda$$

$$= \int_E \boxed{\left(\frac{d\nu}{d\lambda}\right)} d\lambda$$

$\boxed{\text{cor}}$ If $\mu \ll \lambda$ $\lambda \ll \mu$ $\Rightarrow$ $\left(\frac{d\lambda}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) = 1$ a.e.

$\boxed{\text{Example}}$ (non) μ - $\mathcal{L}$ ν ~~point mass~~ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

$\nu \perp \mu$ RN derivative $d\nu/d\mu$ not exists

$\boxed{\text{EX}}$ 3.2.11. a. finite case, trivial

b. $f_n \xrightarrow{L^1} f$

$$n \gg 0. \int |f_n - f| < \varepsilon/3$$

$$\boxed{\int_E |f_n - f|} \geq \int_E |f_n| - \int_E |f|$$

$$\int_E |f_n| \leq \int_E |f_n - f| + \boxed{\int_E |f|} < \varepsilon/3 + \varepsilon/3 \quad (n > N)$$

$\square$

$\boxed{\text{Ex}}$ 3.2.17. ~~in~~ $d\rho = f\,d\nu \Rightarrow \rho \ll \nu$ in $\mathcal{N}$

$$g = \frac{d\rho}{d\nu}$$

$$\int_E g\,d\nu = \int_E \frac{d\rho}{d\nu}\,d\nu = \rho(E) = \int_E f\,d\nu \qquad \square$$

Why do we study R-N derivatives? Here we shall some simple applicati

$\boxed{\text{Def}}$ complex measure : $\nu : \mathcal{M} \to \underset{\text{finite.}}{\underline{\mathbb{C}}}$ ~~scca~~ free

- $\nu(\phi) = 0$
- If $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$. then $\nu(\bigcup_{n=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$

where the series $\boxed{\text{converges absolutely}}$.

$\nu \ll \mu$.

the total variation $|\nu|$: $\boxed{d|\nu| = |f|\,d\mu}$ It's true? (where $d\nu = f d\mu$).

positive (well-defined)

Since $\nu \ll \mu := |\nu_r| + |\nu_i|$

$\Rightarrow d\nu = f\,d\mu$

If $d\nu = f_1 d\mu_1 = f_2 d\mu_2$. (We aim to show $|f_1|\,d\mu_1 = |f_2|\,d\mu_2$)

let $\rho = \mu_1 + \mu_2 \Rightarrow \begin{cases} \mu_1 \ll \rho \\ \mu_2 \ll \rho \end{cases}$

(Since here we consider complex measure, then f is naturally integrable.)

$$f_1 \frac{d\mu_1}{d\rho}\,d\rho = f_2 \frac{d\mu_2}{d\rho}\,d\rho = d\nu.$$

$$\left(\int_E f_i \frac{d\mu_i}{d\rho}\,d\rho = \int_E f_i\,d\mu_i = \nu(E) \right).$$

$$\Rightarrow f_1 \frac{d\mu_1}{d\rho} = f_2 \frac{d\mu_2}{d\rho} \quad \rho\text{-a.e.}$$

$$\frac{d\mu_j}{d\rho} \geq 0 \Rightarrow |f_1| \frac{d\mu_1}{d\rho} = \left| f_1 \frac{d\mu_1}{d\rho} \right| = \cdots = |f_2| \frac{d\mu_2}{d\rho} \quad \rho\text{-a.e.}$$

$$\Rightarrow |f_1| \frac{d\mu_1}{d\rho}\,d\rho = |f_2| \frac{d\mu_2}{d\rho}\,d\rho$$

$$\Rightarrow |f_1|\,d\mu_1 = |f_2|\,d\mu_2 \qquad \square$$

Now we specialize our study in Euclidean space and obtain N-L formula once again.

$\boxed{\text{lem}}$ (covering) $\mathcal{C}$ is a collection of open balls in $\mathbb{R}^n$. let $U = \bigcup_{B \in \mathcal{C}} B$. If $c < m(U)$, there exists disjoint $B_1, \cdots B_k \in \mathcal{C}$ s.t. $\sum_{j=1}^{k} m(B_j) > 3^{-n} c$.

Idea: $c < \underset{\text{compact}}{m(K)} < m(U)$ $\quad \bigcup_{n=1}^{k} B_n \supseteq K$. choose biggest one by one. $\square$

And take a estimation by geometry.

$\boxed{\text{Def}}$ A measurable function $f: \mathbb{R}^n \to \mathbb{C}$ is called locally integrable. write (L_{loc}).

if $\int_K |f| dx < \infty$ for every bounded measurable set $K \subseteq \mathbb{R}^n$

$f \in L^1_{loc}$, $x \in \mathbb{R}^n$ Average value : $A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dy$.

$\boxed{\text{Lem}}$ If $f \in L^1_{loc}$, $A_r f(x)$ is jointly continuous in r & x $r > 0$, $x \in \mathbb{R}^n$

Idea $A_r = \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dy$ is jointly continuous in (x_0, r_0)

$= \int \chi_{B(x,r)} f / m B(x,r) \, dy$

$\left| \chi_{B(x,r)} f / \cancel{m(B(x,r))} r^n \right| \le \chi_{B(x_0, r_0+1)} |f| / (r_0/2)^n$

(x,r) : $\begin{cases} |x - x_0| < \frac{1}{2} \\ |r - r_0| < \frac{1}{2} \end{cases} \cap |r_0| > \frac{1}{2}|r_0|$

By DCT. $\square$

$\boxed{\text{Def}}$ Hardy-littlewood maximal function

$Hf = \sup_{r>0} A_r |f|(x) \; \sup_{B_0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dy$

$\boxed{\text{Thm}}$ (1,1)-weak type for $\forall f \in L'$. $\exists C > 0$

$m(\{ Hf > \alpha \}) \le \frac{C}{\alpha} \|f\|_1$

Idea. $E_\alpha = \{ Hf > \alpha \}$ $\forall x \in E_\alpha$. $\exists r_x$ s.t. $A_{r_x} f(x) > \alpha$

$\Rightarrow \mathcal{C} = \{ B_x \}$ $\bigcup_{\mathcal{C}} B_x \supseteq E_\alpha$

$\Rightarrow m(E_\alpha) \underset{\substack{\uparrow \\ \text{Covering}}}{\le} 3^n \sum_{\text{finite}} m(B_x) = \frac{3^n}{\alpha} \sum \alpha \, m(B_x)$

$\le \frac{3^n}{\alpha} \sum \int_{B_x} |f| \le \frac{3^n}{\alpha} \int |f|$ $\square$

$\boxed{\text{Thm}}$ $f \in L^1_{loc}$ $\lim_{r \to 0} A_r f(x) = f(x)$ for a.e. x (Differentiation theorem ver 1)

$g \in C \cap L'$ $g \xrightarrow{L'} f \chi_{N(1)}$ (σ-division WLOG. $f \in L'$)

technique : $\lim_{r \to R} \phi(r) = c \Leftrightarrow \limsup_{r \to R} |\phi(r) - c| = 0$

$\underset{r \to 0}{|A_r f(x) - f(x)|} \le \underset{\downarrow H(f-g)}{|A_r(f - g)|} + \underset{\downarrow 0}{|A_r g(x) - g(x)|} + \underset{\downarrow \text{a.e.}}{|g(x) - f(x)|}$

$\cancel{|A_r f - f| > \alpha}$

$E_\alpha = \{\limsup_{r\to 0} |A_r f - f| > \alpha\}$ $\quad F_\alpha = \{|f-g|\} > \alpha$

$E_\alpha \subseteq F_\alpha \cup \{ H(f-g) > \alpha/2 \}$

$\alpha/2\, m(F_\alpha) \le \int_{F_\alpha} |f-g| \le \int |f-g| < \varepsilon$

$m(\{H(f-g) > \alpha/2\}) \le \frac{2C}{\alpha} \| f-g \| < \frac{2C}{\alpha}\varepsilon$

$\Rightarrow m(E_\alpha) < \frac{2(C+1)}{\alpha}\varepsilon \quad \varepsilon \to 0$

$\Rightarrow m(E_\alpha) = 0$

$\boxed{\text{Rmk}}$ Using maximal function, we lift the estimation from almost one point to the integral nearby, and we shall see integral is easier to estimate much easier.

Assume ~~more~~ a little stronger. Lebesgue set $L_f = \{x : \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| = 0\}$

$\boxed{\text{Thm}}$ $m(L_f^c) = 0$

Idea $\quad c \in \mathbb{C}$. $\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c| \longrightarrow |f(x) - c|$ a.e. except E_c with measure zero

$\mathbb{C}$ is separable. let $D \subseteq \mathbb{C}$ & D is countable

$\forall \varepsilon > 0.\ \exists c \in D.\ |f(x) - c| < \varepsilon$

$\limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \le 2|f(x) - c| < 2\varepsilon. \quad \varepsilon \to 0.$

$B(x,r)$ can be generalized to "nicely shrink sets"

$\boxed{\text{Def}}$ A Borel ~~set is~~ measure ν on $\mathbb{R}^n$ is called regular if

- $\nu(K) < \infty$ for every compact K
- $\nu(E) = \inf\{\nu(U) : U \text{ open } E \subseteq U\}$ for $\forall E \in \mathcal{B}_{\mathbb{R}^n}$

signed measure or complex measure is regular if $|\nu|$ is.

$\boxed{\text{Thm}}$ ν is regular $\quad d\nu = d\lambda + f\,dm$ then for m-a.e. $x \in \mathbb{R}^n$.

$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x) \qquad E_r \xrightarrow[\text{shrink}]{\text{nicely}} x$

$\boxed{\text{EX}}$ 3.4.26. $\lambda, \mu \ge 0 \quad \lambda \perp \mu$. $\quad \lambda + \mu$ is regular $\Rightarrow \lambda, \mu$ is regular

$\lambda(K) + \mu(K) < \infty \Rightarrow \lambda(K),\ \mu(K) < \infty$

$E \subseteq U.\quad \lambda(E) + \mu(E) + \varepsilon \ge \lambda(U) + \mu(U) \xRightarrow{\lambda \perp \mu}$ $\qquad \square$

13. $\boxed{\text{Thm}}$ Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f\,dm$ be its LRN decomposition. Then for almost $\forall x \in \mathbb{R}^n$.

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x) \quad \text{for every family } \{E_r\}_{r>0} \text{ that shrinks nicely to } x.$$

proof.
$$\begin{aligned} d\nu^+ &= d\lambda_1 + f_1\,dm \\ d\nu^- &= d\lambda_2 + f_2\,dm \end{aligned} \Rightarrow d|\nu| = d(\lambda_1+\lambda_2) + (f_1+f_2)\,dm = \underbrace{d|\lambda|}_{\text{regular}} + \underbrace{|f|\,dm}_{\text{regular}}$$

$\Rightarrow f \in L^1_{loc}$ ($f\,dm$ regular)

need to show $\frac{\lambda(E_r)}{m(E_r)} \to 0$ for a.e. x.

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \le \frac{|\lambda|(E_r)}{m(E_r)} \le \frac{|\lambda|(B(x,r))}{m(E_r)} \le \frac{|\lambda|\,B(x,r)}{\alpha\, m(B(x,r))}.$$

Assume $\lambda \ge 0$ $A \in \mathcal{B}_{\mathbb{R}^n}$ $\lambda(A) = m(A^c) = 0$

let $F_k = \left\{x \in A : \limsup \frac{\lambda(B(x,r))}{m(B(x,r))} > \frac{1}{k}\right\}$

By regularity of λ, $\forall \varepsilon > 0$, $\exists U_\varepsilon \supseteq A$, $\lambda(U_\varepsilon) < \varepsilon$. $\forall x \in F_k$,

is center of $B_x \subseteq U_\varepsilon \Rightarrow \lambda(B_x) > \frac{1}{k} m(B(x,r))$

$U_\varepsilon \supseteq V_\varepsilon = \bigcup_{x\in F_k} B_x \supseteq F_k$ $m(V_\varepsilon) > c$

$$\Rightarrow c < 3^n \sum_{\text{finite}} m(B(x_j, r_{x_j})) < 3^n k \sum_{\text{finite}} \lambda(B_x) < 3^n k\, \lambda(V_\varepsilon)$$
$$< 3^n k\, \lambda(U_\varepsilon) \le 3^n k\, \varepsilon \Rightarrow \lambda(V_\varepsilon) \le 3^n k\, \varepsilon$$

$\Rightarrow m(F_k) \le 3^n k\, \varepsilon \to 0$ $\square$

~~$\nu = \lambda + f\,dm \Rightarrow |\nu| = |\lambda| + |f|\,dm$ $\perp$~~

~~proof of: lem. $\mu \perp \nu$ (A, B) $|\mu+\nu|(E) = |\mu|(E) + |\nu|(E)$~~
~~$|\mu(E\cap A) + \nu(E\cap B)|$~~

I don't ~~totally~~ understand all ~~the~~ details in ~~the proof~~, but I guess its not so necessary as I thought. and I have understood ~~the~~ key with certitude.

$\boxed{\text{Thm}}$ $F: \mathbb{R} \to \mathbb{R}$ is increasing. $G(x) = F(x+)$

a. The set of points at which F is discontinuous is countable

b. F and G are differentiable a.e. & $F' = G'$ a.e.

proof. a. trivial

b. $G \nearrow$ trivial

G is right continuous

$0 \le G(x+h) - G(x) = F(x+h+) - F(x+)$

$h \to 0$ $F(x+h+) \to F(x+)$ ✓

$G(x+h) - G(x) = \mu_G((x, x+h])$ $h>0$

Using last thm to μ_G & μ_G is regular

$$\frac{\mu_G(E_r)}{m(E_r)} = \text{difference quotient of } G = G' \text{ a.e.}$$

Now, let $H = G - F$. We aim to show H' a.e. exists and equals zero.

It's not hard to see that, if $\{x_j\}$ is an enumeration of points at which $H \neq 0$. $H(x_j) > 0$ (of course countable)

$\Rightarrow \sum_{|x_j|<N} H(x_j) < \infty$ let μ be a new measure $\mu = \sum_j H(x_j)\delta_j$

where δ_j is dirac function.

$\Rightarrow \mu$ is bounded in cpt set $\Rightarrow \mu$ is regular

$\Rightarrow$ let $E = \{x_j\}$ $\Rightarrow \mu \perp m$

$$\left|\frac{H(x+h) - H(x)}{h}\right| \le \frac{H(x+h) + H(x)}{|h|} \le 4\frac{\mu((x-2h, x+2h))}{4|h|} \xrightarrow{h\to 0} 0 \text{ a.e.} \quad \square$$

$\boxed{\text{Def}}$ (total variation without smoothness hypotheses)

$F: \mathbb{R} \to \mathbb{C}$. $x \in \mathbb{R}$. $T_F = \sup \{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \}$

T_F is called the total variation function of F

If $T_F(\infty) = \lim_{x\to\infty} T_F(x)$ is finite, we say F is of Bounded Variation.

$\boxed{\text{BV}}$

$\boxed{\text{Example}}$
1. $F \nearrow$ Bounded and increasing
2. linear combination
3. F' is bounded $F' \in BV[a,b]$
4. $F(x) = x\sin\frac{1}{x}$ $F(0) = 0$ $F \notin BV$ contains 0.

5. **Lem** $F \in BV$ is real-valued. $T_F + F$ & $T_F - F \nearrow$

Proof. $x<y$. $\sum_1^n |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon$

$$T_F(y) \pm F(y) \geq |F(y) - F(x)| + \sum_1^n |F(x_j) - F(x_{j-1})| \pm F(y)$$
$$\geq T_F(x) - \varepsilon + |F(y) - F(x)| \pm [F(y) - F(x)] \pm F(x)$$
$$\geq T_F(x) - \varepsilon \pm F(x)$$

let $\varepsilon \to 0$ then $T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$. $\square$

Rmk $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$. That is to say, Any BV function can be represented as the difference of two increasing functions.

Thm a. $F \in BV$ iff $\mathrm{Re}\,F$, $\mathrm{Im}\,F \in BV$

b. $F: \mathbb{R} \to \mathbb{R}$. F is BV iff F is the difference of two bounded increasing functions

c. If $F \in BV$, then $F(x+) = \lim_{y \searrow x} F(y)$ and $F(x-) = \lim_{y \nearrow x} F(y)$ exist for all $x \in \mathbb{R}$, as do $F(\pm\infty)$

d. If $F \in BV$, $G(x) = F(x+)$ then F' & G' exist and equals a.e.

e. If $F \in BV$, the discontinuous points of F is countable.

Proof. a. trivial

b. ($\Leftarrow$) linearity ($\Rightarrow$) $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$. $|F(y) - F(x)| \leq T_F(y) - T_F(x) < \infty$.

c. limit does exists. It does not mean continuity.

d. last theorem showed.

e. ...

$\square$

$NBV = \{F \in BV : F \text{ is right continuous } \& \ F(-\infty) = 0\}$

$G(x+) = F(x+) - F(-\infty)$
$= F_1(x+) - [F_2(x+) + F(-\infty)]$ is the difference of two increasing functions.

So G is NBV

Lem If $F \in BV$, then $T_F(-\infty) = 0$. If F is right continuous, so is T_F.

proof. $\sum_1^n |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon$

$\Rightarrow T_F(x) - T_F(x_0) \geq T_F(x) - \varepsilon$

$\Rightarrow T_F(y) \leq \varepsilon$ for $y \leq x_0$

$\Rightarrow T_F(-\infty) = 0$

let $\alpha = T_F(x+) - T_F(x)$ for a given $x \in \mathbb{R}$. We aim to show $\alpha = 0$. $\forall \varepsilon > 0$, $\exists \delta > 0$, $F(x+h) - F(x) < \varepsilon$, for $0 < h < \delta$. We can take $x = x_0 < x_1 < \cdots x_n = x+h$ s.t. $\sum_1^n |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}[T_F(x+h) - T_F(x)] \geq \frac{3}{4}\alpha$

$\sum_2^n |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}\alpha - |F(x_1) - F(x_0)| \geq \frac{3}{4}\alpha - \varepsilon$

$T_F(x+h) - T_F(x+) < \varepsilon$

likewise $\sum_1^m |F(t_j)-F(t_{j-1})| \geq \frac{3}{4}[T_F(x_1)-T_F(x_0)] \geq \frac{3}{4}\alpha$

$$\alpha+\varepsilon > T_F(x+h)-T_F(x) \geq \sum_2^n |T(x_j)-T(x_{j-1})| + \sum_1^m |T(t_j)-T(t_{j-1})|$$
$$\geq \frac{3}{2}\alpha - \varepsilon$$

$\Rightarrow \alpha < 4\varepsilon \quad \Rightarrow \alpha = 0$ □

[Rmk] What's the idea of the proof? We must estimate the value of α

First, loosen it to distance h, by definition $\begin{cases} T_F(x+h)-T_F(x+) < \varepsilon \\ |F(x+h)-F(x)| < \varepsilon \end{cases}$

$\Rightarrow \alpha+\varepsilon > T_F(x+h)-T_F(x)$

However, in a certain sense, we need to show the reversed ~~inequality~~ inequality

so ...

[Thm] μ is a complex Borel measure on $\mathbb{R}$ & $F(x)=\mu((-\infty,x])$, then F is NBV.

Conversely, if $F \in NBV$, there is a unique complex Borel measure μ_F such that $F(x)=\mu_F((-\infty,x])$; moreover $|\mu_F| = \mu_{T_F}$.

proof. the result is not hard except $|\mu_F| = \mu_{T_F}$.

but it's not in the main line ...

Question. what's the correspondence between NBV and LRN decomposition.

[prop] If $F \in NBV$, then $F' \in L'(m)$. $\mu_F \perp m$ iff $F'=0$ a.e.

$\mu_F \ll m$ iff $F(x)=\int_{-\infty}^x F'(t)\,dt$.

proof. $F'(x) = \lim_{r\to 0} \frac{\mu_F(E_r)}{m(E_r)}$ ... □

μ_F is obviously regular

[Def] A function $F:\mathbb{R}\to\mathbb{C}$ is called absolutely continuous if for $\forall \varepsilon>0$, there exists a $\delta>0$ such that for any finite disjoint intervals $(a_1,b_1)\dots$

$$\sum_1^N (a_j,b_j) < \delta \Rightarrow \sum_1^N |f(b_j)-f(a_j)| < \varepsilon.$$

Example. $|F'| \leq M + C' \Rightarrow AC \Rightarrow$ uniform continuous.

7. $\boxed{\text{prop}}$ If $F \in NBV$, then F is AC iff $\mu_F \ll m$

proof using the regularity of μ_F & ε-δ condition of $\mu_F \ll m$ □.

$\boxed{\text{Cor}}$ If $f \in L^1$, $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is NBV & AC.

$f = F'$ a.e. Conversely, if $F \in NBV$ is AC $\Rightarrow F' \in L^1(m)$, $F(x) = \int_{-\infty}^{x} F'(t)\,dt$

$\boxed{\text{lem}}$ If f is AC on $[a,b]$, then $F \in BV([a,b])$

$\boxed{\text{Thm}}$ The fundamental theorem of calculus for Lebesgue integrals.

If $-\infty < a < b < \infty$ and $F : [a,b] \to \mathbb{C}$, TFAE

a. F is AC on $[a,b]$

b. $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a,b], m)$

c. F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], m)$ and $F(x) - F(a) = \int_a^x F'(t)\,dt$

proof: $a \Rightarrow c$. set $F(a) = 0$ and extend it $\Rightarrow F \in NBV$

$\Rightarrow F(x) = \int_{-\infty}^{x} F'(t)\,dt = \int_a^x F'(t)\,dt$.

$c \Rightarrow b$ ✓

$b \Rightarrow a$ AC of integral. □

We now introduce an interesting result in calculus to end the chapter.

$\boxed{\text{Thm}}$ If f & g are in NBV and at least one of them is cts, then

for $-\infty < a < b < +\infty$

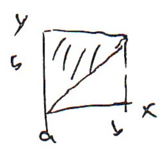
$$\int_{[a,b]} f\,dg + \int_{[a,b]} g\,df = f(b)g(b) - f(a)g(a)$$

proof. It suffices to show f & g is increasing. WLOG g is cts

By Fubini thm: $\Omega = a \le x \le y \le b$

$$\mu_f \times \mu_g(\Omega) = \int_{[a,b]} \int_{[a,y]} df(x)\,dg(y) = \int_{[a,b]} (f(y) - f(a))\,dg(y)$$

$$= \int_{[a,b]} f(y)\,dg(y) - f(a) \overbrace{\int_{[a,b]} d}^{(g(b)-g(a))}$$



$$= \int_{[a,b]} \int_{[x,b]} dg(y)\,df(x) = \int_{[a,b]} (g(b) - g(x))\,df(x)$$

$$= g(b)(f(b) - f(a)) - \int_{[a,b]} g(x)\,df(x)$$

$\Rightarrow$ LHS $= f(a)(g(b) - g(a)) + g(b)(f(b) - f(a)) = g(b)f(b) - g(a)f(a)$. □

Add a definition at the end of the chapter, The definition is a special case of Borel measure on $\mathbb{R}^n$, and it was use in "probability theory" lesson.

$\boxed{\text{Def}}$ A complex Borel measure on $\mathbb{R}^n$ is called discrete, if there is a countable set $\{x_j\} \subseteq \mathbb{R}^n$ and complex numbers c_j such that $\sum |c_j| < \infty$ and $\mu = \sum c_j \delta_{x_j}$ where δ_x is the point mass at x. On the other hand, μ is continuous if $\mu(\{x\}) = 0$ for $\forall x \in \mathbb{R}^n$.

$\mu = \mu_d + \mu_c$ where we denote $E = \{x : \mu(\{x\}) \neq 0\}$

and $\mu_d(A) = \mu(A \cap E)$, $\mu_c(A) = \mu(A \setminus E)$.

if μ is discrete, $\underset{E}{\mu} \perp \underset{E^c}{m}$.

if μ is continuous, $\mu \ll m$

$\Rightarrow \mu = \underset{\perp}{\mu_d} + \underset{\perp}{\mu_{sc}} + \underset{\ll}{\mu_{ac}}$

$\downarrow$

like C-L function