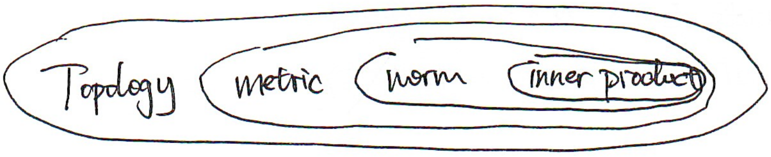
Chapter 4. Elements of functional analysis. (Brief version)

$\boxed{\text{Def 4.1}}$ (Semi norm) • $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \cancel{\mathfrak{X}}\, \mathcal{X}$

• $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in \mathcal{X}$ and $\lambda \in K$

(norm) Semi norm + $\|x\| = 0 \iff x = 0$

Where $\mathcal{X}$ is a vector space over $K (= \mathbb{R}$ or $\mathbb{C})$



A relation for geometric view.

$\boxed{\text{Def 4.2}}$ A normed vector space is complete with respect to norm metric is called **Banach Space**.

cauchy sequence converges.

$\boxed{\text{Def 4.3}}$ If $\{x_n\}$ is a sequence in $\mathcal{X}$, the series $\sum_1^\infty x_n$ is said to

convergent, if $\sum_1^N x_n \to x$ as $N \to \infty$

absolutely converge if $\sum_1^\infty \|x_n\| < \infty$.

$\boxed{\text{Thm 4.4}}$ A normed vector space $\mathcal{X}$ is complete iff every absolutely convergent series in $\mathcal{X}$ converge.

proof) If $\mathcal{X}$ is complete. let. $S_N = \sum_1^N x_n$ where $\sum_1^\infty \|x_n\| < \infty$

$\| S_M - S_N \| \leq \sum_{N+1}^M \|x_n\| \to 0$ as $M, N \to \infty$

$\Rightarrow \{S_N\}$ is a cauchy sequence $\Rightarrow S_N \to x$.

If ..., let $\{x_n\}$ be a cauchy sequence.

choose $n_1 < n_2 < \cdots$ s.t $\|x_m - x_n\| \leq 2^{-j}$ if $m, n > n_j$

let $y_1 = x_1$, $y_n = x_n - x_{n-1}$ for $n > 1$

$\Rightarrow x_{n_k} = \sum_1^k y_n$

$\Rightarrow \sum_1^\infty \|y_n\| \leq \|y_1\| + 1 < \infty \Rightarrow \{\sum_1^k y_n\} = \{x_{n_k}\}$ converges to x.

Since $\{x_n\}$ is a Cauchy sequence, they have the same limit. $\square$

$\boxed{\text{EX 4.1}}$ $L^1(\mu)$ is a Banach Space with L^1 norm

If $\sum_1^\infty \|f_n\|_1 < \infty$ $f = \sum_1^\infty f_n$ exists a.e. by DCT

and $\|f - \sum_1^N f_n\|_1 \leq \|\sum_{N+1}^\infty f_n\|_1 \leq \sum_{N+1}^\infty \|f_n\|_1 \to 0$ as $N \to \infty$

$\boxed{\text{Def 4.5}}$ A linear map $T: \mathcal{X} \to \mathcal{Y}$ between two normed vector spaces is called <u>Bounded</u> if ~~there exists~~ $C \geq 0$ s.t.

$$\|Tx\| \leq C\|x\| \text{ for all } x \in \mathcal{X}$$

$\boxed{\text{Rmk}}$ the definition is different from "Bounded function", and actually means that T is bounded on bounded subset of $\mathcal{X}$.

$\boxed{\text{Prop 4.6}}$ If $\mathcal{X}$, $\mathcal{Y}$ are normed vector space and $T: \mathcal{X} \to \mathcal{Y}$ is a linear map. ~~tfae~~ TFAE:

1. T is continuous ; 2. T is continuous at 0 ; 3. T is bounded.

proof: $1 \Rightarrow 2$

$2 \Rightarrow 3$: There is a neighborhood U of 0 s.t. $T(U) \subseteq \{y \in \mathcal{Y} : \|y\| \leq 1\}$

then $\exists\, \delta > 0$. s.t "$\{x : \|x\| < \delta\} \subseteq U \Rightarrow \|Tx\| \leq 1$" ~~$\Rightarrow \|T\lambda x\| = |\lambda|\,\|Tx\| \leq |\lambda|$~~

Whenever $\|x\| \leq a$ $\|Tx\| = \|T\frac{x\delta}{a} a \cdot \delta^{-1}\| \leq a \cdot \delta^{-1}$

$\Rightarrow \|Tx\| \leq \delta^{-1}\|x\|$ $\left(\begin{array}{l} \text{If not, } \exists x, \|Tx\| > \delta^{-1}\|x\| \text{, let } \|\overset{\tilde{x}}{\overbrace{\lambda x}}\| = a \\ \Rightarrow \|T\tilde{x}\| > \delta^{-1} a \quad \text{↯} \end{array} \right)$

$3 \Rightarrow 1$: $\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq C\|x_1 - x_2\|$ □

We denote the space of all <u>bounded linear maps</u> from $\mathcal{X}$ to $\mathcal{Y}$ by $L(\mathcal{X}, \mathcal{Y})$.

We always assume $L(\mathcal{X}, \mathcal{Y})$ to be equipped with <u>operator norm</u>

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} = \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}$$

Note that if $T \in L(\mathcal{X}, \mathcal{Y})$, $S \in L(\mathcal{Y}, \mathcal{Z})$, then

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\| \Rightarrow \|ST\| \leq \|S\|\|T\|$$

So that $ST \in L(\mathcal{X}, \mathcal{Z})$

$\boxed{\text{Prop 4.7}}$ If Y is complete, then $L(\mathcal{X}, \mathcal{Y})$ is complete.

proof: Let $\{T_n\}$ be a Cauchy sequence in $L(\mathcal{X}, \mathcal{Y})$. If $x \in \mathcal{X}$, then we have $\|T_m x - T_n x\| \leq \|T_m - T_n\|\|x\| \to 0$ as $m, n \to \infty$

$\Rightarrow \{T_n x\}$ is Cauchy sequence $\Rightarrow T_n x \to Tx$. where $Tx \overset{\text{def}}{=} \lim\limits_{n\to\infty} T_n x$.

? $\Rightarrow \|Tx - T_n x\| \to 0$ as $N \to \infty$ $\Rightarrow \|T - T_n\| \to 0$ as $N \to \infty$

$T(x+y) = \lim\limits_{n\to\infty} T_n(x+y) = \lim\limits_{n\to\infty} T_n x + \lim\limits_{n\to\infty} T_n y = Tx + Ty \Rightarrow T \in L(\mathcal{X}, \mathcal{Y})$

$\Rightarrow T$ is what we want. □

1 $\boxed{\text{Def 5.8}}$ • If $T \in L(\mathcal{X}, \mathcal{Y})$, T is said ~~to~~ be <u>invertible</u> if T is bijective and T^{-1} is bounded (i.e. $\|Tx\| \geq C\|x\|$ for some $C>0$)

• T is called an <u>isometry</u> if $\|Tx\| = \|x\|$ for $\forall x \in \mathcal{X}$.

$\boxed{\text{Rmk}}$ An isometry is injective but not necessarily surjective.

$\boxed{\text{Def 5.9}}$ A linear map from $\mathcal{X}$ to K is called <u>a linear functional</u> on $\mathcal{X}$. If $\mathcal{X}$ is a normed vector space, the space $L(\mathcal{X}, K)$ of bounded linear functional on $\mathcal{X}$ is call the dual space of $\mathcal{X}$ and is deonoted by $\mathcal{X}^*$.

$\boxed{\text{Rmk}}$ $\mathcal{X}^*$ is a Banach space with normed operator.

What's the relationship ~~between~~ $L(\mathcal{X}, \mathbb{R})$ and $L(\mathcal{X}, \mathbb{C})$.

$\boxed{\text{prop 5.10}}$ Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. If f is a complex linear functional on $\mathcal{X}$ and $u = \operatorname{Re} f$, then u is a real linear functional. and $f(x) = u(x) - iu(ix)$. for $x \in \mathcal{X}$. Conversely, if u is a real linear functional on $\mathcal{X}$ and $f: \mathcal{X} \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In that case, if $\mathcal{X}$ is normed, we have $\|u\| = \|f\|$

proof.

$\operatorname{Im} f \; \cancel{\operatorname{Re} if} = -\operatorname{Re}(if(x)) = -u(ix) \Rightarrow f(x) = u(x) - iu(ix)$.

On the other hand, if u is real linear and $f(x) = u(x) - iu(ix)$

$f(ix) = u(ix) - iu(-x) = i(u(x) - iu(ix)) = if(x)$

$|u(x)| = |\operatorname{re} f(x)| \leq |f(x)|$

if $f(x) \neq 0$. $\overline{\operatorname{sgn} f(x)} = \alpha$. $|f(x)| = \underbrace{\boxed{\alpha f(x)}}_{}^{\text{real}} = f(\alpha x) = u(\alpha x)$

$\Rightarrow \|u\| = \|f\|$

It's not obvious that there are any non-zero bounded linear functionals on an arbitrary $\boxed{\text{normed vector space}}$

$\boxed{\text{Def 5.11}}$ A sublinear functional on $\mathcal{X}$ is a map $p: \mathcal{X} \to \mathbb{R}$ s.t.

$p(x+y) \leq p(x) + p(y)$. $p(\lambda x) = \lambda p(x)$ for all $x, y \in \mathcal{X}$. $\lambda \geq 0$.

$\boxed{\text{Rmk}}$ Why do we introduce such a definition? Indeed, any semi-norm is a sub-linear functional.

$\boxed{\text{Thm 5.12}}$ (Hahn–Banach Theorem) Let $\mathcal{X}$ be a real vector space, p a sublinear functional on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \le p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{M}$ s.t. $F(x) \le p(x)$ for all $x \in \mathcal{X}$, and

$F|_{\mathcal{M}} = f$.

Idea: extend carefully, and use Zorn's lem to find the maximal one.

proof: If $x \in \mathcal{X} \setminus \mathcal{M}$, we're showing f can be extended to a linear functional g on $\mathcal{M} + \mathbb{R}x$ satisfies $g(y) \le p(y)$.

If $y_1, y_2 \in \mathcal{M}$, $f(y_1) + f(y_2) = f(y_1+y_2) \le p(y_1+y_2) \le \cancel{p(y_1)+p(y_2)}$ $p(y_1-x)+p(x+y_2)$.

$$f(y_1) - p(y_1 - x) \le p(x+y_2) - f(y_2)$$

$$\Rightarrow \sup\{f(y) - p(y-x) : y \in \mathcal{M}\} \underbrace{\le \inf}_{\le \alpha \le} \{p(x+y) - f(y) : y \in \mathcal{M}\}$$

define $g(y+\lambda x) = f(y) + \lambda\alpha$

$$g(y+\lambda x) = f(y) + \lambda\alpha = \lambda\left(f(y/\lambda) + \alpha\right) \le \lambda\left(f(y/\lambda) + p(x + y/\lambda) - f(y/\lambda)\right)$$

if $\lambda > 0$

$$= p(y+\lambda x)$$

likewise for $\lambda < 0$ $\Rightarrow g(z) \le p(z)$ for $z \in \mathcal{M} + \mathbb{R}x$.

Let $\mathcal{F}$ denote the family of all linear extension F of f that satisfies $F \le p$.
We shall see that $\mathcal{F}$ has natural partial ~~ordered~~ structure by set inclusion, and the union of increasing domain should be the maximal element

By Zorn' lemma, done! □

By argument appeared in $\boxed{\text{Prop 5.10}}$, we can easily obtain the complex version of the $\boxed{\text{Thm 5.12}}$

For convenience, we shall assume the scalar field is $\mathbb{C}$.

So, how can we apply the theorem?

$\boxed{\text{Thm}}$ 5.13 Let $\mathcal{X}$ be a normed vector space

a. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{M}$, there exists $f \in \mathcal{X}^*$ such $f(x) \ne 0$ and $f|_{\mathcal{M}} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$, f can be taken to satisfy $\|f\|$ & $f(x) = \delta$. (think of projection).

b. If $x \ne 0 \in \mathcal{X}$, then there exists $f \in \mathcal{X}^*$ such that $\|f\| = 1$ & $f(x) = \|x\|$.

3.

c. The bounded linear functional on $\mathcal{X}$ separate points

d. If $x \in \mathcal{X}$, define: $\hat{x} : \mathcal{X}^* \to \mathbb{C}$, by $\hat{x}(f) = f(x)$. then the map $x \mapsto \hat{x}$ is a linear isometry ~~for~~ from $\mathcal{X}$ into $\mathcal{X}^{**}$.

proof: a. define f on $\mathcal{M} + \mathbb{C}x$ by $f(y + \lambda x) = \lambda \delta$

$|f(y+\lambda x)| = |\lambda||\delta| \le |\lambda| \|\lambda^{-1} y + x\| = \|y + \lambda x\|$

$\Rightarrow$ BH Thm

b. in a., $\mathcal{M} = 0$

c. $x - y \ne 0$, then use b.

d. $|\hat{x}(f)| = |f(x)| \le \|f\| \|x\| \Rightarrow \|\hat{x}\| \le \|x\|$

b. $\Rightarrow \|\hat{x}\| \ge \|x\| \Rightarrow \|\hat{x}\| = \|x\|$ $\square$

$\boxed{\text{Rmk}}$ $\hat{\mathcal{X}} := \{\hat{x} : x \in \mathcal{X}\}$. $\mathcal{X}^{**} = L(\mathcal{X}^*, \mathbb{C})$ is complete.

$\hat{\mathcal{X}} \subseteq \overline{\hat{\mathcal{X}}} \xhookrightarrow{\subseteq} \mathcal{X}^{**}$

$x \longmapsto \hat{x} \in \overline{\hat{\mathcal{X}}}$

$\mathcal{X} \xhookrightarrow[\text{dense}]{} \overline{\hat{\mathcal{X}}}$ (Banach.) So we can denote $\overline{\hat{\mathcal{X}}}$ the completion of $\mathcal{X}$

If $\mathcal{X}$ is a Banach space, we can see them the same, i.e. $\mathcal{X} = \overline{\hat{\mathcal{X}}}$.

If $\mathcal{X}$ is of finite dim, $\hat{\mathcal{X}} = \mathcal{X}^{**}$; But for infinite case,

$\hat{\mathcal{X}} \overset{?}{=} \mathcal{X}^{**}$. If so, we shall call $\mathcal{X}$ is reflexive.

Since we usually identify $\mathcal{X}$ and $\hat{\mathcal{X}}$, so reflexivity means $\mathcal{X} = \mathcal{X}^{**}$.

Now we focus on Compact metric space, and introduce an important result

$\boxed{\text{Thm 5.14}}$ Baire Category theorem

Let X be a complete metric space

1. If $\{U_n\}$ is a sequence of open dense sets, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.
2. X is not a countable ~~the~~ union of nowhere dense sets

proof. 1. $W \overset{\ne \emptyset}{}$ is open in X. It suffices to show $W \cap \left(\bigcap_{n=1}^{\infty} U_n\right) \ne \emptyset$

$U_1 \cap W \ne \emptyset$. $B(r_0, x_0) \subseteq U_1 \cap W_1$ $(0 < r_0 < 1)$. then $\overline{B(r_n, x_n)} \subseteq B(r_{n-1}, x_{n-1}) \cap U_n$

where $r_n < 2^{-n}$. It leads a Cauchy Sequence $\Rightarrow x = \lim x_n$

$x \in \overline{B(r_N, x_N)} \subseteq U_N \cap B(r_0, x_0) \subseteq U_N \cap W \Rightarrow \cdots$

2. E_n is a nowhere dense set in X. $\overline{E_n}^c$ is an open dense set $\left({}^{\circ} = {}^{c\,-\,c} \right)$

~~$X = \bigcup E_n$~~ ~~$\bigcap (\overline{E_n}^c) = (\bigcup \overline{E_n})^c$~~ $\Rightarrow \bigcap_{n=1} \overline{E_n}^c$ = ~~$(\bigcup E_n)^c \ne X$~~

Since E_n^c is an open dense set

$\Rightarrow \bigcap E_n^c \neq \phi \Rightarrow (\bigcup E_n)^c \neq \phi \Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n} \neq X$ $\square$

$\boxed{\text{Def 5.15}}$ If X is a topological space, a set $E \subseteq X$ is of the first category if E is a countable union of nowhere dense sets. otherwise E is of second category.

$\boxed{\text{Rmk}}$ By Baire Category theorem, every complete metric space is of the second category in itself. There is a descriptive word to "~~of~~ the set of the first category", meager.

$\boxed{\text{Thm 5.16}}$ The open mapping theorem

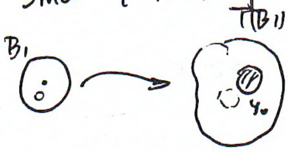
Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T \in L(\mathcal{X}, \mathcal{Y})$ is surjective, then T is open

proof: By standard argument, it suffices to show $B(0,r) \subseteq T(B_1)$, $B_1 = B(0,1)$.

$\mathcal{X} = \bigcup_{n=1}^{\infty} B_n \Rightarrow \mathcal{Y} = \bigcup_{n=1}^{\infty} T(B_n)$. But $T(B_i) \simeq T(B_j)$, & $\mathcal{Y}$ is complete

$\Rightarrow T(B_1)$ can't be nowhere dense $\Rightarrow \exists\, y_0 \in \mathcal{Y}$, s.t. $B(y_0, 4r) \subseteq \overline{T(B_1)}$

Since T is surjective, pick $y_1 \in B(y_0, 4r)$ s.t. $B(y_1, 2r) \subseteq B(y_0, 4r)$.



So $\|y_0 - y_1\| < 2r$, suppose $y_1 = Tx_1 \in T(B_1)$

$y_0 = -Tx_1 + (y_1 + y) \subseteq \overline{T(-x_1 + B_1)} \subseteq \overline{T(B_2)}$

(Here our idea is to translate the interior point to the origin). In a certain sense, the technique here is similar to Steinhaus's theorem ?

~~$\Rightarrow \exists r>0$. if $\|y\| < r$ $y \in \overline{T(B_1)}$. $x_1 \in B_{1/2}$ $\|y_1 - Tx_1\| < \frac{r}{4}$~~

~~$\Rightarrow$ if $\|y\| < r/2^n$ $y \in \overline{T(B_{2^{-n}})}$ $x_n \in B_{2^{-n}}$~~

~~$\|y - \sum_{i=1}^{n} Tx_i\|$~~ Now, we shall show $y \in T(B_1)$, instead of $\overline{T(B_1)}$.

If $\|y\| < r/2$ $\exists\, x_1 \in B_{1/2}$ s.t. $\|y - Tx_1\| < r/4$.

$\Rightarrow \|y - \sum_{j=1}^{n} Tx_j\| < \frac{r}{2^{n+1}}$ $\Rightarrow$ ~~if~~ $x = \sum_{i=1}^{\infty} x_i$. $y = Tx$.

However, $\sum \|x_i\| < 1 \Rightarrow y \in T(B_1)$. $\square$

$\boxed{\text{Def 5.17}}$ Graph of T: $\Gamma(T) = \{ (x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx \}$.

T is closed if $\Gamma(T)$ is closed subspace.

$\boxed{\text{Rmk}}$ If T is continuous, then T is closed.

(x, y) is limit point of $\Gamma(T)$.

$(x_n, y_n) \to (x, y) \in X \times Y$ $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tx$

$\Rightarrow (x, y) \in \Gamma(T)$ ✓.

However, what about the reverse?

55. $\boxed{\text{Thm 5.18}}$ The closed graph theorem

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T: \mathcal{X} \to \mathcal{Y}$ is a closed map, then T is bounded.

proof: $\pi_1(x, Tx) = x$. $\quad \pi_1 \in L(\Gamma(T), \mathcal{X})$

$\pi_2(x, Tx) = Tx$. $\quad \pi_2 \in L(\Gamma(T), \mathcal{Y})$

Since $\mathcal{X}$, $\mathcal{Y}$ is complete $\Rightarrow$ so is $\mathcal{X} \times \mathcal{Y}$.

$\Rightarrow \Gamma(T)$ is complete.

Since $\pi_1: \Gamma(T) \to \mathcal{X}$ is a bijection, by open mapping theorem,

π_1^{-1} is bounded $\Rightarrow T = \pi_2 \circ \pi_1^{-1}$ is bounded. $\square$.

$\boxed{\text{Rmk}}$ continuity: $x_n \to x \Rightarrow Tx_n \to Tx$

closedness: $x_n \to x_n$, $Tx_n \to y$ then $y = Tx$.

$\boxed{\text{Thm 5.19}}$ Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $\mathcal{A}$ is a subset of $L(\mathcal{X}, \mathcal{Y})$.

a. If $\sup\limits_{T \in \mathcal{A}} \|Tx\| < \infty$ for all x in some nonmeager subset of $\mathcal{X}$.

then $\sup\limits_{T \in \mathcal{A}} \|T\| < \infty$

b. If $\mathcal{X}$ is a Banach space and $\sup\limits_{T \in \mathcal{A}} \|Tx\| < \infty$ for all $x \in \mathcal{X}$.

then $\sup\limits_{T \in \mathcal{A}} \|T\| < \infty$.

$\boxed{\text{Rmk}}$ The theorem is not so obvious as I thought.

An ~~ex~~ interesting example collected from ~~MSE~~ Deepseek.

$V = \{ x = (x_k)_{k \geq 1} : \sup\limits_{k \geq 1} k|x_k| < \infty \}$. equip V with uniform norm.

Then V is not complete.

$\mathcal{A}$: $F_k(x) = k x_k \Rightarrow \mathcal{F} = \{ F_k : k \in \mathbb{N} \}$ is pointwise bounded.

$F_k((0 \cdots 1 \cdots 0 \cdots)) = k. \Rightarrow \|F_k\| \geq k$

$\Rightarrow \sup\limits_{\mathcal{F}} \|T\| = +\infty$. $\quad \zeta$

$V = \{$ polynomials on $[0,1]$ $\}$ with L^∞ norm (or ~~max~~)

$\mathcal{A} = \{$ ~~differential operator $\frac{d^n}{(dx)^n}$~~, $n \in \mathbb{N}$ $\}$ at $x = 0 \}$.

derivative of order n

V is not complete by ~~Weierstrass's theorem~~

V is pointwise bounded $\checkmark$

However $\sup\limits_{T \in \mathcal{A}} \|T\| \geq \left(\frac{d}{dx}\right)^n x^n \Big|_{x=0} = n! \to \infty$.

proof: Let $E_n = \{x \in \mathcal{X} : \sup_{T \in \mathcal{A}} \|Tx\| \le n\} = \bigcap_{T \in \mathcal{A}} \{\|Tx\| \le n\}$.

$\Rightarrow E_n$ is closed. Since $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for x in some nonmeager subset of $\mathcal{X}$

$\Rightarrow \exists n.\ E_n \supseteq \overline{B(x_0, r)}$ where $r > 0$.

If $\|x\| \le r$. $\|Tx\| \le \|T(\underbrace{x+x_0}_{\in E_n})\| + \|Tx_0\| \le 2n$

$\Rightarrow \overline{B(0,r)} \subseteq E_{2n} \Rightarrow \|Tx\| \le 2n$ for $\forall x \le r \Rightarrow \sup_{T\in\mathcal{A}} \|T\| \le \frac{2n}{r}$.

$\mathcal{X}$ is Banach space $\Rightarrow$ $\mathcal{X}$ is nonmeager. □

To introduce weak topology, we need recall the knowledge of net.

"Net is a generalization of sequence"

Net converge: $\langle x_\alpha \rangle_{\alpha \in A}$ A is an index set.

in E eventually: $\exists \alpha_0 \in A$ $x_\alpha \in E$ if $\alpha \gtrsim \alpha_0$

frequently: $\forall \alpha \in A$ $\exists \beta \gtrsim \alpha$. $x_\beta \in E$

$x_\alpha \to x$ $\exists U \in \mathcal{N}(x)$. $\langle x_\alpha \rangle$ is eventually in U.

x is a cluster point of $\langle x_\alpha \rangle$ if $\forall U \in \mathcal{N}(x)$. $\langle x_\alpha \rangle$ is frequently in U.

Since Net consider more information than a sequence, which only possess countable information, net converge has better property.

$\boxed{\text{prop}}$ x is an accumulation point of E. iff there is a net in $E \setminus \{x\}$ that converges to x

$x \in \overline{E}$ iff there is a net $\langle x_\alpha \rangle$ in E that converges to x.

$\boxed{\text{prop}}$ $f: X \to Y$ is continuous at x iff for every net $\langle x_\alpha \rangle$ converges to x, $\langle f(x_\alpha) \rangle$ converges to y.

subnet: $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of net $\langle x_\alpha \rangle_{\alpha \in A}$ with map $\beta \mapsto \alpha_\beta$. if

1. $\forall \alpha_0 \in A$. $\exists \beta_0 \in B$. $\alpha_\beta \gtrsim \alpha_0$ whenever $\beta \gtrsim \beta_0$
2. $y_\beta = x_{\alpha_\beta}$.

$\boxed{\text{prop}}$ x is a cluster point of $\langle x_\alpha \rangle$ iff $\langle x_\alpha \rangle$ has a subnet that converges to x.

$\boxed{\text{Def 5.20}}$ A topological vector space is a ~~topology~~ vector space endowed with a topology s.t. $(x,y) \mapsto x+y$ $(\lambda, x) \mapsto \lambda x$ are continuous.

$\boxed{\text{Def 5.21}}$ the definition of locally convex ~~topological vector~~ space seems subtle.

Here we alter the simple version in Folland: exist a base for topology consisted of convex sets.

57 In fact, Folland has proved the equivalence of different definitions.

$\boxed{\text{Thm 5.22}}$ Let $\{p_\alpha\}_{\alpha \in A}$ be a family of seminorms on the vector space $\mathcal{X}$. If $x \in \mathcal{X}$, $\alpha \in A$ and $\varepsilon > 0$. let

$$U_{x\alpha\varepsilon} = \{ y \in \mathcal{X} : p_\alpha(y-x) < \varepsilon \}$$

and $\mathcal{T}$ be the topology generated ~~of~~ by the sets $U_{x\alpha\varepsilon}$. (generated by finite intersections)

1. For each $x \in \mathcal{X}$, the finite intersection of the set $U_{x\alpha\varepsilon}$ $\{\alpha \in A, \varepsilon > 0\}$ form a neighborhood base at x
2. if $\langle x_i \rangle_{i \in I}$ is a net in $\mathcal{X}$, then $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ for $\forall \alpha \in A$.
3. $(\mathcal{X}, \mathcal{T})$ is a locally convex topological vector space.

proof. 1. It suffices to show $x \in \bigcap_{j=1}^{k} U_{x\alpha_j\delta_j} \subseteq \bigcap_{j=1}^{k} U_{x_j\alpha_j\varepsilon_j}$.

We have $x \in U_{x_j\alpha_j\varepsilon_j}$: $p_{\alpha_j}(x - x_j) < \varepsilon_j$. let $\delta_j = \varepsilon_j - p_{\alpha_j}(x - x_j)$

$x \quad x_j$

$\underbrace{\delta_j \; p_{\alpha_j}}_{\varepsilon_j}$

$\Rightarrow$ if $p_\alpha(y - x_j) \le p_\alpha(y - x) + p_\alpha(x_j - x)$

$y \in U_{x\alpha_j\delta_j}$ $\quad < \delta_j + p_\alpha(x - x_j) = \varepsilon_j$

$\Rightarrow x \in \bigcap_{j=1}^{k} U_{x\alpha_j\delta_j} \subseteq \bigcap_{j=1}^{k} U_{x_j\alpha_j\varepsilon_j}$.

2. $x_i \to x \Leftrightarrow x_i$ is ~~frequently~~ eventually in $U_{x\alpha_j\varepsilon}$ $\forall \varepsilon > 0$, $\alpha_j \in A$.

$\Leftrightarrow p_\alpha(x_i - x) \to 0 \quad \forall \alpha \in A$. ◻

3. It's a topological vector space :

$x_i \to x$, $y_i \to y$ (net)

$p_\alpha((x_i + y_i) - (x+y)) \le p_\alpha(x_i - x) + p_\alpha(y_i - y) \to 0$.

$\Rightarrow x_i + y_i \to x + y$.

$\lambda_i \to \lambda \quad p_\alpha(\lambda_i x_i - \lambda x) \le p_\alpha(\lambda_i(x_i - x)) + p_\alpha((\lambda_i - \lambda)x)$

$= \underbrace{|\lambda_i|}_{\text{bounded}} p_\alpha(x_i - x) + \underbrace{|\lambda_i - \lambda|}_{\to 0} p_\alpha(x) \to 0$

locally convex : $\forall y, z \in U_{x\alpha\varepsilon}$

$$p_\alpha(ty + (1-t)z - x) \le p_\alpha(t(y-x)) + p_\alpha((1-t)(z-x)) < \varepsilon$$

◻

So, weak topology is not so abstract as I thought, but it's determined by a family of semi norm.

The following proposition is an analogue of $\boxed{\text{prop 5.3}}$

Prop 5.23 Suppose $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces with the topology defined by the families $\{p_\alpha\}_{\alpha\in A}$, $\{q_\beta\}_{\beta\in B}$ of seminorms. and $T: \mathcal{X}\to\mathcal{Y}$ is a linear map. Then T is continuous iff for each $\beta\in B$, there exists $\alpha_1, \cdots \alpha_k \in A$, and $C>0$ such that $q_\beta(Tx) \le C\sum_{j=1}^{k} p_{\alpha_j}(x)$.

proof: $(\Leftarrow)$ $\langle x_i\rangle \to x$ (net) $\Leftrightarrow$ $p_\alpha(x_i - x)\to 0$ for $\forall \alpha$

$$q_\beta(Tx - Tx_i) \le C\sum_{j=1}^{k} p_{\alpha_j}(x - x_j) \to 0 \quad \text{for } \forall\beta$$

i.e. $\langle Tx_i\rangle \to Tx$ (net) $\Leftrightarrow$ T is continuous.

$(\Rightarrow)$ T is continuous. $\Rightarrow \forall\beta>0$. $\exists\,\mathcal{U}$ is a neighborhood of 0 s.t. $q_\beta(Tx)<1$ for $\forall x\in\mathcal{U}$.

Assume $\mathcal{U} = \bigcap_{j=1}^{k} U_{0\alpha_j\varepsilon_j}$, then let $\varepsilon = \min(\varepsilon_1, \cdots \varepsilon_k)$

$\Rightarrow$ $\boxed{q_\beta(Tx)<1 \text{ whenever } p_\alpha(x)<\varepsilon}$ for all j

If $p_{\alpha_j}(x) > 0$. for some j $\Rightarrow$ let $y = \dfrac{\varepsilon x}{\sum_{j=1}^{k} p_{\alpha_j}(x)}$ $\Rightarrow p_{\alpha_j}(y) \le \varepsilon$.

$$q_\beta(Tx) = q_\beta\Big(Ty \cdot \sum_{j=1}^{k} p_{\alpha_j}(x)\big/\varepsilon\Big) = \sum_{j=1}^{k} \varepsilon^{-1} p_{\alpha_j}(x)\, q_\beta(Ty) \le \varepsilon^{-1}\sum_{j=1}^{k} p_{\alpha_j}(x).$$

If $p_{\alpha_j}(x) = 0$ $\Rightarrow$ $p_{\alpha_j}(rx) = 0$, $r>0$ $\forall j$

$r q_\beta(Tx) = q_\beta(T(rx)) < 1$ $\Rightarrow$ $q_\beta(Tx) = 0$ ✓ $\square$

The following results. I would not prove now, and the conclusion suits our intuition.

Prop 5.24 a. $\mathcal{X}$ is hausdorff iff for each $x\neq 0$ there exists $\alpha\in A$ such that $p_\alpha(x)\neq 0$

b. If $\mathcal{X}$ is hausdorff and A is countable, then $\mathcal{X}$ is metrization with a translation-invariant metric (i.e. $\rho(x,y) = \rho(x+z, y+z)$ for all $x,y,z\in\mathcal{X}$).

def 5.25 A net is called Cauchy if net $\langle x_i - x_j\rangle_{(i,j)\in I\times I} \to 0$.

$\mathcal{X}$ is called ~~Cauchy~~ Complete if every Cauchy net converges.

A complete Hausdorff topological vector space whose topology is defined by a countable family of seminorms is called Fréchet space.

Well, the content above seems not logical or understandable, but I think it's not a big deal. We are going to discuss a concrete procedure below.

Suppose $\mathcal{X}$ is a vector space, $\mathcal{Y}$ is a normed linear space, and $\{T_\alpha\}_{\alpha\in A}$ is a collection of linear functionals from $\mathcal{X}$ to $\mathcal{Y}$. Then the weak topology makes $\mathcal{X}$ into a locally convex topological vector space.

Well, indeed, $\mathcal{T}$ generated by $\{x : \|T_\alpha x - y_0\| < \varepsilon\}$ coincides with $\mathcal{T}'$ generated by $\{x : \|T_\alpha x - T_\alpha x_0\| < \varepsilon\}$ ...

59. Now, first suppose $\mathcal{X}$ be a normed vector space.

The weak topology generated by $\mathcal{X}^*$ is the "weak topology" on $\mathcal{X}$.

The convergence ~~with respect to the topology~~ is known as <u>weak convergence</u>.

By what we have discussed, $\langle x_\alpha \rangle \to x$ weakly iff $f(x_\alpha) \to f(x)$ for all $f \in \mathcal{X}^*$

Repeat the process! The weak topology on $\mathcal{X}^*$ as defined above is the topology generated by $\mathcal{X}^{**}$; of more interest is the <u>topology generated by</u> $\mathcal{X}\ (= \hat{\mathcal{X}} \subseteq \mathcal{X}^{**})$ which is call $\boxed{\text{weak}^* \text{ topology}}$

[Rmk] from the process, we shall feel "weak topology" and "weak* topology" are not so hard to deal with as we thought before, and at least, the definitions of them are not to lead ...

$f_\alpha \to f$ in weak* iff $f_\alpha(x) \to f(x)$ for $\forall x \in \mathcal{X}$.

"the weak* topology on $\mathcal{X}^*$ is even weaker ~~than the~~ the weak topology on $\mathcal{X}^*$, and ~~they~~ coincides precisely when $\mathcal{X}$ is reflexive."

One more thing!

Let $\mathcal{X}$ & $\mathcal{Y}$ be Banach spaces. The topology on $L(\mathcal{X},\mathcal{Y})$ generated by the evaluation map $T \mapsto Tx$ $(x \in \mathcal{X})$ is called the <u>strong operator topology</u> on $L(\mathcal{X},\mathcal{Y})$, and the topology generated by the linear functionals $T \mapsto f(Tx)$ $(x \in \mathcal{X}, f \in \mathcal{Y}^*)$ is called weak operator topology on $L(\mathcal{X},\mathcal{Y})$

$\mathcal{X},\mathcal{Y}$ Banach $\begin{cases} T_\alpha \to T \text{ strongly iff } T_\alpha x \to Tx \text{ in norm topology } \forall x \in \mathcal{X} \\ T_\alpha \to T \text{ weakly iff } T_\alpha x \to Tx \text{ in weak topology of } \mathcal{Y}\ \forall x \in \mathcal{X}. \end{cases}$

$\mathcal{X}$ Normed $\begin{cases} x_\alpha \to x \text{ in weak topology iff } f(x_\alpha) \to f(x) \text{ in norm topology } \forall f \in \mathcal{X}^* \\ f_\alpha \to f \text{ in weak}^* \text{ topology iff } f_\alpha(x) \to f(x) \text{ in norm topology } \forall x \in \mathcal{X}. \end{cases}$

Their comparision may be written later.

[prop 5.26] $\{T_n\} \subseteq L(\mathcal{X},\mathcal{Y})$, $\sup_n \|T_n\| < \infty$ and $T \in L(\mathcal{X},\mathcal{Y})$

If $\|T_n x - Tx\| \to 0$ for all x in a dense subset ~~of~~ D of $\mathcal{X}$,

then $T_n \to T$ strongly.

proof. $C = \sup\{\|T_0\|, \|T_1\|, \cdots, \|T_n\|, \cdots\}$ $x \in \mathcal{X}$ $\varepsilon > 0$. $\|x - x'\| < \varepsilon/3C$ $\|T_n x' - Tx'\| < \varepsilon/3$.

$\Rightarrow \|T_n x - Tx\| \le \underbrace{\|T_n x - T_n x'\|}_{\varepsilon/3} + \underbrace{\|T_n x' - Tx'\|}_{\varepsilon/3} + \underbrace{\|Tx' - Tx\|}_{\varepsilon/3}$. $\square$

$\boxed{\text{Thm 5.27}}$ If $\mathcal{X}$ is a normed vector space, the closed unit ball $B^* = \{f \in \mathcal{X}^*, \|f\| \le 1\}$ in $\mathcal{X}^*$ is compact in the weak* topology.

proof. Idea: Cube embedding. If we can prove that B^* is a closed subset of ~~some~~ certain compact, then ...

$D_x = \{z \in \mathbb{C} : \|z\| \le \|x\|\}$ $\forall x \in \mathcal{X}$. $\Rightarrow D_x$ is compact.

$\Rightarrow D = \prod_{x \in \mathcal{X}} D_x$ is compact

$\Rightarrow B^* \subseteq D$. Regard D's element as functions ~~for~~ $|\phi(x)| \le \|x\|$

So the topology on D is pointwise convergence topology

~~to~~ the ~~sub~~ inherit topology on B^* then becomes weak*-topology

It suffices to show B^* is closed

$\langle f_\alpha \rangle \to f$ in D : $f(ax+by) = \lim f_\alpha(ax+by) = af(x) + bf(y)$. $\square$

$\boxed{\text{WARNING}}$

Alaoglu's theorem does not imply $\mathcal{X}^*$ is locally compact in weak* topology.

Intuition: Weak* topology is not a norm topology, hence ~~the~~ the open set may not contain "open ball", but I'm not very sure about it...

Many results in Hilbert space ~~is suite~~ agrees with what ~~I~~ we have learnt in Linear algebra, which ~~is~~ mainly concerns ~~of~~ space of finite dims.

$\boxed{\text{Def 5.28}}$ Inner product $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that.
$(x, y) \mapsto \langle x, y \rangle$

1. $\langle ax+by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$
2. $\langle y, x\rangle = \overline{\langle x, y\rangle}$
3. $\langle x, x\rangle \in (0, \infty)$ for $x \ne 0$

A complex vector space with an inner product is called a pre-Hilbert space.

$\|x\| := \sqrt{\langle x, x\rangle}$ for a pre-Hilbert space.

$\boxed{\text{prop 5.29}}$ $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$ for $\forall x, y \in \mathcal{H}$, with equality ~~holds~~ iff x, y are linear ~~in~~independent.

$\boxed{\text{prop 5.30}}$ $x \mapsto \|x\|$ is a norm on $\mathcal{H}$

$\boxed{\text{Def 5.31}}$ A pre-Hilbert space that is complete wrt the norm $\|x\| = \sqrt{\langle x, x\rangle}$ is called a Hilbert space.

$\boxed{\text{Example}}$ $L^2(\mu)$ $|f\bar{g}| \le \frac{1}{2}(|f|^2 + |g|^2) \Rightarrow |f\bar{g}| \in L^1(\mu)$

In the remainer of the section, $\mathcal{H}$ will denote a Hilbert space.

$\boxed{\text{Prop 5.32}}$ If $x_n \to x$, $y_n \to y$ $\Rightarrow$ $\langle x_n, y_n\rangle \to \langle x, y\rangle$

proof
$$\left|\langle x,y\rangle - \langle x_n,y_n\rangle\right| = \left|\langle x, y-y_n\rangle + \langle x-x_n, y_n\rangle\right|$$
$$\leq \|x\|\,\|y-y_n\| + \|x-x_n\|\,\|y_n\| \qquad \square$$

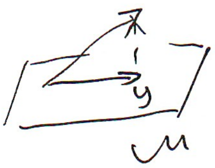
$\boxed{\text{Prop 5.33}}$ (The parallelogram law) $x, y \in \mathcal{H}$
$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

$\boxed{\text{Def 5.34}}$ If $x, y \in \mathcal{H}$, we say x is orthogonal to y if $\langle x,y\rangle = 0$, and write $x \perp y$. If $E \subseteq \mathcal{H}$, we ~~write~~ define
$$E^\perp = \{x \in \mathcal{H} : \langle x, y\rangle = 0 \text{ for all } y \in E\}.$$
It's not hard to see $E^\perp$ is closed by $\boxed{\text{Prop 5.32}}$.

$\boxed{\text{Prop 5.35}}$ (勾股定理) If $x_1, \cdots x_n \in \mathcal{H}$ $x_j \perp x_k$ $j \neq k$.
$$\left\|\sum_1^n x_j\right\|^2 = \sum_1^n \|x_j\|^2 .$$

$\boxed{\text{Thm 5.36}}$ If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$. ~~then~~ $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$
$$x \overset{!}{=} \underset{\mathcal{M}}{y} + \underset{\mathcal{M}^\perp}{z}$$
. y, z are the unique elements of $\mathcal{M}$ & $\mathcal{M}^\perp$ whose distance to x is maximal.

proof. (The proof is necessary, since a space of infinite dims can't be treated casually like finite dims)



let $\delta = \inf\{\|x-y\| : y \in \mathcal{M}\}$
$\Rightarrow$ y_n s.t. $\|x-y_n\| \to \delta$
$$\|y_m - y_n\|^2 = 2\left(\|y_m - x\|^2 + \|y_n - x\|^2\right) - \|y_m + y_n - 2x\|^2$$
$$= 2\left(\|y_m - x\|^2 + \|y_n - x\|^2\right) - 4\left\|\frac{y_m+y_n}{2} - x\right\|^2$$
$$\leq 2\left(\|y_m - x\|^2 + \|y_n - x\|^2\right) - 4\delta$$
$$\to 0 \quad \text{as} \quad m, n \to \infty$$

$\Rightarrow \{y_n\}$ is a Cauchy sequence $\Rightarrow$ $y_n \to y \in \underset{\text{closed}}{\mathcal{M}}$.

let $z = x - y$. If $u \in \mathcal{M}$, assuming $\langle u, z\rangle \in \mathbb{R}$.
$$f(t) = \|z + tu\|^2 = \|z\|^2 + 2t\langle u, z\rangle + t^2\|u\|^2.$$
$$\underset{\|x + (tu - y)\|}{\|} \Rightarrow f(t) \text{ attains minimum at } t=0 \Rightarrow f'(t) = 0$$
$$\Rightarrow \langle u, z\rangle = 0$$

If $z' \in \mathcal{M}^\perp$

$$\|x-z'\|^2 = \|\underset{\mathcal{M}}{\underset{\cap}{x-z}}\|^2 + \|\underset{\mathcal{M}^\perp}{\underset{\cap}{z-z'}}\|^2 \geq \|x-z\|^2 \quad \ldots \text{Uniqueness.}$$

$\square$

$\boxed{\text{Thm 5.37}}$ If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ s.t. $f(x) = \langle x, y\rangle$ for all $x \in \mathcal{X}$.

proof. Uniqueness: $f(x) = \langle x, y\rangle = \langle x, y'\rangle$ for $\forall x$

$\Rightarrow \langle y-y', y-y'\rangle = \|y-y'\|^2 = 0 \Rightarrow y = y'$.

Existence: If $f \equiv 0$, $y = 0$

If $f \not\equiv 0$. $\mathcal{M} = \{x : f(x) = 0\}$. Then $\mathcal{M}$ is a closed subspace of $\mathcal{X}$, $\mathcal{M}^\perp \neq \{0\}$. Pick $z \in \mathcal{M}^\perp$ with $\|z\| = 1$

If $u = f(x)z - f(z)x$, then $u \in \mathcal{M}$

$0 = \langle u, z\rangle = f(x)\|z\|^2 - f(z)\langle x, z\rangle = f(x) - \langle x, \overline{f(z)}z\rangle$

$\Rightarrow f(x) = \langle x, \overline{f(z)}z\rangle$. $\square$

Question; What's the idea of the construction of y?

Review the version of this theorem in linear algebra, there we can treat the similar conclusion by dual basis. However, here $\mathcal{H}$ is of infinite dimensions..

Fortunately, we have another method in LA:

$\dim \operatorname{Im} f = 1 \Rightarrow \dim \ker f = n-1 \quad \exists\, v \in (\ker f)^\perp \quad f(v) = \langle v, v\rangle$

$\forall x \in V$, $f(x) = f(\underset{\ker f}{x_0} + \alpha v) = \alpha\|v\|^2$

$\langle x_0 + \alpha v, v\rangle = \alpha\|v\|^2$ So $\underline{v}$ is what we want.

So, here our idea is similar to the second method!

$z \in \mathcal{M}^\perp$. $\|z\| = 1$ $f(\tilde{z}) = \|\tilde{z}\|^2$?

$\tilde{z} = \alpha z$. $f(\tilde{z}) = \alpha f(z) = |\alpha|^2\|z\|^2 = |\alpha|^2$

$\Rightarrow \alpha = \overline{f(z)}$

$\Rightarrow$ Considering $y = \overline{f(z)}z$ is Ok!

Thus, $\mathcal{H}$ is reflexive in a very strong sense: $\mathcal{H}$ is naturally isomorphic to $\mathcal{H}^{**}$.

A subset $\{u_\alpha\}_{\alpha \in A}$ is called orthonormal if $\|u_\alpha\| = 1$ $\forall \alpha$, $u_\alpha \perp u_\beta$ $\alpha \neq \beta$

By standard "Gram-Schmidt process", we can always convert a sequence of linear independent vector $\{x_n\}$ into orthonormal one $\{u_n\}$, what's more,

$\operatorname{span}(u_1, \ldots, u_n) = \operatorname{span}(x_1, \ldots, x_n)$ $\forall n \in \mathbb{N}$.

$\boxed{\text{prop 5.38}}$ If $\{u_\alpha\}_{\alpha\in A}$ is an orthnormal set in $\mathcal{H}$, then for any $x \in \mathcal{X}$

$$\sum_{\alpha\in A} |<x, x_\alpha>|^2 \leq \|x\|^2$$

In particular, $\{\alpha : <x, x_\alpha> \neq 0\}$, is countable.

$\boxed{\text{NOTE}}$ $\{\alpha : <x, x_\alpha> > 0\} = \bigcup_{n=1}^{\infty} \{\alpha : <x, x_\alpha> > \frac{1}{n}\}$. If uncountable,

$$\sum_{\alpha\in A} |<x, x_\alpha>|^2 = \infty$$

$\boxed{\text{Thm 5.39}}$ If $\{x_\alpha\}_{\alpha\in A}$ is an $\boxed{\text{orthnormal}}$ set in $\mathcal{H}$, TFAE:

1. (completeness) $<x, x_\alpha> = 0 \quad \forall \alpha \Rightarrow x = 0$
2. (Parseval's identity) $\|x\|^2 = \sum_{\alpha\in A} \|<x, x_\alpha>\|^2$ for $\forall x \in \mathcal{H}$
3. For each $x \in \mathcal{H}$, $x = \sum_{\alpha\in A} <x, x_\alpha> x_\alpha$, where the sum on the right has only countably many non-zero terms and converges in the norm topology no matter how these terms are ordered.

proof $1 \Rightarrow 3$. $\alpha_1, \cdots \alpha_n, \cdots$ s.t. $<x, x_\alpha> \neq 0$

$\Rightarrow \sum |<x, x_\alpha>|^2 < \infty \qquad \sum_m^n \|<x, x_\alpha> x_\alpha\|^2 = \sum_m^n |<x, x_\alpha>|^2 \to 0$

as $m, n \to \infty$

By completeness $x - \sum_{\alpha\in A} <x, x_\alpha> x_\alpha = 0$

$3 \Rightarrow 2$. $\left\|x - \sum_1^n <x, x_{\alpha_j}> x_{\alpha_j}\right\|^2 = \|x\|^2 - \sum_1^n |<x, x_{\alpha_j}>|^2 \to 0.$

$\square$

$2 \Rightarrow 1$

$\boxed{\text{prop 5.40}}$ Every Hilbert space has an ~~oth~~ orthnormal basis.

$\boxed{\text{Zorn's lemma}}$.

$\boxed{\text{prop 5.41}}$ A Hilbert space $\mathcal{H}$ is separable iff it has a countable orthnormal basis, in which case every orthonormal basis for $\mathcal{H}$ is countable.

proof. Separable $\Rightarrow$ trivial

$\Leftarrow$ Coefficients.

$\square$

$\boxed{\text{Def 5.42}}$ If $\mathcal{H}_1$ & $\mathcal{H}_2$ are two Hilbert space with inner product $<\cdot,\cdot>_1$ $<\cdot,\cdot>_2$

a Unitary map from $\mathcal{H}_1$ to $\mathcal{H}_2$ is an ~~Hilbert~~ invertible linear map ~~from~~

$U : \mathcal{H}_1 \to \mathcal{H}_2$ that preserves inner product: $<x, y>_1 = <Ux, Uy>_2$ for all $x, y \in \mathcal{H}$

$\boxed{\text{prop 5.43}}$ Unitary $\Leftrightarrow$ isometry + surjective (think about LA)

$\boxed{\text{prop 5.44}}$ Let $\{e_\alpha\}_{\alpha\in A}$ be an orthnormal basis for $\mathcal{X}$. Then the correspondence $x \mapsto \hat{x}$ defined by $\hat{x}(\alpha) = <x, x_\alpha>$ is a Unitary map from $\mathcal{H}$ to $l^2(A)$