Chapter 5 L^p space.

Basic Back~~grop~~ground: $(X, \mathcal{M}, \mu)$ f is measurable. $0<p<\infty$

$\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$ allowing $\|f\|_p = \infty$

$L^p \cancel{(f)} = \{ f : X \to \mathbb{C} : f \text{ is measurable}, \|f\|_p < \infty \}$

$|f+g|^p \le (2\max(|f|, |g|))^p \le 2^p(|f|^p + |g|^p)$

$\hookrightarrow$ L^p is a vector space.

Since ~~triag~~ triangle inequality fails for the case $p<1$, we shall always require $p \ge 1$ later.

The following inequality is the cornstone of L^p theory, and we are quite familar with it.

$\boxed{\text{lem 5.1}}$ $a \ge 0$, $b \ge 0$ $0<\lambda<1$, then

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda) b \qquad \text{"=" holds for } a=b$$

$\boxed{\text{Thm 5.2}}$ Hölder's inequality $1<p<\infty$ & $\frac{1}{p}+\frac{1}{q}=1$ $\left(q = \frac{p}{p-1}\right)$

If f & g are measurable functions on X, then we have

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

"=" holds iff $\alpha |f|^p = \beta |g|^q$ a.e. for $(\alpha, \beta) \ne (0,0)$.

proof. $f, g \ne 0$ a.e. $\|f\|_p, \|g\|_q \ne \infty$

Suppose $\|f\|_p = \|g\|_q = 1$ $\left(\tilde{f} = \frac{f}{\|f\|_p} \quad \tilde{g} = \frac{g}{\|g\|_q}\right)$

$|fg| \le \frac{1}{p}|f|^p + \frac{1}{q}|g|^q$

$\Rightarrow \|fg\|_1 \le \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = 1 = \|f\|_p \|g\|_q$

"=" : $\left(\frac{|f|}{\|f\|_p}\right)^p = \left(\frac{|g|}{\|g\|_q}\right)^q \iff \alpha |f|^p = \beta |g|^q$ a.e. $\square$

$q = \frac{p}{p-1}$ for $p \ge 1$ is called congruent exponent to p.

$\boxed{\text{Thm 5.3}}$ (Minkowski's inequality) If $1 \le p < \infty$, $f, g \in L^p$, then

$\|f+g\|_p \le \|f\|_p + \|g\|_p$

proof. $p=1$, $f+g=0$ a.e. trivial

$|f+g|^p \le |f| |f+g|^{p-1} + |g| |f+g|^{p-1} \Rightarrow \|f+g\|_p^p \le \|f\|_p \| |f+g|^{p-1} \|_q + \cdots$

$= (\|f\|_p + \|g\|_p)\left(\int |f+g|^p\right)^{\frac{1}{q}}$

$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$ $\square$

If we consider $L^p = \mathcal{L}^p/\sim$, then L^p is a normed vector space. What's more, it's complete!

$\boxed{\text{Thm 5.4}}$ For $1 \le p < \infty$, L^p is a Banach space.

proof: By $\boxed{\text{Thm 4.1}}$, it suffices to show $\{f_n\} \subset L^p$, with ~~which~~ $\sum_{k=1}^{\infty} \|f_k\|_p = B < \infty$ is convergent. Let $G_n = \sum_{k=1}^{n} |f_k| \Rightarrow \|G_n\|_p \le \sum_{k=1}^{n} \|f_k\|_p < \infty$

By MCT, $\int G^p = \lim_n \int G_n^p \le B^p$

$\Rightarrow G \in L^p \Rightarrow G$ is finite a.e.

$\Rightarrow \sum_{k=1}^{\infty} f_k$ is convergent a.e.

$F = \sum_{k=1}^{\infty} f_k$ $\quad \|F - \sum_{k=1}^{n} f_k\|_p^p = \int |F - \sum_{k=1}^{n} f_k|^p \to 0$ □

$|F| \le G$ $\quad |F - \sum_{k=1}^{n} f_k|^p \le (2G)^p \in L^1$

We should develop approximation theory, using functions with good properties to approach a bad function and making ~~bad~~ good functions inherit more ~~good~~ properties from the original function.

$\boxed{\text{Thm 5.5}}$ For $1 \le p < \infty$, the set of simple function $f = \sum_{j=1}^{n} a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for j, is dense in L^p

proof. $f \in L^p$ $\quad$ a sequence $\{f_n\}$ of simple functions s.t. $f_n \to f$ a.e. and $|f_n| \nearrow |f|$

$\Rightarrow f_n \in L^p$ $\quad |f_n - f|^p \le 2^p |f|^p \in L^1$

$\lim_n \int |f_n - f|^p \to 0 \Rightarrow \|f_n - f\|_p \to 0$ □

$\boxed{\text{Def 5.6}}$ $\|f\|_\infty = \inf\{a \ge 0 : \mu(\{x : |f(x)| > a\}) = 0\}$, where f is measurable.

$\inf \emptyset := \infty$ $\quad \|f\|_\infty := \operatorname{ess\,sup}_{x \in X} |f(x)|$ is called essential supremum.

$\mathcal{L}^\infty = \{f : X \to \mathbb{C} : f$ is measurable $\&$ $\|f\|_\infty < \infty\}$.

$(L^\infty = \mathcal{L}^\infty/\sim)$

$\boxed{\text{Thm 5.7}}$ Some properties of L^∞.

1. f & g are measurable. $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. If $f \in L^1$, $g \in L^\infty$, "=" holds true iff $g(x) = \|g\|_\infty$ a.e. on $\{f \ne 0\}$.
2. $\|\cdot\|_\infty$ is a norm on L^∞
3. $\|f_n - f\|_\infty \to 0$ iff $\exists E \in \mathcal{M}$ s.t. $\mu(E^c) = 0$ $f_n \rightrightarrows f$ uniformly on E.
4. L^∞ is a Banach space
5. simple functions are dense in L^∞.

The ~~~~ third one is a little bit subtle, since $\|\cdot\|_\infty \neq \|\cdot\|_\mu$

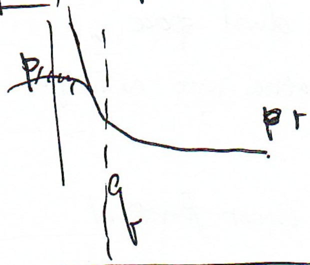
But the proof of the third one is almost obvious.

A natural question is that, what's the relation between L^p & L^q $(p\neq q)$.

A basic observation is that, if p is ~~large~~ small, then L^p allows functions which are locally singular to appear; if p is ~~small~~ large, then L^p allows functions which decay slowly.

Then, the result following is not ~~so~~ surprising...

$\boxed{\text{prop 5.8}}$ If $0<p<q<r\leq\infty$ then $L^q \subseteq L^p + L^r$



	1 ✓	2 ✗	3 ✗
	$x^{-0.5}$		$x^{-0.3}$
decay	fast		slow
singular	big		small
$\mu(x)<\infty$	L^1 ✓		L^1 ✓
	$\frac{1}{\sqrt{x}}$		$x^{0.7}$
	L^2		L^2
	$\ln x$		$x^{a}x$ ✓

$\boxed{\text{prop 5.9}}$ If $0<p<q<r\leq\infty$, then $L^p\cap L^r \subseteq L^q$ & $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ where $\lambda\in(0,1)$ is defined by

$q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$, that is, $\lambda = \dfrac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$.

$= \dfrac{q^{-1}-p^{-1}}{p^{-1}-r^{-1}} + 1 \Rightarrow 1-\lambda = \dfrac{p^{-1}-q^{-1}}{p^{-1}-r^{-1}}$

$\left(\frac{p}{\lambda q}\right)^{-1} + \left(\frac{r}{(1-\lambda)q}\right)^{-1}$

$= \frac{q}{p}(1-\lambda) + \frac{q}{r}(1-\lambda)$ ~~$=$~~

$= (1-\lambda)\left(\frac{q}{r} - \frac{q}{p}\right) + \frac{q}{p}$

$= \frac{qr-pr}{\cancel{qr-pq}} \cdot \frac{\cancel{pq-rq}}{rp} + \frac{q}{p}$

$= -\frac{q-p}{p} + \frac{q}{p} = 1$

proof. If $r<\infty$

$$\int |f|^q = \int |f|^{\lambda q} |f|^{(1-\lambda)q} \overset{\text{Hölder}}{\leq} \cancel{\| |f|^{\lambda q}\|}$$

$$\| |f|^{\lambda q}\|_{\frac{p}{\lambda q}} \cdot \| |f|^{(1-\lambda)q}\|_{\frac{r}{(1-\lambda)q}}$$

$$= \left(\int |f|^p\right)^{\frac{\lambda q}{p}} \cdot \left(\int |f|^r\right)^{\frac{(1-\lambda)q}{r}} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q} \quad \square$$

$r=\infty$ is easier.

$\boxed{\text{prop 5.10}}$ If A is any set & $0<p<q$, then $\ell^p(A) \subset \ell^q(A)$ & $\|f\|_q \leq \|f\|_p$

proof ($\ell^p(A) = L^p(A)$, μ is counting measure.)

$$\|f\|_\infty^p = \sup_{\alpha\in A} |f(\alpha)|^p \leq \sum_{\alpha\in A} |f(\alpha)|^p \Rightarrow \|f\|_\infty \leq \|f\|_p$$

if $q<\infty$. $\Rightarrow$ $\|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda}$ $\quad \lambda = \frac{p}{q}$

$\leq \|f\|_p^\lambda \cdot \|f\|_p^{1-\lambda} = \|f\|_p \quad \square$

$\boxed{\text{prop 5.11}}$ If $\mu(X)<\infty$. & $0<p<q\leq\infty$ ~~then~~ $L^p(\mu) \supseteq L^q(\mu)$ & $\|f\|_p \leq \|f\|_q \,\mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

proof: ~~condition $\mu(X)<\infty$ omit the impact of decay rate, so $L^q(\mu)$~~

$$\|f\|_p^p = \int |f|^p \leq \| |f|^p\|_{\frac{q}{p}} \| 1 \|_{\frac{q}{q-p}} = \left(\int |f|^q\right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{q-p}{q}} \quad \square$$

$q=\infty$ ...

the conclusion is not surprising ...

$\mu(X)<\infty \Rightarrow$ we shall only consider the singularity of functions.

$p \searrow$ singularity $\nearrow$

$\boxed{\text{Rmk}}$ L^1 integral

L^2 Hilbert space

L^∞ uniform

$\Rightarrow$ pathlogical $\Rightarrow$ consider dual space.

Many operators are bounded in L^p $1<p<\infty$ but L^1 & L^∞.

Now, let's consider the dual of L^p.

Indeed, I don't understand the reason why we shall study the dual space.

Maybe by ~~Riesz~~ Riesz represent theorem, we can change an equation into an integral equation?

Suppose p, q are congruent exponents. $\forall g \in L^q$, define a linear functional

$$\phi_g : L^p \to \mathbb{C} \qquad \phi_g(f) = \int fg \qquad \int |fg| \le \|f\|_p \|g\|_q < \infty.$$

(In L^2, we shall define it as $\int f\bar{g} = \phi_g(f)$).

We need some basic properties to understand the dual of L^p.

$\boxed{\text{prop 5.12}}$ p, q are congruent. $1 \le q < \infty$. If $g \in L^q$, then

$$\|g\|_q = \|\phi_g\| = \sup\{|\int fg| : \|f\|_p = 1\}.$$

If μ is semifinite, this result holds also for $q=\infty$.

any ~~subset~~ of infinite measure has a subset of positive ~~finite~~ measure.

proof. $|\int fg| \le \int |fg| \le \|f\|_p \|g\|_q = \|g\|_q$

$\Rightarrow \|\phi_g\| \le \|g\|_q$

$$f = \frac{|g|^{q-1} \cdot \overline{\operatorname{sgn} g}}{C} \qquad \left(\int \frac{|g|^{(q-1)p}}{C^p}\right)^{\frac{1}{p}} = \frac{1}{C}\left(\int |g|^q\right)^{\frac{1}{p}}$$

$$C = \|g\|_q^{q-1}$$

$$|\int fg| = \left|\int \frac{|g|^q \cdot \cancel{\operatorname{sgn} g}}{\|g\|_q^{q-1}}\right| = \|g\|_q$$

$\Rightarrow \|\phi_g\| = \|g\|_q$.

If $q=\infty$ $\|g\|_\infty < \infty$. $|\int fg| \le \int |f| \, \|g\|_\infty = \|g\|_\infty$.

$A_\varepsilon = \{x : |g(x)| > \|g\|_\infty - \varepsilon\}$ $\mu(A_\varepsilon) > 0 \Rightarrow \exists B_\varepsilon \subseteq A_\varepsilon$, $\mu(B_\varepsilon) < \infty$

$f = \frac{1}{\mu(B)} \chi_B \overline{\operatorname{sgn} g}$ $|\int fg| = \frac{1}{\mu(B)} \int_B |g| > \|g\|_\infty - \varepsilon$

$\Rightarrow \|\phi_g\| = \|g\|_\infty$ when μ is semi-finite. $\square$

$g \in L^q \Rightarrow \phi_g$ is a ~~bo~~ bounded linear functional $\in (L^p)^*$.

If ~~g~~ $f \mapsto \int fg$ is a bounded linear functional on L^p

then $g \in L^q$ in almost all cases.

$\boxed{\text{Thm 5.13}}$ Let p and q are congruent. Suppose that g is a measurable function on X s.t. $\int fg \in L^1$ for all $f \in \Sigma$ of simple functions that vanish outside a set of finite measure, and the quantity

$$M_q(g) = \sup\{|\int fg| : f \in \Sigma \;\&\; \|f\|_p = 1\} < \infty.$$

Also, suppose either that $S_g = \{x : g(x) \neq 0\}$ is σ-finite or that μ is semifinite. $\boxed{\text{Then } g \in L^q \;\&\; M_q(g) = \|g\|_q.}$

proof. ($f \in \Sigma$. $\|f\|_p = \left(\sum_1^n |a_n|^p \mu(\chi_{E_n})\right)^{\frac{1}{p}}$).

~~$\boxed{\text{lem}}$ μ is semifinite $\Rightarrow S_g = \{x : g(x) \neq 0\}$ is σ-finite.~~

~~proof of lem.~~

$\boxed{\text{EX 17}}$ if μ is semifinite. $q < \infty$ $M_q(g) < \infty$ then $\{x : |g(x)| > \varepsilon\}$ has finite measure for all and hence S_g is σ-finite.

proof. ~~$E_\varepsilon = \{|g| > \varepsilon\}$ for $B \subseteq E_\varepsilon$ $\exists B \subseteq E_\varepsilon$ with $\infty > \mu(B) > 0$~~

~~Let $f = \left(\frac{1}{\mu(B)}\right)^{\frac{1}{p}} \chi_B \,\overline{\mathrm{sgn}\, g}$~~

~~$\Rightarrow C > \int \left(\frac{1}{\mu(B)}\right) \chi_B |g| > \frac{\varepsilon}{\mu(B)} \Rightarrow \mu(B) > \frac{\varepsilon}{C}$.~~

assume $E_\varepsilon = \{|g| > \varepsilon\}$ with $\mu(E_\varepsilon) = \infty$ for some ε.

We shall see an important fact that, $\forall N > 0$. $\exists B \subseteq E_\varepsilon$ s.t. $N \leq \mu(B) < \infty$. Since we can do an iteration: $B_1 \subseteq E_\varepsilon$. $\mu(E_\varepsilon \setminus B_1) = \infty$. $0 < \mu(B_1) < \infty$. $B_2 \subseteq E_\varepsilon \setminus B_1$. $\mu(E_\varepsilon \setminus \bigcup_1^2 B_i) = \infty$. $0 < \mu(B_2) < \infty$

... repeat it! If $\mu(\bigcup_1^\infty B_j) = \infty$ ✓

If $\mu(\bigcup_1^\infty B_j) < \infty$. $\Rightarrow \mu(E_\varepsilon \setminus \bigcup_1^\infty B_j) = \infty$

we can get $\bigcup_1^\infty B_j \cup \tilde{B}_j \cdots$

$\|f\|_p = 1$: $f = \left(\frac{1}{\mu(B)}\right)^{\frac{1}{p}} \chi_B \,\overline{\mathrm{sgn}\, g}$ $M_q(g) = m$

~~$|\int fg| = \int \frac{|g|}{(\mu(B))^{\frac{1}{p}}}$.~~

~~$m = \varepsilon \frac{m}{\varepsilon} \Leftarrow g\,\mu(B)$~~

~~$\mu(B)^{-\frac{1}{p}}$~~ $\int fg = \int \left(\frac{1}{\mu(B)}\right)^{\frac{1}{p}} g \cdot \chi_B$

$\Rightarrow \varepsilon \cdot \mu(B)$

let $\mu(B) > \left(\frac{m}{\varepsilon}\right)^{\frac{1}{q}}$

✓

69. Hence we shall just consider ~~the~~ condition that S_g is σ-finite.

$S_g = \bigcup_1^\infty E_n \quad E_n \nearrow \quad \mu(E_n) < \infty.$

g is not easy to deal with, let $|\phi_n| \nearrow g$ (ϕ_n simple), $\phi_n \to g \quad g_n = \phi_n \chi_{E_n}$

then $g_n \to g$ pointwise

$$f_n = \frac{|g_n|^{q-1}\ \mathrm{sgn}\, g}{\|g_n\|_q^{q-1}} \leftarrow \text{Bounded, supported} \quad \|f_n\|_p = 1. \quad f_n \notin \Sigma.$$

However ψ_k simple $\to f_n \quad |\psi_k| \le |f_n| \quad |\psi_k| \le \|f_n\|_\infty \chi_{E_n}$

$$\left|\int f_n g\right| \overset{DCT}{=} \lim_k \left|\int \psi_k g\right| \le M_q(g)$$

$$\|g\|_q \overset{\text{fatou}}{\le} \liminf \|g_n\|_q = \liminf \int |f_n g_n|$$

$$\le \liminf \int |f_n g| = \liminf \int f_n g = M_q(g)$$

$\Rightarrow \|g\|_q = M_q(g)$ Total, it's to say " A Bounded measurable function that vanishes E of finite measure & $\|f\|_p=1$ then $|\int f g| \le M_q(g)$.

Now, $q = \infty$. for given $\varepsilon > 0$. $A = \{x : |g(x)| \ge M_\infty(g) + \varepsilon\}$.

$\boxed{\text{if } \mu(A) > 0.}$ $\exists B \subseteq A.\ 0 < \mu(B) < \infty$ (semifinite or $A \subseteq S_g$)

$f = (\mu(B))^{-1} \chi_B\ \mathrm{sgn}\, g \qquad \|f\|_1 = 1.$

$\int fg = \mu(B)^{-1} \int_B g \ge M_\infty(g) + \varepsilon \quad \lightning$

$\Rightarrow \|g\|_\infty \le M_\infty(g)$ The reverse it's trivial $\square$

Now, we have already shown two things :

1. $g \in L^q \rightsquigarrow \phi_g(f) = \int fg,\ \phi_g \in (L^p)^*$
2. ϕ_g is bounded on $\Sigma \rightsquigarrow g$ is L^q.

$\boxed{\text{Thm 6.14}}$ p & q are congruent exponents. If $1 < p < \infty$, for each $\phi \in (L^p)^*$ there exists ~~a~~ $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, and hence L^q is isometricly isomorphic to $(L^p)^*$. The same condusion for $p=1$ provided μ is σ-finite.

Before proving it, we shall see the result an interesting one. It's a form of "Representation theorem", and the natural extension of familiar theorem makes me happy.

proof. (STEP 1). μ is finite. Then simple functions are in L^p.

(The most interesting ~~thing~~ here is ~~that~~ the proof is also based on a representation ~~theorem~~ of another form).

$\nu(E) := \phi(\chi_E)$. We shall prove ν is a complex measure and AC to μ.

$\{E_j\}$ ~~is~~ a sequence of disjoint sets, $E = \bigcup_{n=1}^{\infty} E_n$. $\chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$

$$\| \chi_E - \sum_{k=1}^{n} \chi_{E_k} \|_p = \| \sum_{k=n+1}^{\infty} \chi_{E_k} \|_p = \mu\left(\bigcup_{k=n+1}^{\infty} E_k\right)^{\frac{1}{p}} \to 0 \quad \text{as } n \to \infty$$

since $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \mu(X) < \infty$.

$\Rightarrow$ Since $\phi \in (L^p)^*$ (linear & $\boxed{\text{continuous}}$)

$$\cancel{\phi(E)}\ \nu(E) = \phi\left(\sum_{n=1}^{\infty} \chi_{E_n}\right) = \sum_{n=1}^{\infty} \nu(E_n).$$

Here note ϕ is continuous. since linear $\neq$ σ-additive.

If $\mu(E) = 0$, χ_E can be regarded as 0 in $L^p (= \mathcal{L}^p/\sim)$.

$\Rightarrow \nu(E) = \phi(\chi_E) = 0. = \mu(E) \Rightarrow \nu \ll \mu$.

By LRN decomposition, we have $d\nu = g\, d\mu$

$\Rightarrow \phi(f) = \int f g\, d\mu$ for all simple functions f

By $\boxed{\text{Thm 5.13}}$ $\Rightarrow g \in L^q$. Since simple functions are dense in L^p $\boxed{\text{Thm 5.5}}$

by continuity. $\phi(f) = \int f g\, d\mu$ for $\forall f \in L^p$.

STep 2. Suppose μ is σ-finite. Let $\{E_n\}$ is an increasing sequence of sets such that $0 \leq \mu(E_n) < \infty$ & $X = \bigcup_{n=1}^{\infty} E_n$

Without confusion, we can identify $L^q(E_n)$ & $(L^p(E_n))^*$

Now, for each n, there exists $g_n \in L^q(E_n)$ such $\phi(f) = \int f g_n$ for all $f \in L^p(E_n)$

and $\|g_n\|_q = \| \phi|_{L^p(E_n)} \| \leq \|\phi\|$

g_n is unique modulo alteration on null sets so $g_m = g_n$ a.e. on E_n for $m > n$.

So we can define g a.e. on X by setting $g = g_n$ on E_n. By MCT

$\|g\|_q = \lim \|g_n\|_q \leq \|\phi\|$, so $g \in L^q$. Moreover, if $f \in L^p$. $f\chi_{E_n} \to f$

$\phi(f) = \lim \phi(f\chi_{E_n}) = \lim \int_{E_n} fg = \int fg$ by DCT.

71. STEP 3. Suppose μ is arbitrary & $p>1 \Rightarrow q<\infty$.

$\forall E \subseteq X$ (σ-finite) $\longrightarrow \exists g_E \in L^q(E)$ s.t. $\phi(f)=\int f g_E$ for $f\in L^p(E)$

$\|g_E\| \le \|\phi\|$

If F σ-finite $\supseteq E \Rightarrow g_F = g_E$ a.e. on E

Let $M = \sup\{\|g_E\|_q : E\subseteq X,\ E$ is σ-finite$\}$. $\Rightarrow M\le\|\phi\|$

take $\{E_n\}$ s.t. $\|g_{E_n}\|_q \to M$ set $F=\bigcup_{n=1}^{\infty} E_n \Rightarrow \|g_F\|_q \ge \|g_E\|_q$

$\Rightarrow \|g_F\|_q = M$

If A is σ-finite. $A\supseteq F$

$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \le M^q = \int |g_F|^q$

$\Rightarrow g_{A\setminus F}=0$ a.e. $\Rightarrow g_A = g_F$ a.e. $(q<\infty)$

If $f\in L^p$. ~~Let~~ $A = F\cup\{f(x)\ne 0\}$ is σ-finite $f\in L^p(A)$ exactly

$\Rightarrow \phi(f) = \int f g_A = \int f g_F$. So, let $g = g_F$ $\square$

$\boxed{\text{Cor 5.15}}$ If $1<p<\infty$, L^p is reflexive.

$(L^p)^* = L^q \quad (L^q)^* = L^p$.

$\boxed{\text{EX 7}}$ L^∞ ensures ~~that~~ f is not locally singular. L^p ensures its decay speed.

~~$\int |f|^q = \int_E |f|^q + \int_F |f|^q$ $E=\{|f|\ge 1\}$ $F=\{|f|<1\}$~~

$\le \int_E$

$\|f\|_q \le \|f\|_p^{\lambda}\|f\|_\infty^{1-\lambda} \Rightarrow f\in L^q$ for $\forall q>p$.

$= \|f\|_p^{\frac{p}{q}}\|f\|_\infty^{1-\frac{p}{q}}$ $\quad q\to\infty$ $\Rightarrow \overline{\lim_{q\to\infty}}\|f\|_q \le \|f\|_\infty$

~~$\Rightarrow \|f\|_q \le \|f\|_\infty$.~~

$\|f\|_\infty \le \|f\|_q$: $\quad A_\lambda = \{|f|\ge\lambda\}$ $\quad 0<\lambda \le \|f\|_\infty$

$\left(\int |f|^q\right)^{\frac{1}{q}} \ge \left(\int_{A_\lambda} \lambda^q\right)^{\frac{1}{q}} = (\mu(A_\lambda))^{\frac{1}{q}}\lambda$

$\Rightarrow \underline{\lim_{q\to\infty}}\|f\|_q \ge \lambda \Rightarrow \underline{\lim_{q\to\infty}}\|f\|_q \ge \|f\|_\infty$.

take lim $\quad \overline{\lim_{q\to\infty}}\|f\|_q \le 1\cdot\|f\|_\infty \le \underline{\lim_{q\to\infty}}\|f\|_q$. $\square$

$\boxed{EX 9}$ $f_n \xrightarrow{L^p} f \Rightarrow f_n \xrightarrow{\mu} f \quad 1 \le p < \infty$

$\left(\int |f-f_n|^p\right)^{\frac{1}{p}} \to 0$ Let $E_{\varepsilon,n} = \{ |f-f_n| \ge \varepsilon \}$

$\left(\int |f-f_n|^p\right)^{\frac{1}{p}} \ge \left(\int_{E_{\varepsilon,n}} |f-f_n|^p\right)^{\frac{1}{p}} \ge \varepsilon \cdot \mu(E_{\varepsilon,n})^{\frac{1}{p}} \to 0$ as $n \to \infty$

Since ε is given $\Rightarrow \mu(E_{\varepsilon,n}) \to 0$ as $n \to \infty$.

$f_n \xrightarrow{\mu} f$ + dominate $\Rightarrow f_n \xrightarrow{L^p} f$.

~~$\forall \varepsilon > 0 \quad \mu(E_n) = \{ |f_n - f| \ge \varepsilon \} \to 0$ as $n \to \infty$.~~

~~$\left(\int |f_n - f|^p\right)^{\frac{1}{p}} \overset{\text{Minkowski}}{\le} \left(\int_{E_n} |f_n - f|^p\right)^{\frac{1}{p}} + \left(\int_{E_n^c} |f_n - f|^p\right)^{\frac{1}{p}}$~~

~~$\left(\int_{E_n^c} |f_n - f|^p\right)^{\frac{1}{p}} < \varepsilon \mu(E_n^c)^{\frac{1}{p}}$~~

$f_n \xrightarrow{\mu} f \Rightarrow f_{n_k} \xrightarrow{a.e.} f \Rightarrow |f| \le g$ a.e. $\Rightarrow f \in L^p$

If $f_n \overset{L^p}{\not\to} f \quad \exists \varepsilon \ \{g_n\} \subseteq \{f_n\}$ s.t. $\|g_n - f\|_p \ge \varepsilon$

But $g_n \xrightarrow{\mu} f \Rightarrow \{h_n\} \subseteq \{g_n\} \quad h_n \xrightarrow{a.e.} f$

By L^p DCT $\Rightarrow \|h_n - f\|_p \to 0 \quad \Downarrow \qquad \square$

$\boxed{EX 10}$ $\underbrace{f_n \to f}_{L^p}$ a.e. then $\|f_n\|_p \to \|f\| \Leftrightarrow \|f_n - f\|_p \to 0$

proof: $(\Leftarrow)$ $\left| \|f_n\| - \|f\|_p \right| \le \|f_n - f\|_p \to 0$

$(\Rightarrow)$ $\underline{|f_n - f|^p} \le (2|f_n| + |f|)^p = g_n$ A skill in proof of generalized DCT.

$g_n \to 4^p |f|^p \qquad \int g_n = \int 2^p(|f_n| + |f|)^p \overset{DCT}{\Rightarrow} \int g$

$\Rightarrow \int |f_n - f|^p \to 0 \qquad \square$

$\boxed{EX 15}$ $(\Rightarrow)$ $\{f_n\}$ is Cauchy in $L^p \qquad \int |f_m - f_n|^p \to 0$ as $m,n \to \infty$

$\hookrightarrow$ Cauchy in measure

$\{f_n\}$ Cauchy in L^p + L^p Banach $\Rightarrow f_n \xrightarrow{L^p} f \Leftrightarrow |f_n|^p \xrightarrow{L^1} |f|^p$

$\Rightarrow \{|f_n|^p\}$ is uniformly integrable.

~~Let $E_m = \{|f_n| \ge \frac{1}{m}\}$ $E_m \nearrow$ $\bigcup^\infty E_m = \{|f| > 0\} = E$. $\mu(E_m) < \infty$.~~

~~$|f_n \chi_{E_m^c}| \le |f_n| \in L^p \Rightarrow \lim_m \int |f_n \chi_{E_m^c}|^p = \int |f_n \chi_{E^c}|^p = 0$~~

73. $f_n \xrightarrow{L^p} f$ completely

$\forall \varepsilon > 0.\ \exists N.\ \|f_n - f\|_p < \varepsilon/3$ for $n > N$

$E_m = \bigcup_{n=1}^{N} \{|f|_n \geq \frac{1}{m}\} \cup \{|f| \geq \frac{1}{m}\}$. $\mu(E_m) < \infty$. $E_m \nearrow E = \bigcup E_m = X$.

$$|f_n \chi_{E_m^c}| \leq |f_n| \quad \lim \int |f_n \chi_{E_m^c}|^p = \int |f \chi_{E^c}|^p = 0$$

$$\Rightarrow \exists \tilde{E}.\ \text{s.t}\ f_1 \cdots f_N,\ f \quad \cdots \quad \int_{E^c} |f|^p < \varepsilon.$$

$$\|f_m\|_p^p \leq \|f_m - f\|_{p_{E^c}}^p + \|f\|_{p_{E^c}}^p < 2\varepsilon/3 < \varepsilon.$$

$m > N$

Sufficiency: (E) $\mu(E) < \infty$. $\int_{E^c} |f|^p < \varepsilon$.

$A_{mn} = \{x \in E : |f_m - f_n| \geq \varepsilon\}$ $\mu(A_{mn}) \to 0$ as $m,n \to \infty$.

$$\int |f_n - f_m|^p = \int_{E \setminus A_{mn}} |f_m - f_n|^p + \underline{\int_{A_{mn}} |f_m - f_n|^p} + \int_{E^c} |f_m - f_n|^p$$

$$\leq \varepsilon^p \cdot \underset{\text{finite}}{\mu(E \setminus A_{mn})} \quad \underset{AC}{2^p \int_{A_{mn}} |f_m|^p + 2^p \int_{A_{mn}} |f_n|^p} \quad 2\varepsilon.$$

$\to 0$ as $m, n \to \infty$. $\square$

$\boxed{\text{EX 20}}$ $\sup_n \|f_n\|_p < \infty$, $f_n \to f$ a.e.

a. $f_n \to f$ weakly in L^p: By $\boxed{\text{Thm 5.14}}$ $\forall\ \phi \in (L^p)^*$ $\phi(f) = \int fg$.

where $g \in L^q$

It suffices to show $\phi(f_n) \to \phi(f) \iff \phi(f_n - f) = \int (f_n - f) g \to 0$ as $n \to \infty$

(1) $\mu(A) < \infty$. $A \subseteq X$. $\int_{X \setminus A} |g|^q < \varepsilon$ (DCT) or simple functions' denseness. L^q

(2) $B \subseteq A$ egoroff δ $f_n \rightrightarrows f$ on B

$\mu(A \setminus B) < \delta$

(3) $\int_\delta |g|^q < \varepsilon$.

$$\Rightarrow \int |(f_n - f) g| = \int_B |(f_n - f) g| + \int_{A \setminus B} |(f_n - f) g| + \int_{A^c} |(f_n - f) g|$$

$$\overset{\text{Hölder}}{\leq} \|f_n - f\|_{p,B} \|g\|_{q,B} + \underset{\text{finite}}{\|f_n - f\|_p} \underset{\varepsilon}{\|g\|_q} + \underset{\text{finite}}{\|f_n - f\|} \underset{\varepsilon}{\|g\|} \to 0.$$

$n \to \infty$

$\sup\{\|f_n\|, \|f\|\} < \infty$.

b.

1) For convenience, consider X is a σ-finite space where $L(X)$ is reflexible. like. $(\mathbb{R}, m)$.

$\phi(f) = \int fg$ g is essentially bounded.

Condition: $f_n \to f$ a.e. $\sup \|f_n\|_1 < \infty$

$f_n = \cancel{n\chi_{(0, 1/n)}}\ \chi_{[n, n+1)}$ $f_n \to 0$ a.e.

$g = 1$

$\int fg = 1 \not\to 0$.

2) $\sup \|f\|_\infty < \infty$ $f_n \to f$ a.e. $\Rightarrow f_n \to f$ weak*

$\int |f - f_n| g \le (\|f_n\|_\infty + \|f\|_\infty) \int |g| < \infty$

$\boxed{L^\infty = (L^1)^*}$ $f \in (L^1)^*$ $\forall g \in L^1$. $\perp$.

Learn more Functional Analysis.

$\boxed{\text{EX21}}$ $1 < p < \infty$. $f_n \to f$ weakly in $\ell^p(A)$ iff $\sup \|f_n\|_p < \infty$ & $f_n \to f$ p.c..

$(\Leftarrow)$ ✓

$(\Rightarrow)$ $\chi_{\{a\}} \in \ell^q$ $\Rightarrow \int f_n \chi_a = f_n(a) \to \int f \chi_{\{a\}} = f(a)$

$\Rightarrow f_n \to f$ p.c.

$f_n \in (\ell^{p'})^*$. for $g \in \ell^{p'}$ $\int f_n g \to \int fg$

$\Rightarrow \sup_n \|Tf_n(g)\|_q < \infty \xRightarrow{\text{resonance}} \sup_n \|Tf_n\| < \infty \Rightarrow \sup \|f_n\|_p < \infty$ $\square$

$\boxed{\text{EX22}}$ 1) $\forall g \in L^2$ $\int f_n g \to 0$ R.L lemma

2) $f_n \overset{m}{\not\to} 0$ easy

$f_n \overset{a.e.}{\not\to} 0$ If so, By DCT $\lim \int_0^1 \cos^2 2\pi x = \lim \frac{1}{2} + \int_0^1 \cos 4\pi x$

$\| \qquad\qquad = \frac{1}{2}$

$0 \qquad\qquad \not\|$ $\square$

75.

Some L^p ineq

$\boxed{\text{Thm 5.16}}$ (Chebyshev) $f \in L^p$ $(0<p<\infty)$, then for $\forall \alpha>0$

$$\mu(\{|f(x)|>\alpha\}) < \left(\frac{\|f\|_p}{\alpha}\right)^p$$

proof. $E_\alpha = \{|f|>\alpha\}$

$$\left(\int |f|^p\right) > \int_{E_\alpha} |f|^p > \alpha^p \mu(E_\alpha) \Rightarrow \mu(E_\alpha) < \left(\frac{\|f\|_p}{\alpha}\right)^p$$

$\|$

$\|f\|_p^p$

$\square$

$\boxed{\text{Thm 5.17}}$ Let $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ be σ-finite measure space.

K $\mathcal{M}\otimes\mathcal{N}$ measurable on $X \times Y$

Suppose $\int |K(x,y)| d\mu(x)$. $\int |K(x,y)| d\nu(y) \leq C$ a.e.

$1 \leq p \leq \infty$. If $f \in L^p$

$\Rightarrow Tf(x) = \int K(x,y) f(y) d\nu(y)$ converges a.e. for $x \in X$.

$Tf \in L^p(\mu)$ $\|Tf\|_p \leq C \|f\|_p$.

proof. $|K(x,y) f(y)| = |K(x,y)|^{\frac{1}{q}} \, |K(x,y)^{\frac{1}{p}} f(y)|$

$$\int |K(x,y) f(y)| \cdot d\nu(y) \leq \left(\int \left(|K(x,y)|^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} \left(\int \left(|K(x,y)|^{\frac{1}{p}} |f(y)|\right)^p\right)^{\frac{1}{p}}$$

$$\left(\int |K|\right)^{\frac{1}{q}} \left(\int |K| |f|^p\right)^{\frac{1}{p}}$$

$\|$

$$\Rightarrow Tf(x) \leq C^{\frac{1}{q}} \left(\int |K| |f(y)|^p d\nu(y)\right)^{\frac{1}{p}}$$

$$\|Tf\|_p^p \leq \int C^{\frac{p}{q}} \int |K| |f(y)|^p d\nu(y)$$

$$= C^{p/q} \int\int |K| |f(y)|^p d\nu(y) d\mu(x)$$

$$\overset{\text{Tonelli}}{=} C^{\frac{p}{q}} \int\int |K| |f(y)|^p d\mu(x) d\nu(x)$$

$$= C^{\frac{p}{q}+1} \int f^p d\nu(y)$$

$$\Rightarrow \|Tf\|_p \leq C \|f\|_p$$ $\square$

For case $p=1$ or ∞, it seems easier.

In RA note, I had treated an ineq with chaos, and to relieve my self, I will deal the ineq below carefully.

$\boxed{\text{Thm 5.18}}$ (Minkowski) $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are both σ-finite

Let f be a $\mathcal{M} \otimes \mathcal{N}$ - measurable function on $X \times Y$.

a. If $f \geq 0$, $1 \leq p < \infty$

$$\left(\int \left(\int f(x,y) d\nu(y) \right)^p d\mu(x) \right)^{\frac{1}{p}} \leq \int \left[\int f(x,y)^p d\mu(x) \right]^{\frac{1}{p}} d\nu(y)$$

b. If $1 \leq p \leq \infty$, $f(\cdot, y) \in L^p(\mu)$ for a.e. y & $y \mapsto \| f(\cdot,y) \|_p$ is $L^1(\nu)$, then $f(x, \cdot) \in L^1(\nu)$ for a.e. x, the function $x \mapsto \int f(x,y) d\nu(y)$ is in $L^p(\mu)$, and

$$\left\| \int f(\cdot, y) d\nu(y) \right\|_p \leq \int \| f(\cdot, y) \|_p \, d\nu(y).$$

proof. a. $p = 1$: tonelli.

$p > 1$ $\quad g \in L^q$

$$\int \left[\int |f(x,y)| d\nu(y) \right] |g(x)| \, d\mu(x)$$

$$= \iint |f(x,y)| |g(x)| \, d\nu(y) \, d\mu(x)$$

$$= \iint f(x,y) \, g(x) \, d\mu(x) \, d\nu(y)$$

$$\leq \int \|g\|_q \, \| f(x,y) \|_p \, d\nu(y)$$

$$= \underbrace{\|g\|_q}_{} \int \left(\int |f(x,y)|^p \right)^{\frac{1}{p}} d\nu(y).$$

$$\left\| \frac{g}{\|g\|_q} \right\|_q = 1 \qquad \sup \text{LHS} = \left\| \int |f(x,y)| \, d\nu(y) \right\|_p \leq \text{RHS}.$$

~~Thm 5.19 Let K be~~ $\quad$ b. fubini ... $\square$

The last topic is about Hardy inequality. Sometimes we can observe the special case of a theorem to understand it, so here we do.

" T is 'average operator' $\Rightarrow \|Tf\|_p \leq \frac{p}{p-1} \|f\|_p$. i.e. average operator would not enlarge its singularity ~~proper~~. $\longrightarrow$ We can adjust the weight of different points $\Rightarrow$ generalize ! "

77 $\boxed{\text{Thm 5.19}}$ Let K be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x,\lambda y)=\lambda^{-1}K(x,y)$ for all $\lambda>0$ & $\int_0^\infty |K(x,1)|\,x^{-\frac{1}{p}}dx=C<\infty$ for some $p\in[1,\infty]$, and let q be the congruent exponent to p. For $f\in L^p$ & $g\in L^q$, let

$$Tf(y)=\int_0^\infty K(x,y)f(x)\,dx,\qquad Sg(y)=\int_0^\infty K(x,y)g(y)\,dy.$$

proof. Note condition, let $x=yz$ $dx=y\,dz$

$$\int_0^\infty |K(x,y)f(x)|\,dx=\int_0^\infty |K(yz,y)f(yz)|\,y\,dz=\int_0^\infty |K(z,1)f(yz)|\,dz$$

$$\underline{\underline{f_z(y)=f(yz)}}\ \int_0^\infty |K(z,1)f_z(y)|\,dz$$

$$\cancel{\|Tf\|_p=\left[\int_0^\infty\left(\int_0^\infty K(x,y)f(x)\,dx\right)^p dy\right]^{\frac{1}{p}}\overset{\text{Minkowski}}{\leq}\int_0^\infty\left(\int_0^\infty |K(x,y)f(x)|^p dx\right)^{\frac{1}{p}}dy}$$

$$\cancel{\overset{x=yz}{=}\int_0^\infty\left(\int_0^\infty |K(yz,y)f(yz)|^p\,y\,dz\right)^{\frac{1}{p}}dy}$$

$$\cancel{=\int_0^\infty}$$

$$\|Tf\|_p=\left\|\int_0^\infty K(x,y)f(x)\,dx\right\|_p\leq\int_0^\infty\|K(x,y)f(x)\|_p\,dx$$

$$=\int_0^\infty\|K(yz,y)f(yz)\|_p\,y\,dz$$

$$=\int_0^\infty\|K(z,1)f(yz)\|_p\,dz=\int_0^\infty |K(z,1)|\,\|f_z\|_p\,dz$$

f_z ? : $\|f_z\|_p=\left(\int_0^\infty |f(yz)|^p dy\right)^{\frac{1}{p}}=z^{-\frac{1}{p}}\|f\|_p$

$$\|Tf\|_p\leq\int_0^\infty z^{-\frac{1}{p}}|K(z,1)|\,\|f\|_p\,dz\leq C\|f\|_p.\qquad\square$$

$\boxed{\text{Cor 5.20}}$ ~~$K(x,y)=\frac{1}{y}$ $\int_0^\infty K(x,1)x^{-\frac{1}{p}}dx$~~

Let $Tf(y)=\frac{1}{y}\int_0^y f(x)\,dx$. $Sg(x)=\int_x^\infty y^{-1}g(y)\,dy$

Then for $1<p\leq\infty$ & $1\leq q<\infty$

$\|Tf\|_p\leq\frac{p}{p-1}\|f\|_p$, $\|Sg\|_q\leq q\|g\|_q$

proof. Let $K(x,y)=y^{-1}\chi_{x<y}$ $\int_0^\infty |K(x,1)|\,x^{-\frac{1}{p}}dx=\int_0^1 x^{-\frac{1}{p}}dx$

$$=\frac{1}{1-\frac{1}{p}}=\frac{p}{p-1}=q$$

$\square$

$\boxed{\text{Def } 5.23}$ If f is measurable on X, $0<p<\infty$. We define

$$[f]_p = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{\frac{1}{p}}$$

$$\text{Weak-}L^p = \{ f : [f]_p < \infty \}$$

Why do we say $-$weak$-L^p$? It shall imply ~~the~~ inclusion relationship

$$L^p \subseteq \text{Weak-}L^p$$

By chebyshev ineq. we see

$$\frac{\|f\|_p^p}{\alpha^p} \geq \mu(\{|f|>\alpha\}) = \lambda_f(\alpha)$$

$$\Rightarrow \quad [f]_p \leq \|f\|_p$$

If we replace $\lambda_f(\alpha)$ by $\left(\frac{[f]_p}{\alpha}\right)^p$ in the integral $p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha$ which equals $\|f\|_p^p$, we obtain

$\|f\|_p^p = p\int_0^\infty [f]_p^p \, \alpha^{-1} d\alpha$. which diverges at 0 & ∞.

This ~~phen~~ phenomenon seems ~~to~~ imply ~~that~~ we need some stronger bounds near 0 & ∞ to make a function be in L^p instead of weak-L^p.

Our standard example of a function that is in weak-L^p but not in L^p is $f(x) = \cancel{x} \; x^{-\frac{1}{p}}$ on $(0,\infty)$ with m.

$$\left(\alpha^p \mu\{x^{-\frac{1}{p}} > \alpha\} = \alpha^p \, m\left(\frac{1}{\alpha^p} > x\right) = 1 < \infty. \right).$$

Frequently it is convenient to express a function as the sum of a small part and a big part, as what we did in probability theory —— truncation.

$\boxed{\text{Prop } 5.24}$ If f is a measurable function, $A>0$. Let $E(A) = \{|f|>A\}$.

and set
$$h_A = f\chi_{X\setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)} \quad \leftarrow \text{fat}$$
$$g_A = f - h_A = (\operatorname{sgn} f)(|f|-A)\chi_{E(A)} \quad \leftarrow \text{tall}$$

Then
$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha+A) \qquad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha<A. \\ 0 & \alpha \geq A \end{cases}$$

§1. Interpolation of L^p spaces.

Before further discussion, let's recall some useful tools.

$\boxed{\text{prop } 5.9}$ $0<p<q<r\le\infty$, then $L^p\cap L^r \subseteq L^q$ and $\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$.

where $\lambda\in(0,1)$ is defined by

$$q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1} \qquad \lambda = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$$

$\boxed{\text{Thm } 5.13}$ Let p & q be the congruent exponents. Suppose that g is a measurable function on X s.t. $fg\in L^1$ for all f in the space Σ of simple functions that vanish outside a set of finite measure, and the quantity

$$M_q(g) = \sup\{ |\int fg| : f\in\Sigma \ \& \ \|f\|_p = 1\} \quad \text{is finite.}$$

Also, suppose either $S_g = \{g(x)\ne 0\}$ is σ-finite or μ is semifinite.

Then $g\in L^q$ & $M_q(g) = \|g\|_q$

The first time I saw the two results, I thought they are strange, and I don't understand their motivation totally. But now, maybe a little better?

$\boxed{\text{Thm } 5.25}$ (Riesz-Thorin Interpolation) Suppose that $(X,\mathcal{M},\mu)$ & $(Y,\mathcal{N},\nu)$ are measure spaces and $p_0, p_1, q_0, q_1 \in [1,\infty]$. If $q_0=q_1=\infty$, suppose also that ν is semi-finite. For $0<t<1$, define p_t & q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

If T is a linear map from $L^{p_0}(\mu)+L^{p_1}(\mu)$ into $L^{q_0}(\nu)+L^{q_1}(\nu)$ such that $\|Tf\|_{q_0} \le M_0\|f\|_{p_0}$ for $f\in L^{p_0}(\mu)$ and

$$\|Tf\|_{q_1} \le M_1 \|f\|_{p_1} \text{ for } f\in L^{p_1}(\mu), \text{ then}$$

$$\|Tf\|_{q_t} \le M_0^{1-t} M_1^{t} \|f\|_{p_t} \text{ for } L^{p_t}(\mu), \ 0<t<1.$$

$\boxed{\text{Motivation}}$ · We have the relation of L^p spaces: $1\le p<q\le r<\infty$

" $L^p\cap L^r \subseteq L^q \subseteq L^p + L^r$ "

extend such relation to operator!

1. $p_0 = p_1 = p$

$$\|Tf\|_{q_t} \underset{\substack{\uparrow \\ \boxed{\text{prop } 6.9}}}{\le} \|Tf\|_{q_0}^{1-t} \|Tf\|_{q_1}^{t} \le M_0^{1-t} M_1^{t} \|f\|_p$$

$$\left(t = \frac{q_t^{-1} - q_0^{-1}}{q_1^{-1} - q_0^{-1}} = \frac{\frac{1-t}{q_0} + \frac{t}{q_1} - \frac{1}{q_0}}{\frac{1}{q_1} - \frac{1}{q_0}} \right) \checkmark$$

2. $p_0 \ne p_1$ $\boxed{\forall f \xrightarrow{\text{dense.}} \Sigma X}$ It suffices to show that $\Rightarrow p_t < \infty$

$\|Tf\|_{q_t} \le M_0^{1-t} M_1^{t} \|f\|_{p_t}$ for all $f \in \Sigma_X$

$$\|Tf\|_{q_t} \underset{\substack{\uparrow \\ \boxed{\text{Prop } 6.13}}}{=} \sup\left\{ \left| \int (Tf) g \, d\nu \right| : g \in \Sigma_Y \ \& \ \|g\|_{q_t'} = 1 \right\}.$$

$\{Tf \ne 0\}$ must be σ-finite ~~when~~ ~~since~~ since $Tf \in L^{q_0} \cap L^{q_1}$, unless $q_0 = q_1 = \infty$

(for the case $q_0 = q_1 = \infty$. we have assumed ν is semi-finite).

Moreover, we may assume ~~that~~ $f \ne 0$ and rescale f so that $\|f\|_{p_t} = 1$.

We therefore wish to establish the following claim:

" If $f \in \Sigma_X$ & $\|f\|_{p_t} = 1$. then $\left| \int (Tf) g \, d\nu \right| \le M_0^{1-t} M_1^{t}$ for all $g \in \Sigma_Y$ such $\|g\|_{q_t'} = 1$ ".

Let $f = \sum_1^m c_j \chi_{E_j}$, $g = \sum_1^n d_k \chi_{F_k}$. where the E_j's & the F_k's are disjoint in X & Y and the c_j's and d_k's are non-zero.

Write

$$\begin{cases} c_j = |c_j| e^{i\theta_j} \quad d_k = |d_k| e^{i\psi_k} \\ \alpha(z) = (1-z)p_0 + z p_1 \quad \beta(z) = (1-z) q_0 + z q_1 \end{cases} \Rightarrow \begin{cases} \alpha(t) = p_t^{-1} \\ \beta(t) = q_t^{-1} \end{cases} \text{ for } t \in (0,1)$$

Fix t. We have known $\alpha(t) > 0$.

$$f_z = \sum_1^m |c_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{i\theta_j} \chi_{E_j}$$

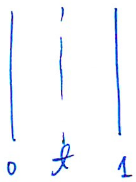
$$g_z = \begin{cases} \sum_1^n |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{i\psi_k} \chi_{F_k} & \beta(t) < 1 \\ g. & \beta(t) = 1 \end{cases}$$

Finally, set $\phi(z) = \int T(f_z) g_z \, d\nu$

$$\phi(z) = \sum_{j,k} A_{jk} |c_j|^{\alpha(z)/\alpha(t)} |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} \qquad A_{jk} = e^{i(\theta_j + \psi_j)} \int (T\chi_{E_j}) \chi_{F_k} \, d\nu.$$

So ϕ is an entire holomorphic function of z that is bounded in the strip $0 \le \mathrm{Re}\, z \le 1$

$\boxed{\text{Lem } 5.26}$ (Hadamard three-lines) let ϕ be a bounded continuous function on the strip $0 \le \mathrm{Re}\, z \le 1$ that is holomorphic on the interior of the strip. If $|\phi(z)| \le M_0$ for $\mathrm{Re}\, z = 0$ and $|\phi(z)| \le M_1$ for $\mathrm{Re}\, z = 1$, then $|\phi(z)| \le M_0^{1-t} M_1^{t}$ for $\mathrm{Re}\, z = t$



$$\phi_\varepsilon(z) = \phi(z)\, M_0^{z-1} M_1^{-z} \exp(\varepsilon z (z-1))$$

$$\begin{cases} \alpha(is) = p_0^{-1} + is(p_1^{-1} - p_0^{-1}) \\ 1-\beta(is) = (1-q_0^{-1}) - is(q_1^{-1} - q_0^{-1}) \end{cases}$$

$$|f_{is}| = \cancel{\sum_1^m |c_j|}\, |f|^{\mathrm{Re}\frac{\alpha(is)}{\alpha_t}} = |f|^{\frac{p_t}{p_0}}$$

$$|g_{is}| = |g|^{\frac{q_t'}{q_0'}}$$

$$|\phi(is)| \overset{\text{Hölder}}{\le} \|Tf_{is}\|_{q_0} \|g_{is}\|_{q_0'} \le M_0 \|f_{is}\|_{p_0} \|g_{is}\|_{q_0'}$$
$$= M_0 \|f\|_{p_t} \|g\|_{q_t'} = M_0$$

So $|\phi(1+is)| \le M_1$.

$\Rightarrow \|Tf\|_{q_t} \le M_0^{1-t} M_1^{t} \|f\|_{p_t}$ for $f \in \Sigma_X$

$\forall f \in L^{p_t}$. $\underset{\in \Sigma_X}{f_n} \xrightarrow{p.c.} f$ $\quad |f_n| \nearrow |f|$

$E = \{|f| > 1\}$ $\quad g_n = f_n \chi_E$ $\quad h_n = f_n - g_n$.

$g = f\chi_E$ $\quad h = f - g$ $\quad$ WLOG, suppose $p_0 < p_t < p_1$

$(g_n \in L^{p_0} \quad h_n \in L^{p_1})$

$$\text{DCT} \Rightarrow \begin{cases} \|f_n - f\|_{p_t} \to 0 \\ \|g_n - g\|_{p_0} \to 0 & \Rightarrow \|Tg_n - Tg\|_{q_0} \to 0 \\ \|h_n - h\|_{p_1} \to 0. & \Rightarrow \|Th_n - h\|_{q_1} \to 0. \end{cases}$$

Sub seq. $Tf_n \xrightarrow{a.e.} Tf$

$$\|Tf\|_{q_t} \overset{\text{Fatou}}{\le} \liminf \|Tf_n\|_{q_t} \le \liminf M_0^{1-t} M_1^{t} \|f_n\|_{p_t}$$
$$= M_0^{1-t} M_1^{t} \|f\|_{p_t}.$$

□

$\boxed{\text{Thm 5.27}}$ (Marcinkiewicz) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces; $p_0, p_1, q_0, q_1 \in [1, \infty]$ s.t. $p_0 \le q_0$, $p_1 \le q_1$, and $q_0 \ne q_1$; and $\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$, $\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$ where $0 < t < 1$.

If T is sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) & (p_1, q_1), then T is strong type (p, q). More precisely, if $[Tf]_{q_j} \le C_j \|f\|_{p_j}$ $(j = 0, 1)$; then $\|Tf\|_q \le B_p \|f\|_p$ where B_p depends only on p_j, q_j, C_j in addition to p; ~~& for~~ and for $j = 0, 1$, $B_p |p - p_j|$ resp. B_p remains bounded as $p \to p_j$ if $p_j < \infty$ (resp. $p_j = \infty$).

Φ: some vector space of measurable functions. $T : \Phi \subseteq (X, \mathcal{M}, \mu) \to (Y, \mathcal{N}, \nu)$

$\boxed{\text{Def 5.28}}$ (1) T is sublinear if $\begin{cases} |T(f+g)| \le |Tf| + |Tg| \\ |T(cf)| = c|Tf| \end{cases}$ for $\forall f, g \in \Phi$, $c > 0$

(2) A sublinear map T is strong type (p, q) $(1 \le p, q \le \infty)$ if $L^p(\mu) \subseteq \Phi$, $T : L^p(\mu) \to L^q(\nu)$, $\exists\, C > 0$ s.t. $\|Tf\|_q \le C\|f\|_p$ for $\forall f \in L^p(\mu)$

(3) A sublinear map T is weak type (p, q) $(1 \le p \le \infty, 1 \le q < \infty)$ if $L^p(\mu) \subseteq \Phi$, $T : L^p(\mu) \to \text{weak-}L^q(\nu)$ & $\exists\, C > 0$ s.t. $[Tf]_q \le C\|f\|_p$ for $\forall f \in L^p(\mu)$

$\boxed{\text{Rmk}}$ weak (p, ∞) = strong (p, ∞)

sublinear $\supsetneq$ linear

such as H-L.

Proof of $\boxed{\text{thm 5.27}}$

Case $p_0 < p_1$: Assume $q_1, q_0 < \infty$. Given $f \in L^p(\mu)$ & $A > 0$.

$$\int |g_A|^{p_0} d\mu = p_0 \int_0^\infty \beta^{p_0 - 1} \lambda_{g_A}(\beta)\, d\beta = p_0 \int_0^\infty \beta^{p_0-1} \lambda_f(\beta + A)\, d\beta$$

$$= p_0 \int_A^\infty (\beta - A)^{p_0 - 1} \lambda_f(\beta)\, d\beta \le p_0 \int_A^\infty \beta^{p_0-1} \lambda_f(\beta)\, d\beta$$

$$\int |h_A|^{p_1} d\mu = p_1 \int_0^A \beta^{p_1 - 1} \lambda_f(\beta)\, d\beta$$

$$\int |Tf|^q d\nu = q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha)\, d\alpha = 2^q q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(2\alpha)\, d\alpha$$

$$\lambda_{Tf}(2\alpha) \le \lambda_{g_A}(\alpha) + \lambda_{h_A}(\alpha)$$

85. What ~~do~~ we did above holds true for $\forall$ $\alpha, A > 0$. In Folland's Book, he lets σ denote the common value of the following equation

$$\sigma = \frac{p_0(q_0-p)}{q_0(p_0-p)} = \frac{p_1(q_1-q)}{q_1(p_1-p)} \quad \& \quad A = \alpha^{\sigma}.$$

I'm not sure about ~~the~~ motivation now...

$$\|Tf\|_q^q = q\int \alpha^{q-1}\lambda_{Tf}(\alpha)\,d\alpha \le 2^q q\int_0^\infty \alpha^{q-1}\left(\lambda_{Tg_A}(\alpha)+\lambda_{Th_A}(\alpha)\right)d\alpha.$$

$$\left(\cancel{\lambda_{Tg_A}(\alpha) \le \frac{[Tg]_p^p}{\alpha^p} \le C_{p_0}^p \|g\|_{p_0}^{p_0}}\right.$$

$$\left(\begin{aligned}\lambda_{Tg_A}(\alpha) &\le \frac{[Tg_A]_{p_0}^{q_0}}{\alpha^{q_0}} \le \frac{C_{p_0}^{q_0}\|g_A\|_{p_0}^{q_0}}{\alpha^{q_0}}\\ \lambda_{Th_A}(\alpha) &\le \frac{C_{p_1}^{q_1}\|Th_A\|_{p_1}^{q_1}}{\alpha^{q_1}}\end{aligned}\right)$$

$$\Rightarrow \text{LHS} \le 2^q q\int_0^\infty \alpha^{q-1}\left(\left(\frac{C_0\|g_A\|_{p_0}}{\alpha}\right)^{q_0} + \left(\frac{C_1\|h_A\|_{p_1}}{\alpha}\right)^{q_1}\right)$$

$$\le 2^q q\int_0^\infty \alpha^{q-1}\left(\frac{C_0^{q_0}}{\alpha^{q_0}}\left(p_0\int_A^\infty \beta^{p_0-1}\lambda_f(\beta)\,d\beta\right)^{\frac{q_0}{p_0}} + \frac{C_1^{q_1}}{\alpha^{q_1}}\left(p_1\int_0^A \beta^{p_1-1}\lambda_f(\beta)\,d\beta\right)^{\frac{q_1}{p_1}}\right)d\alpha$$

$$= \sum_0^1 2^q q\, C_j^{q_j} p_j^{q_j/p_j}\int_0^\infty\left(\int_0^\infty \phi_j(\alpha,\beta)\,d\beta\right)^{q_j/p_j} d\alpha$$

$$\phi_j = \chi_j(\alpha,\beta)\,\alpha^{\frac{(q-q_j-1)p_j}{q_j}}\beta^{p_j-1}\lambda_f(\beta)$$

$$\int_0^\infty\left(\int_0^\infty \phi_j(\alpha,\beta)\,d\beta\right)^{\boxed{q_j/p_j}\ge 1} d\alpha \overset{\text{Minkowski}}{\le} \left(\int_0^\infty\left(\int_0^\infty \phi_j^{q_j/p_j}(\alpha,\beta)\,d\alpha\right)^{\frac{p_j}{q_j}} d\beta\right)^{\frac{q_j}{p_j}}$$

Let $\tau = \frac{1}{\sigma}$. If $q_1 > q_0$

$$\int_0^\infty\left(\int_0^\infty \phi_0(\alpha,\beta)^{q_0/p_0}\,d\alpha\right)^{\frac{p_0}{q_0}} d\beta = \int_0^\infty\left(\int_0^{\beta^\tau}\alpha^{q-q_0-1}\,d\alpha\right)^{\frac{p_0}{q_0}}\beta^{p_0-1}\lambda_f(\beta)\,d\beta$$

$$= |q-q_0|^{-\frac{p_0}{q_0}}p^{-1}\|f\|_p^p$$

If $q_1 < q_0$ ...

Similarly ... $= |q-q_1|^{-p_1/q_1}p^{-1}\|f\|_p^p$

$$\Rightarrow \sup\{\|Tf\|_q : \|f\|_p^p = 1\} \le \underset{B_p}{\underset{\shortparallel}{2q^{\frac{1}{q}}\left(\sum C_j^{q_j}\left(p_j/p\right)^{q_j/p_j}|q-q_j|^{-1}\right)^{\frac{1}{q}}}}.$$

What has been remained?

Case II $\begin{cases} p_0 < p_1 = \infty \\ q_0 < q_1 = \infty \end{cases}$ $\cancel{A = \left(\frac{\alpha}{d}\right)^{\sigma} \quad d = C_1 \left(p_1 \|f\|_p^{p}\right)^{\frac{1}{p_1}}}$

$\cancel{\sigma = \frac{p_1}{p_1 - p}}$

$\cancel{\|Th_A\|_\infty^{p} \le C_1^{p_1} \|h_A\|_{p_1}}$

Case III $\begin{cases} p_0 < p_1 < \infty \\ q_0 < q_1 = \infty \end{cases}$

Case IV $\begin{cases} p_0 < p_1 < \infty \\ q_1 < q_0 < \infty. \end{cases}$ It's not the key point though. $\square$

Maybe I will write out the proof ~~of~~ two special ~~case~~ cases

that (1) $p_0 = q_0 = 1$. $p_1 = q_1 = 2$

(2) $p_0 = q_0 = 1$, $p_1 = q_1 = \infty$

Marcinkiewicz V.s Riesz-Thorin

sub linear linear

weak estimation at endpoints strong.

T is ~~bound~~ bounded ~~T is bo~~ much sharpen

Example — H-L

$\begin{cases} \|Hf\|_\infty \lesssim \|f\|_\infty \\ [Hf]_1 \lesssim \|f\|_1 \end{cases} \Rightarrow \|Hf\|_p \le \frac{cp}{p-1} \|f\|_p$

Special Case of Marcinkiewicz Interpolation thm

(1) ~~$p_0 = p_0 = 1$ $q_0 = q_1 =$~~ $\begin{cases} [Tf]_1 \lesssim \|f\|_1 \\ [Tf]_2 \le C_2 \|f\|_2 \end{cases}$

$p_0 = q_0 = 1$ $p_1 = q_1 = 2$

Cut by distribution functions $f_1 = f \chi_{\{|f| < st\}}$ $f_2 = f \chi_{\{|f| \ge st\}}$

$\|Tf\|_q^q = \int |Tf|^q d\nu = \int q t^{q-1} \underline{\lambda_{Tf}(t)}\, dt$ (1)

$\lambda_{Tf}(\alpha) \le \lambda_{Tf_1}(t/2) + \lambda_{Tf_2}(t/2).$ (2)

$$\lambda_{Tf_1}(t/2) \le \left(\frac{2C_1\|f\|_2}{t}\right)^2 = \frac{4C_1^2}{t^2}\int_{|f|<st} |f|^2 dx$$

$$\lambda_{Tf_2}(t/2) \le \frac{2C_1}{t}\|f\|_1 = \frac{2C_1}{t}\int_{|f|\ge st} |f|\, dx$$

$$\|Tf\|_q^q \le q\int_0^\infty t^{q-1}\left(\frac{4C_1^2}{t^2}\int_{|f|<st}|f|^2dx + \frac{2C_0}{t}\int_{|f|\ge st}|f|\,dx\right)dt$$

$$\int_0^\infty \int_{|f|<st} \frac{4C_1^2}{t^2}|f|^2 dx\, dt = \int_{|f|<st}\int_{|f|/s}^\infty \boxed{4C_1^2}\, t^{q-3}|f|^2 dt\, dx$$

$$= \int_{|f|<st} 4C_1^2|f|^2 \frac{(f/s)^{q-2}}{2-q} = \frac{4C_1^2 s^{2-q}}{2-q}\int_{|f|<st}|f|^q$$

...

$$\Rightarrow \|Tf\|_q^q \le \boxed{\frac{4C_1^2}{2-q}s^{2-q}\cdot q}\int_{|f|<st}|f|^q dx + \boxed{\frac{2C_0}{q-1}\cdot s^{1-q}\cdot q}\int_{|f|\ge st}|f|^q dx$$

let $\frac{4C_1^2}{2-q}s^{2-q}\cancel{q} = \frac{2C_0}{q-1}\cancel{s^{1-q}}\cdot\cancel{q}$

$$s = \frac{2C_0(2-q)}{4C_1^2(q-1)}$$

$$B_p = \frac{4C_1^2}{2-q}\left(\frac{2C_0(2-q)}{4C_1^2(q-1)}\right)^{2-q}\cdot q$$

$$(4C_1^2)^{q-1}\cdot(2C_0)^{2-q}\cdot(2-q)^{1-q}\cdot(q-1)^{2-q}\, q.$$

(2) $p_0 = q_0 = 1$. $p_1 = q_1 = \infty$ $\begin{cases}[Tf]_1 \le C_0\|f\|_1 \\ [Tf]_\infty \le C_1\|f\|_\infty \Leftrightarrow \|Tf\|_\infty \le C_1\|f\|_\infty\end{cases}$

$$\|Tf\|_q^q \le \int_0^\infty q t^{q-1}\left[\lambda_{Tf_1}(t/2) + \lambda_{Tf_2}(t/2)\right]dt$$

$f_1 = f\chi_{\{|f|<\alpha\}}$
$f_2 = f\chi_{\{|f|\ge\alpha\}}$

~~λ_{Tf_1}~~

$\|Tf_1\|_\infty \le C_1\|f_1\|_\infty \le C_1\alpha$. $\boxed{\text{If } t > 2C_1\alpha \quad \mu\{|Tf_1| > t/2\} = 0}$

$$\mu\{|Tf_2| > t/2\} \le \frac{2C_0\|f_2\|_1}{t} = \frac{2C_0}{t}\int_{|f|\ge\alpha}|f|\,dx$$

assume f with ~~cc~~ cpt supp first

$$\cancel{\|Tf\|_q^q \le \int_0^\infty q t^{q-1}\left(\mu(E)(2C_1\alpha)+\right.}$$

$$(2C_1\alpha)^q\mu(E) + \frac{2C_0 q}{q-1}(2C_1)$$

Since the estimation in I_∞ is dependent on the measure of supp set, 88
We shall treat it carefully.

Suppose $\|f_n - f\|_p \to 0$.

$$\|Tf_n\|_q^q \le I_\infty + J_1 \le \frac{2A_1^p}{p-1}(2A_\infty)^{p-1}\int |f_n|^p + \mu(E_n)\cdot(2A_0\alpha)^p$$

$$\|Tf_n\|_q^q \le \frac{2A_1^p}{p-1}(2A_\infty)^{p-1}\|f_n\|_p^p + \underline{\mu(E_n)}(2A_0\alpha)^p$$

Left.

$\boxed{\text{EX36}}$

$$\|f\|_q^q = \int |f|^q d\mu = q\int \alpha^{q-1}\lambda_f(\alpha)\,d\alpha$$

$$\left(\sup \alpha^p \lambda_f(\alpha)\right) < \infty$$

$$= q\int \frac{\alpha^p\lambda_f(\alpha)}{\underline{\alpha^{(p-q)+1}}}\,d\alpha < \infty.$$

$$\|f\|_q^q \le \cancel{C\left(q\int \alpha^{q-1}\lambda_{f_1}(\alpha)\,d\alpha + q\int \alpha^{q-1}\lambda_{f_2}(\alpha)\right)}$$

short

$$= q\int_0^c \alpha^{q-p-1}\,\underline{\alpha^p\lambda f(\alpha)\,d\alpha} < \infty.$$

□

$\boxed{\text{EX41}}$ $p < q$

$$\int T(\lambda_1 f_1 + \lambda_2 f_2)\,g = \lambda_1\int Tf_1\,g + \lambda_2\int Tf_2\,g \Rightarrow T \text{ is linear}$$

$$\|Tf\|_q = \sup\left\{\left|\int Tf\,g\right| \mid g\in L^p\ \|g\|_p = 1\right\}$$

$$= \sup\left\{\left|\int f\,Tg\right| : g\in L^p,\ \|g\|_p = 1\right\}$$

$$\le \sup\left\{\|f\|_q \underline{\|Tg\|_p} : g\in L^p,\ \|g\|_p = 1\right\}$$

$$\overset{\text{Bound}}{\le} C\|f\|_q.$$

□

R-T + dense

~~EX42~~

$$\cancel{[Tf]_{q,\infty} \le \|f\|_p}$$

$$\cancel{[Tf]_{q,} \le \|f\|_p}$$