Chapter 6 Radon measure.

Basic setting: X — LCH (topological)

$\mathcal{B}_X$ — the Borel ~~sets~~ σ-algebra on X

$C_c(X)$ — continuous functions on X with compact support.

$\boxed{\text{Def 6.1}}$ A linear functional I on $C_c(X)$ will be called positive if

$$I(f) \geq 0 \quad \text{whenever } f \geq 0.$$

$\boxed{\text{Prop 6.2}}$ If I is a positive linear functional on $C_c(X)$, for each compact $K \subseteq X$ there exists a constant C_K s.t. $|I(f)| \leq C_K \|f\|_u$ for all $f \in C_c(X)$ s.t. $\text{supp}(f) \subseteq K$.

proof. It suffices to consider real-valued f. Using Urysohn lemma, we can construct a function ϕ which $\phi \in C_c(X, [0,1])$ and $\phi = 1$ on K.

Then if $\text{supp}(f) \subseteq K$, we have $|f| \leq \|f\|_u \cdot \phi$.

$$\Rightarrow \begin{cases} \|f\|_u \phi + f \geq 0 \\ \|f\|_u \phi - f \geq 0 \end{cases} \Rightarrow \|f\|_u I(\phi) \pm I(f) \geq 0$$

$$\Rightarrow |I(f)| \leq C_K \|f\|_u. \qquad \square$$

$\boxed{\text{Def 6.3}}$ (Regularity) (i) Outer regular on E

$$\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\}.$$

(ii) inner regular on E

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ cpt}\}.$$

If μ is outer and inner regular on all Borel sets, μ is called regular.

$\boxed{\text{Def 6.4}}$ Radon measure (A Borel measure): (i) finite on all cpt sets

(ii) Outer regular on all Borel sets

(iii) Inner regular on all open sets.

Notation: If U is open in X and $f \in C_c(X)$, we shall write

$$f \prec U$$

to mean that $0 \leq f \leq 1$ and $\text{supp}(f) \subseteq U$.

$\boxed{\text{Thm 6.5}}$ (The Riesz Representation Theorem)

If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f\,d\mu$ for all $f \in C_c(X)$. Moreover, μ satisfies

$$\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\} \text{ for all open } U \subseteq X \quad (*)$$

$$\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\} \text{ for all cpt } K \subseteq X \quad (**)$$

proof: Uniqueness: If μ is a Radon measure s.t. $I(f) = \int f\,d\mu$ for all $f \in C_c(X)$, then clearly $I(f) \leq \mu(U)$ whenever $f \prec U$, U is open.

On the other hand, if $K \subseteq U$ is cpt, by Urysohn there is an $f \in C_c(X)$ s.t. $f \prec U$ and $f = 1$ on K. whence $\mu(K) \leq I(f)$.

$\mu(U) \overset{\text{inner}}{\underset{\text{regular}}{=}} \sup\{\mu(K) : K \subseteq E,\ K \text{ cpt}\}$. $\leq \sup\{I(f) \cdots$

$I(f) \leq \mu(U)$

$= \sup\{I(f) : f \prec U,\ f \in C_c(X)\} \quad (*)$

from ineq. Then the measure of open sets is determined totally by I, and thus on all Borel sets because of outer regularity.

Existence. We begin by defining $\mu(U) = \sup\{I(f) : f \prec U,\ f \in C_c(X)\}$. for all U open. Then we define $\mu^*(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\}$.

Clearly $\mu^*(U) \leq \mu^*(V)$ if $U \subseteq V$ and hence $\mu^*(U) = \mu(U)$ if U is open.

What do we need to prove?

1. μ^* is an outer measure
2. Every open set is μ^*-measurable.

   ~~we have that~~ every Borel set is μ^*-measurable
   If so, then every Borel set is μ^*-measurable from Caratheodory thm. and $\mu = \mu|_{B_X}$ is a Borel measure

   The measure μ is outer regular and satisfies $\mu(U) = \cdots \quad (*)$.

3. μ satisfies $(**)$.

   If so, it clearly implies every compact set is of finite measure. and inner regularity on open sets follows. Indeed, if U is open, and $\alpha < \mu(U)$, then choose $f \prec U$ s.t. $I(f) > \alpha$. and let $K = \operatorname{supp} f$. If $g \in C_c(X)$ and $g \geq \chi_K$, then $g - f \geq 0$ and hence $I(g) \geq I(f) > \alpha \Rightarrow \mu(K) > \alpha \Rightarrow$ Inner regularity.

91. 4. $I(f) = \int f \, d\mu$ for all $f \in C_c(X)$.

proof of 1. : ~~Recalling the relative result shown in Chapter 1, it suffices to show~~ $\cancel{\mu^*(E) = \inf\{\sum_1^\infty \mu(U_j) : U_j \text{ open} ; E \subseteq \bigcup_1^\infty U_j\}}$

If $U = \bigcup_1^\infty U_j$, $f \in C_c(X)$ and $f \prec U$, let $K = \text{supp}(f)$. Since K is cpt, we have $K \subseteq \bigcup_1^n U_j$ for some finite n.

Using partion of unity, there exists $g_1, \cdots g_n \in C_c(X)$ with $g_j \prec U_j$ & $\sum g_j = 1$ on K.

$\Rightarrow f = \sum_1^n f g_j$ & $f g_j \prec U_j$

So $I(f) = \sum_1^n I(f g_j) \le \sum_1^n \mu(U_j) \le \sum_1^\infty \mu(U_j)$

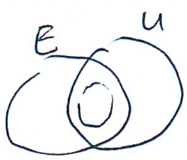
Since it's true for all $f \prec U \Rightarrow \mu(U) \le \sum_1^\infty \mu(U_j)$.

proof of 2. We must show that if U is open and $E \subseteq X$ with finite measure, then $\mu^*(E) \ge \mu^*(U \cap E) + \mu^*(E \setminus U)$.

First, suppose E is open, then $E \cap U$ is open.

$\forall \varepsilon, \exists f \in C_c(X)$, $I(f) > \mu(E \cap U) - \varepsilon$.

$f \prec E \cap U \Rightarrow E \setminus \text{supp}(f)$ is open



$g \prec E \setminus \text{supp} f$ s.t. $I(g) > \mu\, E \setminus \text{supp}(f) - \varepsilon$.

$\Rightarrow f + g \prec E$.

$\mu(E) \ge I(f) + I(g) > \mu(E \cap U) - \varepsilon + \mu(E \setminus \text{supp}(f)) - \varepsilon$

$\ge \mu(E \cap E) + \mu(E \setminus U) - 2\varepsilon$.

For the general case. if $\mu^*(E) < \infty$, by the def of μ^*

$\exists V \overset{\supseteq E}{\text{open}}$ $\mu(V) < \mu^*(E) + \varepsilon$.

$\mu^*(E) > \mu(V) - \varepsilon \ge \mu^*(V \setminus U) + \mu^*(V \cap U) - \varepsilon$

$\ge \mu^*(E \setminus U) + \mu^*(E \cap U) - \varepsilon$. ✓

proof of 3: If K is cpt, $f \in C_c(X)$ and $f \ge \chi_K$. let $U_\varepsilon = \{f > 1 - \varepsilon\}$

U_ε is open, $\forall g \prec U_\varepsilon \Rightarrow (1-\varepsilon)^{-1} f - g \ge 0 \Rightarrow I(g) \le (1-\varepsilon)^{-1} I(f)$.

Thus $\mu(K) \le \mu(U_\varepsilon) \le (1-\varepsilon)^{-1} I(f)$ $\quad \varepsilon \to 0$ $\quad$ Urysohn

$\Rightarrow \mu(K) \le I(f)$ on the other hand, $\forall U \supseteq K$ $\exists f$ s.t. $f \ge \chi_K$ and $f \prec U \Rightarrow I(f) \le \mu(U)$. By outer regularity of μ on K. ✓

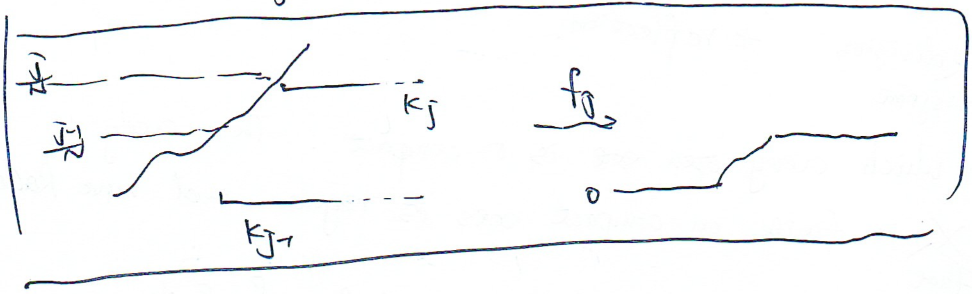
proof of 4. It suffices to show $I(f) = \int f\,d\mu$ if $f \in C_c(X, [0,1])$.

Given $N \in \mathbb{N}$ for $1 \le j \le N$ let $K_j = \{ f \ge \frac{j}{N} \}$.

$K_0 = \operatorname{supp}(f)$. Also, define $f_1, \cdots f_N \in C_c$ by

$f_j(x) = 0$ if $x \notin K_{j-1}$, $f_j(x) = f(x) - \frac{j-1}{N}$ if $x \in K_{j-1} \setminus K_j$

and $f_j(x) = N^{-1}$ if $x \in K_j$.



$\Rightarrow f_j: \frac{\chi_{K_j}}{N} \le f_j \le \frac{\chi_{K_{j-1}}}{N}$

$\Rightarrow \frac{1}{N}\mu(K_j) \le \int f_j\,d\mu \le \frac{1}{N}\mu(K_{j-1})$

$f = \sum_1^N f_j$

$\Rightarrow \frac{1}{N}\sum_1^N \mu(K_j) \le \underline{\int f\,d\mu} \le \frac{1}{N}\sum_0^{N-1}\mu(K_j)$

for $I(f)$. $U \supseteq K_{j-1}$ $Nf_j \prec U \Rightarrow I(f_j) \le \frac{\mu(U)}{N}$

$\xRightarrow{\text{outer}} I(f_j) \le \frac{1}{N}\mu(K_{j-1})$

$I(f_j) \ge \frac{1}{N}\mu(K_j)$

$\Rightarrow \frac{1}{N}\sum_1^N \mu(K_j) \le I(f) \le \frac{1}{N}\sum_0^{N-1}\mu(K_j)$

$\Rightarrow \left| I(f) - \int f\,d\mu \right| \le \frac{\mu(K_0) - \mu(K_N)}{N} \le \frac{\mu(\operatorname{supp}(f))}{N} \to 0$. $<\infty$.

$\Rightarrow I(f) = \int f\,d\mu$. $\square$.

After Representation theorem, I will just state some conclusions and their ideas unless I ~~think~~ find something interesting in this chapter.

As we have ~~sh~~ known, Radon measure is only inner regular for ~~an~~ open sets, but if we add some conditions, ~~then~~ it should regular for all.

$\boxed{\text{Prop 6.6}}$ Every Radon measure is inner regular ~~for~~ on all of its σ-finite sets.

E finite $U \supseteq E$. $F \subset\subset U$ $K = F \setminus V$

$V \supseteq U \setminus E$ $\mu(K) = \mu(F) - \mu(F \cap V) \ge \mu(F) - \varepsilon$

$\ge \mu(U \cap E) - \mu(V) - \varepsilon \ge \mu(E) - 2\varepsilon$

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$\boxed{\text{Cor 6.7}}$ Every σ finite Radon measure is regular.
If X is σ-Compact, every Radon measure on X is regular.

$\boxed{\text{Prop 6.8}}$ μ is a σ-finite Radon measure on X and E is a Borel set in X.
(a) for $\forall \varepsilon>0$, $\exists$ open U, closed F $F \subseteq E \subseteq U$ and $\mu(U \setminus F) < \varepsilon$
(b) $\exists$ F_σ A, G_δ B $A \subseteq E \subseteq B$ and $\mu(B \setminus A) < \varepsilon$.

Idea: σ-finite $\underset{\text{disjoint}}{\Rightarrow}$ division + reflection.

$\boxed{\text{Thm 6.9}}$ X LCH in which every open set is σ-compact. Then every Borel measure on ~~X~~ that is finite on compact sets is regular and hence Radon.

finite on cpt sets $\Rightarrow$ $C_c(X) \subseteq L^1(\mu)$ $\Rightarrow$ $I(f) = \int f \, d\mu$
$\Updownarrow$ associate.
Radon measure
(functional $\Rightarrow$ measure)

$\boxed{\text{Aim: the Radon measure constructed by } I \text{ agrees with } \mu.}$

$U \subseteq X \Rightarrow U = \bigcup_{j=1}^{\infty} K_j$

$f_1 \prec U$ s.t. $f_1 = 1$ on K_1
$\vdots$
$f_n \prec U$ s.t. $f_n = 1$ on $\bigcup_{j=1}^{n} K_j$ (& $\bigcup_{j=1}^{n-1} \operatorname{supp}(f_j)$)
$f_n \nearrow \chi_U$.

Idea: Exhaustion

$\mu(U) = \int \chi_U \, d\mu = \int \lim f_n \, d\mu$
$\underset{\text{MCT}}{=} \lim \int f_n \, d\mu \overset{I(f)}{=} \nu(U) \Longrightarrow \mu$ & ν agree on open sets.
$\left(\overset{\|\ \text{def of } \nu}{\lim \int f \, d\nu} \right)$.

It suffices to μ is a Radon measure $<$ outer regular on $\mathcal{B}$ / inner regular on $\mathcal{T}$.

Then we conclude by uniqueness. Standard measure-approximation argument.. $\square$

$\boxed{\text{Prop 6.10}}$ μ is a Radon measure on X. $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$

Idea: $\boxed{\text{simple} \overset{\text{dense}}{\subseteq} L^p}$
$\Downarrow$
cpt support $\longrightarrow$ E finite
$K \subseteq E \subseteq U$ Urysohn $\chi_K \le f \le \chi_U$ $f \in C_c$. $\square$

$\boxed{\text{Thm 6.11}}$ (Lusin) μ Radon $f: X \to \mathbb{C}$ measurable vanish outside a set of finite measure. $\forall \varepsilon > 0$.
$\exists \phi \in C_c(X)$ s.t. $\phi = f$ except ε-set. If f is bounded $\|\phi\|_u \le \|f\|_u$.

$\boxed{\text{Def 6.12}}$ LSC if $\{f > a\}$ open
USC if $\{f < a\}$ open

$\boxed{\text{prop 6.13}}$ (1) U open $\Rightarrow \chi_U$ LSC ✓

(2) f LSC $\Rightarrow$ $c \ge 0$ cf LSC ✓

(3) G a family LSC $f = \sup\{g(x) : g \in G\}$ is LSC

(4) f_1 f_2 LSC $\Rightarrow f_1 + f_2$ LSC

(5) X LCH + $f \ge 0$ LSC
$\Rightarrow f(x) = \sup\{g(x) : g \in C_c,\ 0 \le g \le f\}$

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(3) $f^{-1}((a,\infty]) = \bigcup_{g \in G} g^{-1}((a,\infty])$

(4) $\{f_1 + f_2 > a\} \supseteq \{f_1 > a - f_2(x_0) + \varepsilon\} \cap \{f_2 > f(x_0) - \varepsilon\}$ ✓
ψ
x_0

(5) ~~Urysohn $0 \le g \le a\chi_U \le f$~~ Urysohn. $\square$.
$f(y) > a$
~~$f(x) > 0 \Rightarrow f(x) > a > 0$~~

$\boxed{\text{prop 6.14}}$ G a family of $\underbrace{}_{\text{non-negative}}$ LSC on LCH that is directed by $\le$
$f = \sup\{g : g \in G\}$ If μ is any Radon measure on X. then
$\int f d\mu = \sup\{\int g d\mu : g \in G\}$

$\int f d\mu \ge \sup\{\int g d\mu : g \in G\}$. ✓

What about the reverse? $U_{nj} = \{f(x) > j/2^n\}$

$\Rightarrow \phi_n = \frac{1}{2^n} \sum_{j=1}^{2^{2n}} \chi_{U_{nj}}$ (Atom decomposition)

$a < \int f d\mu \Rightarrow \int \phi_n = 2^{-n} \sum_{j=1}^{2^{2n}} \mu(U_{nj}) > a$ for n large enough by MCT

$\boxed{\text{But } \phi_n \notin G.}$

$\psi = 2^{-n}\sum_j \chi_{\underset{\text{cpt}}{\underline{k_j}}}$ cc U_{nj} $\quad \cancel{\Rightarrow} \underline{\int \psi > a}$ $\quad k_j$ cc U_{nj}.

For $x \in \bigcup_j k_j \Rightarrow f(x) - \phi_n(x) \geq \psi(x)$.

$\Rightarrow g_x(x) > \psi(x) \quad -\chi_{k_j}$ LSC $\Rightarrow g_x(x) - \psi(x)$ LSC

$V_x = \{y : g_x(y) > \psi(y)\}$ open

$\Rightarrow \underset{\underset{\text{finite}}{=}}{\bigcup_j k_j} \subseteq \{V_x\}$ open cover

$\Rightarrow \bigcup_j k_j \subseteq V_{x_1} \cup \cdots \cup V_{x_m} \quad g \in G$ s.t $g \geq g_{x_i}$ $i = 1, \cdots, m$.

$\Rightarrow g \geq \psi \quad \boxed{\int g > a}$ ✓. compactness argument

$\boxed{\text{Cor 6.15}}$ μ Radon $\quad f \geq 0$ LSC.

$$\int f\,d\mu = \sup\{\int g\,d\mu : g \in C_c(X),\ 0 \leq g \leq f\}.$$

$\boxed{\text{prop 6.16}}$ μ Radon $\quad f \geq 0$.

$$\int f\,d\mu = \inf\{\int g\,d\mu : g \geq f \quad g \text{ LSC}\}$$

If $\{f > 0\}$ σ-finite

$$\int f\,d\mu = \sup\{\int g\,d\mu : 0 \leq g \leq f \quad g \text{ USC}\}.$$

$\phi_n \nearrow f \quad f = \phi_1 + \sum_2^\infty(\phi_n - \phi_{n-1}) \overset{\text{rewrite}}{=\!=} \sum_{j=1}^\infty a_j \chi_{E_j}$

$U_j \supseteq E_j \quad \mu(U_j) \leq \mu(E_j) + \frac{\varepsilon}{2^j a_j}$

$g = \sum_{j=1}^\infty a_j \chi_{U_j}$ LSC $\quad g \geq f \quad$ & $\int g\,d\mu \leq \int f\,d\mu + \varepsilon$ ✓

$a < \int f\,d\mu \quad a < \sum_j^N \underset{\sigma\text{-finite}}{\underline{\mu(E_j)}} \Rightarrow a < \sum_1^N \mu(k_j) = g$. ✓

The Dual of C_0

$\boxed{\text{Def 6.17}}$ $C_0(X) = \cancel{\overline{C_c(X)}^{\text{uni}}}\ \{f \in C(X) : f \text{ vanishes at infinity}\}$

$\boxed{\text{Rmk}}$ $C_0(X) = \overline{C_c(X)}^{\text{uniform}} \quad (X = \text{LCH})$

extend the functional $I(f) = \int f \, d\mu$ to $C_0(X)$ iff it's bounded wrt uni norm.

$\boxed{\text{lem } 6.18}$ $I \in C_0(X_0, \mathbb{R})^*$ $\exists$ $I^{\pm}$ s.t $I = I^+ - I^-$

$I^+(f) := \sup \{ I(g) : g \in C_0, \ 0 \le g \le f \}$

$|I(g)| \le \|I\| \, \|g\|_u \le \|I\| \, \|f\|_u$

$\Rightarrow 0 \le \cancel{|I(g)|} \ I^+(f) \le \|I\| \, \|f\|_u$. $\quad (0 = I(0).)$.

$I^+(f_1 + f_2) \ge I(g_1) + I(g_2)$ $\qquad$ $0 \le g_1 \le f_1$, $0 \le g_2 \le f_2$

$\Rightarrow I^+(f_1+f_2) \ge I^+(f_1) + I^+(f_2)$

$0 \le g \le f_1 + f_2 \quad g_1 = \min(g, f_1) \quad g_2 = g - g_1 \le f_2$

$\Rightarrow I(g) = I(g_1) + I(g_2) \le I^+(f_1) + I^+(f_2)$

$\Rightarrow I^+(f_1 + f_2) \le I^+(f_1) + I^+(f_2)$.

$\Rightarrow I^+$ is linear.

$f \in C_0$. $\quad f = f^+ - f^-$

$I^+(f) = I^+(f^+) - I^+(f^-)$

$f = g - h \quad g, h \ge 0$

$\Rightarrow g + f^- = h + f^+$

$\Rightarrow I^+(g) + I^+(f^-) = I^+(h) + I^+(f^+)$

$\Rightarrow I^+(f) = I^+(g) - I^+(h)$

$\boxed{\text{Def } 6.19}$ Signed Radon measure : A signed Borel measure whose positive and negative variations are Radon.

Complex Radon measure : Real and Imaginary Radon

finite. $\longrightarrow$ Regular

$\boxed{\text{prop } 6.20}$ μ Complex ~~Radon~~ Borel $\Leftrightarrow$ then μ Radon iff $|\mu|$ Radon

9/ $\boxed{\text{Thm 6.20}}$ (Radon Representation thm) X LCH, $\mu \in M(X)$ $f \in C_0(X)$

let $I_\mu(f) = \int f d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isomorphism

from $M(X)$ to $C_0^*(X)$.

proof. By continuity argument, $\forall I \in C_0(X)^*$ is of form I_μ

$$\left|\int f d\mu\right| \leq \|f\|_u \|\mu\|$$

$\Rightarrow \|I_\mu\| \leq \|\mu\| \Rightarrow$

if $h = \frac{d\mu}{d|\mu|}$ $|h| = 1$. By Lusin's thm $\exists f \in C_c$ s.t $\|f\|_u = 1$

$f = \bar{h}$ outside a small set.

$$\|\mu\| = \int |h|^2 d|\mu| = \int \bar{h} d\mu \leq \left|\int f d\mu\right| + \left|\int (f - \bar{h}) d\mu\right|$$

$$\leq \left|\int f d\mu\right| + \underline{\underline{2\varepsilon}} = \|I_\mu\| + 2\varepsilon$$

$\Rightarrow \|\mu\| \leq \|I_\mu\|$ ✓ □

In probability theory, we call weak*-topology by Vague topology in which $\mu_\alpha \to \mu$ iff $\int f d\mu_\alpha \to \int f d\mu$ for $\forall f \in C_0(X)$

<u>I will stop here</u>.