Since I have written a note of Fourier Analysis (though very elementary), many props here will be treated ~~simp~~ briefly.

$\boxed{\text{Def 7.1}}$ $\mathcal{S}$ Schwartz space

$$\mathcal{S} = \{ f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha \}$$

in which $\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|$.

$\boxed{\text{prop 7.2}}$ If $f \in \mathcal{S}$, $\partial^\alpha f \in L^p$ for $\forall \alpha$ and $\forall p \in [1,\infty]$

$\boxed{\text{prop 7.3}}$ $\mathcal{S}$ is a Frechet space with topology defined by the norms $\|\cdot\|_{(N,\alpha)}$.

Complete & Hausdorff + Countable seminorms.

proof: Completeness: $\{f_k\}$ Cauchy i.e. $\|f_j - f_k\|_{(N,\alpha)} \to 0$ for $\forall N, \alpha$.

for each α $\{\partial^\alpha f_k\} \rightrightarrows g_\alpha$.

$$f_k(x+te_j) - f_k(x) = \int_0^t \partial_j f_k(x+se_j)\, ds$$

$$\overset{k\to\infty}{\Rightarrow} g_0(x+te_j) - g_0(x) = \int_0^t \partial_j g_0(x+se_j)\, ds$$

$$\Rightarrow g_{e_j} = \partial_j g_0 \xRightarrow{\text{Induction}} g_\alpha = \partial^\alpha g_0 \Rightarrow \| g_0 - f_k\|_{(N,\alpha)}$$

$f_j \overset{c}{\rightrightarrows} f \Rightarrow f$ ✓ Check f is the limit. ($f \in \mathcal{S}$ & $\|f - f_k\|_{(N,\alpha)} \to 0$)

$$\partial_j f(x) = \lim_t \frac{f(x+te_j) - f(x)}{t} = \lim_t \frac{f(x+te_j) - f_k(x+te_j)}{}$$

Using the uniform convergence of ~~the~~ ~~derivative~~ derivation of high order

$\Rightarrow f_k \to f \in C^\infty$

$$f_k(x+te_j) - f_k(x) = \int_0^t \partial_j f_k(x+se_j)\, ds$$

$$\Rightarrow g_0(x+te_j) - f_0(x) = \int_0^t g_{e_j}(x+se_j)\, ds$$

$$\Rightarrow \partial_j g_0 = g_{e_j}$$

By standard series theory, we can attain a limit function $f \in C^\infty$.

It suffices to show f is in $\mathcal{S}$.

$$P_{N,\alpha}(f) = \underline{P_{N,\alpha}(f - f_i)} + P_{N,\alpha}(f_i)$$

$$P_{N,\alpha}(f - f_i) = \sup \lim (1+|x|)^N \partial^\alpha (f_j - f_i) \le \limsup (1+|x|)^N \partial^\alpha (f_i - f_j) < \varepsilon \quad \square$$

99. $\boxed{\text{prop 7.4}}$ If $f \in C^\infty$, then $f \in \mathcal{S}$ iff $x^\beta \partial_\alpha f < \infty$ for $\forall \alpha, \beta$.
iff $\partial_\alpha (x^\beta f) < \infty$ for $\forall \alpha, \beta$.

$x^\alpha := \prod_1^n x_i^{\alpha_i}$

proof $|x|^\beta \leq (1+|x|)^N$ for $|\beta| \leq N$ ($\mathcal{S} \Rightarrow \cdots$)

($\mathcal{S} \Leftarrow \cdots$)

$$(1+|x|)^N \underset{N}{\lesssim} (1+|x|^N) \underset{\in N}{\lesssim} 1 + \sum |x_i|^N$$
$$\lesssim \sum_{|\beta| \leq N} |x^\beta|$$

$\sum |x_i|^N \neq 0$ if $|x| = 1$

$\delta = \inf \sum |x_i|^N > 0$

$\Rightarrow \boxed{\sum |x_i|^N \geq \delta |x|^N}$ ✓

The second equivalence can be established by product rule. □

The next topic is the ~~invariance~~ continuity of ~~co~~ translation.

$\boxed{\text{Lem 7.5}}$ If $f \in C_c$, then f is uniform continuous.

$\boxed{\tau_y f(x) = f(x-y)}$

$\boxed{\text{prop 7.6}}$ If $1 \leq p < \infty$, ~~then~~ translation is continuous in L^p norm

i.e. $\lim_{y \to 0} \| \tau_{y+z} f - \tau_z f \|_p = 0$ for $f \in L^p$ $z \in \mathbb{R}^n$.

proof: $\tau_{y+z} f - \tau_z f = \tau_z(\tau_y f - f)$

Hence it suffices to show $\lim_{y \to 0} \| \tau_y f - f \|_p = 0$.

$\forall \varepsilon > 0$, let $\| \phi - f \| < \varepsilon/3$ in which ϕ is of compact support.

$$\Rightarrow \| \tau_y f - f \|_p \leq \| \tau_y f - \tau_y \phi \|_p + \| \tau_y \phi - \phi \|_p + \| \phi - f \|_p$$
$$\leq 2\varepsilon/3 + \| \tau_y \phi - \phi \|_p \to 0. \quad \square$$

$\boxed{\text{Rmk}}$ The conclusion ~~of~~ is false for $p = \infty$.

Some results later will concern about periodic functions. For convenience, we can suppose ~~that~~ the period is ~~one~~ $1 \in \mathbb{Z}^n$ and then the original functions become functions on ~~the~~ cube. We will indentify $\mathbb{T}^n$ with the unit cube.

$$f*g(x) = \int f(x-y)\,g(y)\,dy.$$

list some basic properties below.

$\boxed{\text{prop } 7.7}$ (a) $f*g = g*f$

(b) $(f*g)*h = f*(g*h)$

(c) for $z \in \mathbb{R}^n$, $\tau_z(f*g) = (\tau_z f)*g = f*(\tau_z g)$.

(d) $\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$.

proof (a) $(f*g)(x) = \int f(x-y)\,g(y)\,dy \overset{u=x-y}{=\!=} \int f(u)\,g(x-u)\,du = (g*f)(x)$

(b). $(f*g)*h(z) = \cancel{\int\left(\int f(x-y)\,g(y)\,dy\right) h(z-x)}\,dx$

$\cancel{= \iint f(x-y)\,g(y)\,h(z-x)\,dy\,dx}$

$= \cancel{\int\left(\int f(y)\,g(z-x-y\right.}$

$\iint f(y)\,g(z-x-y)\,h(x)\,dy\,dx$

$= \int f(y)\; g*h(z-y)\,dy = f*(g*h)(z).$

(c) $\tau_z\left(\int f(x-y)\,g(y)\,dy\right) = \int f(x-z-y)\,g(y)\,dy$

$= \int \tau_z f(x-y)\,g(y) = (\tau_z f)*g(x)$

(d) If $x \notin \overline{\operatorname{supp} f + \operatorname{supp} g} \Rightarrow x-y \notin \operatorname{supp} f$.

$f(x-y)\,g(y) = 0 \;\; \forall y \Rightarrow f*g(x) = 0$ $\square$

$\boxed{\text{Thm } 7.8}$ If $f \in L^1$, $g \in L^p$ $(1 \le p \le \infty)$, then $f*g(x)$ exists for a.e. x.

$f*g \in L^p$, $\|f*g\|_p \le \|f\|_1 \|g\|_p$

proof. $\|f*g\|_p = \left\|\int f(y)\cdot g(\cdot - y)\,dy\right\|_p \le \int |f(y)|\,\|g\|_p\,dy$ (Minkowski)

$= \|f\|_1 \|g\|_p$

$\boxed{\text{prop } 7.9}$ p, q are congruent exponents. $f \in L^p$, $g \in L^q$, then $f*g(x)\ \exists$ a.e.

$f*g$ is bounded and uniformly continuous, and $\|f*g\|_u \le \|f\|_p \|g\|_q$

If $1 < p < \infty \Rightarrow f*g \in C_0(\mathbb{R}^n)$.

proof. $\|\tau_y f*g - f*g\|_u = \|(\tau_y f - f)*g\|_u \overset{\text{Hölder}}{=} \|\tau_y f - f\|_p \cdot \|g\|_q \to 0$ as $y \to 0$ ($1 \le p < \infty$)

If $p = \infty$, exchange the role of f & g.

$\|f_n - f\|_p \to 0$. $\|g_n - g\|_q \to 0$ $f_n\, g_n$, are of compact supports

$\Rightarrow f_n * g_n \in C_c$

$\Rightarrow \|f_n * g_n - f*g\|_u \leq \|f_n - f\|_p \|g_n\|_q + \|f_n\|_p \|g_n - g\|_q \to 0$

$\Rightarrow f*g \in C_0$ $\square$

For simplicity, I omit the statment of other generalizations.

One of the most important properties of convolution is that, $f*g$ is at least as smooth as either f or g, because formally we have

$\partial^\alpha \int f(x-y)\, g(y)\, dy = \int \partial^\alpha f(x-y)\, g(y)\, dy = (\partial^\alpha f) * g$.

$\boxed{\text{prop 7.10}}$ If $f \in L^1$, $g \in C^k$ and $\partial^\alpha g$ is bounded for $|\alpha| \leq k$, then

$f*g \in C^k$. $\partial^\alpha (f*g) = f*(\partial^\alpha g)$

proof : DCT. $\square$

$\boxed{\text{prop 7.11}}$ If $f, g \in \mathcal{S}$, then $f*g \in \mathcal{S}$

proof : It's trivial that $f*g \in C^\infty$

$1+|x| \leq 1+|x-y|+|y| \leq (1+|x-y|)(1+|y|)$

$(1+|x|)^N |\partial^\alpha (f*g)(x)| \leq \int \underline{(1+|x-y|)^N |\partial^\alpha f(x-y)|} \cdot (1+|y|)^N g(y)\, dy$

$\leq \|f\|_{N,\alpha} \|g\|_{N+n+1,\alpha} \int \frac{1}{y^{n+1}}\, dy < \infty$. $\square$

If $\phi(x)$ is any function, $t>0$, we set

$\phi_t(x) = \frac{1}{t^n} \phi\left(\frac{x}{t}\right)$

If $\phi \in L^1 \Rightarrow \underline{\int \phi_t = \int \phi}$

$\boxed{\text{Thm 7.12}}$ Suppose $\phi \in L^1$ & $\int \phi = a$

(a). If $f \in L^p$ $(1 \leq p < \infty)$, then $f * \phi_t \to af$ in L^p norm as $t \to 0$

(b) If f is bounded and uniform continuous, then $f*\phi_t \to af$ uniformly as $t \to 0$

(c) If $f \in L^\infty$ and f is continuous on an open set U, then $f*\phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

proof: (a) $\cancel{\|f*\phi_t - af\|_p = \|(\phi_t - a)*f\|_p}$

$\cancel{= \int}$ $\qquad tz = y$

$$f*\phi_t(x) - af(x) = \int (f(x-y) - f(x))\phi_t(y)\,dy$$
$$= \int (f(x-tz) - f(x))\,\phi(z)\,dz$$
$$= \int [\tau_{tz} f(x) - f(x)]\,\phi(z)\,dz$$

$$\|f*\phi_t - af\|_p \overset{\text{Minkowski}}{\le} \int \|\tau_{tz} f - f\|_p |\phi(z)|\,dz$$

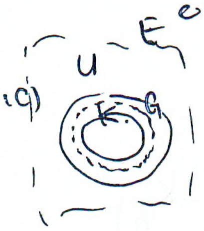
$\|\tau_{tz} f - f\| \le 2\|f\|_p$, $\|\tau_{tz} f - f\| \to 0$ as $t \to 0$. Then use DCT.

(b) $\|f*\phi_t - af\|_u = \sup_x \left| \int [\tau_{tz} f(x) - f(x)]\,\phi(z)\,dz \right|$

$\int_{B^c} |\phi(z)| < \varepsilon$ in B $t \to 0$ $|\tau_{tz} f(x) - f(x)| \overset{\text{uni}}{<} \varepsilon$

$$\le \quad C\varepsilon + C'\varepsilon \quad \to 0.$$

$E \subset\subset \mathbb{R}^n$

(c) 

$\int_{E^c} |\phi| < \varepsilon$. $\quad \int |f(x - tz) - f(x)|\,|\phi(z)|\,dz$

$x \in K$, $z \in E$ $\Rightarrow$ $x - tz \in G$

$K \subseteq G \subseteq \overline{G} \subseteq U$

$\underline{\underline{\text{LCH}}}$ $\searrow$ cpt

$$\overset{t\to 0}{\le} 2\|f\|_\infty \varepsilon + \int_E (|f(x-tz) - f(x)|)|\phi|\,dz$$
$$< 2\|f\|_\infty \varepsilon + \varepsilon \int |\phi|\,dz.$$

The argument in Folland's Book is not perfect. $\square$

$\boxed{\text{Thm 7.13}}$ Suppose $|\phi(x)| \le C \dfrac{1}{(1+|x|)^{n+\varepsilon}}$ for some $C, \varepsilon > 0$ ($\phi \in L^1$) and $\int \phi\,dx = a$. If $f \in L^p$ ($1 \le p \le \infty$), then $f*\phi_t(x) \to af(x)$ as $t \to 0$ for every x in Lebesgue set of f (i.e. a.e. x for f) (every x if f is cts).

Lebesgue point x. $\forall \delta$. $\exists \eta$

proof. $\int_{|y| \le r} |f(x-y) - f(x)|\,dy \le \delta r^n$, for $r \le \eta$

$$|f*\phi_t(x) - af| = \left| \int [f(x-y) - f(x)]\,\phi_t(y)\,dy \right|$$

$$\le \underbrace{\int_{|y| \le \eta} |f(x-y) - f(x)||\phi_t(y)|\,dy}_{I_1 \text{ nontrivial}} + \overbrace{\int_{|y| > \eta} |f(x-y) - f(x)||\phi_t(y)|\,dy}^{I_2 \text{ trivial}}$$

Cannot treat directly: $\delta \eta^n \to ?$ as $\delta \to 0$

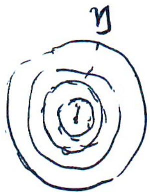
Estimation for $I_1 = \int_{|y|\le\eta} \underbrace{|f(x-y)-f(x)|}_{\le\delta\eta^n}\,\underbrace{|\phi_t(y)|}_{=\frac{1}{t^n}\phi\left(\frac{y}{t}\right)}\,dy$

Cancel

Let K be a integer such that $2^K \le \frac{\eta}{t} \le 2^{K+1}$ if $\frac{\eta}{t} > 1$ (t is small)

$K=0$ if $\eta/t < 1$

$\Rightarrow \quad 2^k \le \frac{|y|}{t} \le 2^{k+1}$



$$\{|y|\le\eta\} = \{|y|\le 2^{-K}\eta\} \cup \bigcup_{k=1}^{K} \{2^{-k}\eta \le |y| \le 2^{1-k}\eta\}$$

on $\{2^{-k}\eta \le |y| \le 2^{1-k}\eta\}$

$$|\phi_t(y)| \le \frac{C}{t^n}\left|\frac{y}{t}\right|^{-(n+\varepsilon)} \le \frac{C}{t^n}\left|\frac{2^{-k}\eta}{t}\right|^{-(n+\varepsilon)}$$

$$I_1 \le \sum_{k=1}^{K} \frac{C}{t^n}\left(\frac{2^{-k}\eta}{t}\right)^{-n-\varepsilon} \int_{2^{-k}\eta\le|y|\le 2^{1-k}\eta} |f(x-y)-f(x)|\,dy$$

$$+\, Ct^{-n}\int_{|y|\le 2^{-K}\eta} |f(x-y)-f(x)|\,dy$$

$$\le C\delta \sum t^{-n}\left(\frac{2^{-k}\eta}{t}\right)^{-n-\varepsilon}\cdot\left(2^{1-k}\eta\right)^n + Ct^{-n}\delta\cdot\left(2^{-K}\eta\right)^n$$

$$= 2^n C\delta \sum_{k=1}^{K} 2^{k\varepsilon}\cdot\left(\frac{\eta}{t}\right)^{-\varepsilon} + C\delta\left(\frac{2^{-K}\eta}{t}\right)^n$$

$$\le (\quad)\,\delta$$

□

For completeness, I add the estimation for I_2 here:

$$I_2 = \int_{|y|\ge\eta} |f(x-y)-f(x)|\,|\phi_t(y)|\,dy \le \int |f(x-y)\,\phi_t(y)|\,dy + \int |f(x)|\,|\phi_t(y)|\,dy$$

$$\le \|f\|_p \|\chi\phi_t\|_q + |f(x)|\,\|\chi\phi_t\|_1$$

$p=1$ ✓ $\quad \|\chi\phi_t\|_\infty \lesssim t^\varepsilon$

$p>1 \quad \|\chi\phi_t\|_q = \int_{|y|\ge\eta} t^{-nq}\left|\phi\left(\frac{y}{t}\right)\right|^q dy$

$$= \int_{|z|\ge\frac{\eta}{t}} t^{n(1-q)}|\phi(z)|^q\,dy$$

$n(1-q)-n+(n+\varepsilon)q$

$$\lesssim t^{n(1-q)}\int_{\frac{\eta}{t}}^{\infty} r^{n-1-(n+\varepsilon)q}\,dr$$

$$\lesssim t^{n(1-q)}\cdot\left(\frac{\eta}{t}\right)^{n-(n+\varepsilon)q} \lesssim t^{q\varepsilon}$$ ✓

$\boxed{\text{prop 7.14}}$ C_c^∞ (and hence $\mathscr{S}$) is dense in L^p $(1 \le p < \infty)$ and C_0.

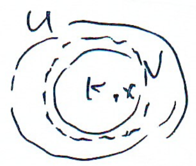
proof. Given $f \in L^p$ and $\varepsilon > 0$, there exists $g \in C_c$ with $\|f-g\|_p < \varepsilon/2$,

Let ϕ be a function in C_c^∞ s.t. $\int \phi = 1$ $\Rightarrow$ $g * \phi_t \in C_c^\infty$

$\|g * \phi_t - g\|_p < \varepsilon/2$ for sufficiently small t.

C_0 : $\|\cdot\|_u$ ✓ $\left(C_0 = \overline{C_c}^{\,uni} \right)$ □

$\boxed{\text{prop 7.15}}$ (C^∞ Urysohn lemma) If $K \subseteq \mathbb{R}^n$ is compact and U is an open set containing K, there exists a $f \in C_c^\infty$ such that $0 \le f \le 1$, $f = 1$ on K and $\operatorname{supp} f \subseteq U$



Proof $\delta = \operatorname{dist}(K, U^c)$

$V = \{x : \operatorname{dist}(x, K) < \delta/3\}$.

$\phi \in C_c^\infty$ $\int \phi = 1$ $\phi = 0$ if $|x| > \delta/3$

$f = \chi_V * \phi$ $\in C_c^\infty$

$\operatorname{supp} f \subseteq U$. & $\underset{K}{f(x)} = \int \chi_V(y) \phi(x-y)\, dy$

$$= \int_{y \in V} \phi(x-y)\, dy = 1.$$
$|x-y| < \delta/3$ □

The arguments in this part ~~is~~ are standard and well-known ...

Fourier transform.

Folland gave an interesting point of view to enter harmonic analysis.

$\boxed{\text{Thm 7.16}}$ If ϕ is a measurable function on $\mathbb{R}^n$ (w.r.t. $\mathbb{T}^n$) such that $\phi(x+y) = \phi(x)\phi(y)$ and $|\phi| = 1$, there exists $\xi \in \mathbb{R}^n$ (w.r.t. $\mathbb{Z}^n$) s.t. $\phi(x) = e^{2\pi i \xi \cdot x}$.

proof: It's a ODE problem.

$\boxed{\text{Thm 7.17}}$ Let $E_k(x) = e^{2\pi i k \cdot x}$. Then $\{E_k\}$ is an ortho~~gonal~~normal basis of $L^2(\mathbb{T}^n)$

proof. $\{E_k\}$ is dense in $C(\mathbb{T}^n)$ in the uni norm, hence L^2 norm.

$L^2(\mathbb{T}^n)$ is complete. □

~~prop 7.18~~ Restate: $f \in L^2(\mathbb{T}^n)$ $\hat{f}(k) := \langle f, E_k \rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x} dx$

$\sum_{k\in\mathbb{Z}^n} \hat{f}(k) E_k$ Fourier Series

We shall observe that $|\hat{f}(k)| \le \|f\|_1$ $\|\hat{f}(k)\|_2 = \|f\|_2$

$\boxed{\text{prop 7.18}}$ Suppose that $1 \le p \le 2$. and q is the congruent exponent to p. If $f \in L^p(\mathbb{T}^n)$, then $\hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \le \|f\|_p$ $\frac{1}{1-\frac{t}{2}}$

proof: $\begin{cases} \|\hat{f}\|_\infty \le \|f\|_1 & \text{for } f \in L^1 \\ \|\hat{f}\|_2 = \|f\|_2 & \text{for } f \in L^2 \end{cases} \xRightarrow{R\text{-}T}$ $\boxed{\begin{aligned} \frac{1}{p_t} &= \frac{1-t}{1} + \frac{t}{2} \\ \frac{1}{q_t} &= \frac{1-t}{\infty} + \frac{t}{2} \end{aligned}}$

$\Rightarrow \|\hat{f}\|_q \le \|f\|_p$

$\boxed{\text{Rmk}}$ Do you understand the use of Riesz-Thorin Interpolation?

$\boxed{\text{Thm 7.19}}$ Suppose $f, g \in L^1(\mathbb{R}^n)$

(a) $\widehat{\tau_y f}(\xi) = e^{-2\pi i \xi\cdot y} \hat{f}(\xi)$, $\tau_y(\hat{f}) = \hat{h}$ in which $h(x) = e^{2\pi i y\cdot x} f(x)$.

(b) If T is an invertible linear transform of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ is its inverse transpose, then $(f\circ T)^\wedge = |\det T|^{-1} \hat{f}\circ S$. In particular, if T is a rotation, then $(f\circ T)^\wedge = \hat{f}\circ T$; and if $Tx = t^{-1}x$ $(t>0)$, then $(f\circ T)^\wedge(\xi) = t^n \hat{f}(t\xi)$ so that $\widehat{f_t}(\xi) = \hat{f}(t\xi)$.

(c) $(f*g)^\wedge = \hat{f}\hat{g}$

(d) If $x^\alpha f \in L^1$ for $|\alpha| \le k$, then $\hat{f} \in C^k$ & $\partial^\alpha \hat{f} = ((-2\pi i x)^\alpha f)^\wedge$

(e) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \le k$, and $\partial^\alpha f \in C_0$ for $|\alpha| \le k-1$

then $(\partial^\alpha f)^\wedge(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.

proof. (a) $\widehat{\tau_y f}(\xi) = \int f(x-y) e^{-2\pi i \xi\cdot x} dx = e^{-2\pi i \xi\cdot y} \hat{f}(\xi)$

(b) $(f\circ T)^\wedge(\xi) = \int f(Tx) e^{-2\pi i \xi\cdot x} dx$

$\xi \cdot T^{-1}x = \cancel{\xi (T^{-1}x)^T}$ $\qquad = \int f(x) e^{-2\pi i \xi\cdot T^{-1}x} |\det T^{-1}| dx$

$= (T^{-1})^* \xi \cdot x$ $\qquad = |\det T|^{-1} \cdot \hat{f}(S\xi)$

Adjoint

ⓒ $\int \left(\int f(x-y)\, g(y)\, dy \right) e^{-2\pi i \xi \cdot x}\, dx$

$= \int \left(\int f(x-y)\, e^{-2\pi i \xi \cdot (x-y)}\, dx \right) g(y)\, e^{-2\pi i \xi \cdot y}\, dy$

$= \hat{f}\,\hat{g}\,(\xi)$

ⓓ $\partial_\xi^\alpha \hat{f}(\xi) = \partial_\xi^\alpha \int f(x)\, e^{-2\pi i \xi \cdot x}\, dx$

$\underset{\text{exchange}}{=} \int f(x)\, (-2\pi i x)^\alpha e^{-2\pi i \xi \cdot x}\, dx$

ⓔ If $f \in C^1 \cap C_c \Rightarrow |\xi|\, \hat{f}(\xi) < \infty \Rightarrow \hat{f}(\xi) \in C_0$

$C^1 \cap C_c \underset{\text{dense}}{\subseteq} L^1 \quad \in C_c.$

$f_n \xrightarrow{L^1} f \Rightarrow \boxed{\hat{f}_n} \xrightarrow{\text{uni}} \hat{f} \quad \left(\|\hat{f}_n - \hat{f}\|_u \le \|f_n - f\|_1 \right)$

$\Rightarrow \hat{f} \in C_0$

$\boxed{\text{Thm 7.20}}$ (R-L lemma) $\mathcal{F}(L^1(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n)$ □

proof:

The proof of ⓔ $\partial \hat{f}(\xi) = \int f'(x)\, e^{-2\pi i \xi x}\, dx \underset{\text{parts}}{\overset{\text{by}}{=}} \cdots$ ✓

induction. □.

$\boxed{\text{Cor 7.21}}$ $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. ctsly

proof: $f \in \mathcal{S} \Rightarrow x^\alpha \partial^\beta f \in \mathcal{S} \subseteq L^1 \cap C_0$ (multiply $\overset{\mathcal{F}}{\Rightarrow}$ derivative, derivative $\overset{\mathcal{F}}{\Rightarrow}$ multiply)

$\Rightarrow (x^\alpha \partial^\beta f)^\wedge = C\, \partial^\alpha (\xi^\beta \hat{f}) = \Sigma * < \infty$,

$\Rightarrow \hat{f} \in \mathcal{S}$

$\|(x^\alpha \partial^\beta f)^\wedge\|_u \le \|x^\alpha \partial^\beta f\|_1 \le C \,\|(1+|x|)^n x^\alpha \partial^\beta f\|_u$ □

Well, some contents are too common and I shall just list them.

① $f(x) = e^{-\pi x^2} \Rightarrow \hat{f}(\xi) = e^{-\pi \xi^2}$

② $f, g \in L^1$ $\boxed{\int \hat{f} g = \int f \hat{g}}$

③ Inversion $f, \hat{f} \in L^1 \Rightarrow f \overset{a.e.}{=} f_0 \in C$ & $(\hat{f})^\vee = (f^\vee)^\wedge = f_0$

107 proof of Inversion thm:

It suffices to show $f = (\hat{f})^\vee$ a.e.

$$(\hat{f})^\vee(x) = \int e^{2\pi i \xi \cdot x} \hat{f}(\xi)\,d\xi \overset{DCT}{=} \lim_{t\to 0} \int \underbrace{e^{-\pi t^2 \xi^2} e^{2\pi i x\xi}}_{\phi(\xi)} \hat{f}(\xi)\,d\xi$$

$$g(x) := e^{-\pi |x|^2} \Rightarrow \hat{\phi}(y) = g_t(x-y)$$

$$= \lim_{t\to 0} \int \hat{\phi}(\xi) f(\xi)\,d\xi$$

It is a classical ~~cost~~ proof which even appears in Bing Wang's Complex Analysis (H).

$$= \lim_{t\to 0} \int g_t(x-\xi) f(\xi)\,d\xi$$

$$= \lim_{t\to 0} g_t * f(x) \xrightarrow{a.e.} f$$ $\boxed{\text{on its lebesgue point set.}}$ $\square$

④ If $f \in L^1$, $\hat{f} = 0 \Rightarrow f = 0$ a.e.

⑤ $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ on to $\mathcal{S}$.

⑥ Plancherel thm: If $f \in L^1 \cap L^2 \Rightarrow \hat{f} \in L^2$; $\mathcal{F}|_{L^1\cap L^2}$ extends unique to an $\boxed{\text{unitary isomorphism on } L^2.}$

proof of Plancherel thm:

$\mathcal{X} := \{f \in L^1 : \hat{f} \in L^1\}$. $\hat{f} \in L^1 \Rightarrow |f| \overset{a.e.}{=} \left|\int \hat{f} e^{2\pi i \xi \cdot x}\right| \le \|\hat{f}\|_1 < \infty$

$\Rightarrow f \in L^\infty \Rightarrow f \in L^2$

$\Rightarrow \mathcal{X} \subseteq L^2$ Note that $\mathcal{S} \subseteq \mathcal{X}$, $\mathcal{S} \underset{dense}{\subseteq} L^2$

$\Rightarrow \mathcal{X} \underset{dense}{\subseteq} L^2$

$f, g \in \mathcal{X}$. Let $h = \bar{\hat{g}}$

$$\hat{h}(\xi) = \int e^{-2\pi i \xi \cdot x} \bar{\hat{g}} = \overline{\int e^{2\pi i \xi x} \hat{g}} \overset{a.e.}{=} \overline{g(\xi)}$$

$\int f \bar{g} = \int f \hat{h} = \int \hat{f} \bar{\hat{g}} \Rightarrow \mathcal{F}|_{\mathcal{X}}$ is an unitary operator.

Let $g = f \Rightarrow \|f\|_2 = \|\hat{f}\|_2$

Since $\mathcal{F}(\mathcal{X}) = \mathcal{X} \Rightarrow \mathcal{F}|_{\mathcal{X}}$ extends by continuity to an unitary isomorphism to L^2.

It remains to show ~~$\mathcal{F}$~~ that this extension ~~agrees~~ agrees with $\mathcal{F}$ on $L^1 \cap L^2$

If $f \in L^1 \cap L^2$, $g = e^{-\pi|x|^2}$

$\Rightarrow f * g_\varepsilon \in L^1$ (By Young's)

$\widehat{f * g_\varepsilon} = \hat{f}\, \hat{g}_\varepsilon = \underbrace{\hat{f}}_{|\hat{f}|_u \le \|f\|_1 < \infty} e^{-\pi^2 \varepsilon^2 |\xi|^2} \in L^1$

$\Rightarrow f * g_\varepsilon \in \mathcal{X}$ $\qquad f * g_\varepsilon \to f$ in L^1, L^2

$\Rightarrow \widehat{f * g_\varepsilon} \to \hat{f}$ uni & L^2

Maybe I need to show that it's "extension"

$\mathcal{X} \to L^2$ $\qquad g \in L^2$

$f \mapsto \hat{f}$ $\qquad f_n \xrightarrow{L^2} g$

$\|f_m - f_n\|_2 = \|\hat{f}_m - \hat{f}_n\|_2$

L^2 is complete

$\lim \hat{f}_n := \hat{g}(g)$

If $h_k \xrightarrow{L^2} g$ $\quad \cancel{\lim_k \hat{h}_k = \lim_n \hat{f}}$

$\lim_{k,n} \|\hat{h}_k - \hat{f}_n\|_2 = 0$?

$= \lim \|h_k - f_n\|_2$

$\le \lim_{k,n} \|h_k - g\| + \|f_n - g\| = 0.$

$\square$

⑦ H-Y Ineq

$1 \le p \le 2$. $\quad f \in L^p(\mathbb{R}^n)$

$\Rightarrow \|\hat{f}\|_q \le \|f\|_p$.

RT-I

⑧ $f \in C(\mathbb{R}^n)$ $\quad |f| \le \frac{C}{(1+|x|)^{n+\varepsilon}}$

$|\hat{f}(\xi)| \le \frac{C}{(1+|\xi|)^{n+\varepsilon}}$

$\Rightarrow \sum_{k \in \mathbb{Z}^n} f(x+k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x}$

both converge absolutely and uniformly on $\mathbb{T}^n$

Summation of Fourier Series and Integrals

Thm 7.22 $\quad f$ periodic & absolutely cts on $\mathbb{R}$.

$f' \in L^p(\mathbb{T})$ $\quad p > 1$

$\Rightarrow \hat{f} \in \ell^1(\mathbb{Z})$

proof: By Lyapunov, $L^p(\mathbb{T}) \subseteq L^2(\mathbb{T})$ for $p \geq 2$

assume $p \le 2$.

$$\sum_k |\hat{f}(k)| \le \left[\sum (2\pi|k|)^{-p}\right]^{\frac{1}{p}} \left[\sum |2\pi k \hat{f}|^q\right]^{\frac{1}{q}}$$

$$= C_p \|\hat{f'}\|_q \overset{H\text{-}Y}{\le} C_p \|f'\|_p. \qquad \square$$

109. How to recover f from $\hat{f}$ with minimal hypotheses?

$\boxed{\text{lem 7.23}}$ $f, g \in L^2$ $(\hat{f}\hat{g})^\vee = f * g$

proof $\|\hat{f}\hat{g}\|_1 \le \|\hat{f}\|_2 \|\hat{g}\|_2 = \|f\|_2 \|g\|_2 < \infty$

$h(y) = \overline{g(x-y)}$ $\hat{h}(\xi) = \overline{\hat{g}(\xi)}\, e^{-2\pi i \xi \cdot x}$.

$f * g(x) = \int f(y) g(x-y) = \int f \bar{h} \xlongequal{\text{Unitary}} \int \hat{f} \bar{\hat{h}}$

$= (\hat{f}\hat{g})^\vee$ $\square$

$\boxed{\text{Thm 7.24}}$ $\Phi \in L^1 \cap C_0$ $\Phi(0) = 1$ $\phi = \Phi^\vee \in L^1$, $f \in L^1 + L^2$ for $t > 0$

$f^t(x) = \int \hat{f}(\xi) \Phi(t\xi) e^{2\pi i \xi \cdot x} d\xi$

a. If $f \in L^p$ $(1 \le p < \infty)$, then $f^t \in L^p$. $\|f^t - f\|_p \to 0$ as $t \to 0$

b. If f is bounded and uniformly continuous, then so is f^t and $f^t \to f$ uniformly as $t \to 0$

c. If $|\phi(x)| \le \frac{C}{(1+|x|)^{n+\varepsilon}}$ for some C, $\varepsilon > 0$. Then $f^t(x) \to f(x)$ for every point in Lebesgue point set of f.

proof: a. $f = f_1 + f_2$. $f_1 \in L^1$ $f_2 \in L^2$

$\cancel{\Phi(t\xi) e^{2\pi i \xi x}}$ $\hat{\phi_t}(\xi) = \Phi(t\xi)$

$\cancel{f^t(x) = (\hat{f}\hat{\phi_t})^\vee = f * \phi_t}$

$f_1^t(x) = \int \hat{f} \hat{\phi_t} e^{2\pi i \xi \cdot x} = \int \widehat{(f * \phi_t)} e^{2\pi i \xi \cdot x}$

Take note of some details.

$\begin{cases} \|f * \phi\|_1 \underset{\text{young}}{\le} \|f\|_1 \|\phi\|_1 \\ \|\hat{f}\hat{\phi}\|_1 \le \|\hat{f}\|_\infty \|\hat{\phi}\|_1 \end{cases}$ $\xlongequal{\text{Inversion}} f * \phi_t$.

$\Phi \in (L^1 \cap C_0) \subset (L^1 \cap L^2)$. ✓

$f_2^t(x) = \int (\hat{f_2}\hat{\phi_t})^\vee = f_2 * \phi_t(x)$

$f^t = f * \phi_t$ $\square$

~~Thm 7.24~~ ~~Suppose~~ Using Poisson Summation formula, we can obtain a similar ~~test~~ result for periodic function, but I'd omit it here.

Folland's book also lists some examples of good kernels, but these contents can be seen in my Fourier Analysis Note so I'd omit ~~them~~ them, too.

$\boxed{\text{Thm 7.26}}$ If $f \in BV(\mathbb{T})$, then $\lim\limits_{m\to\infty} S_m f(x) = \frac{1}{2}[f(x+) + f(x-)]$ for $\forall x$.

proof. WLOG. $x=0$. $\boxed{f \nearrow}$

$S_m f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_m(x)\,dx$

$S_m f := f * D_m$ D_m is even.

$D_m = \frac{\sin (2m+1)\pi x}{\sin \pi x}$

$S_m f(x) - \frac{1}{2}[f(0+) + f(0-)] = \int_0^{\frac{1}{2}} [f(x) - f(0+)] D_m(x)\,dx + \int_{-\frac{1}{2}}^{0} [f(x) - f(0-)] D_m(x)\,dx.$

It suffices to show $\int_0^{\frac{1}{2}} [f(x) - f(0+)] D_m(x)\,dx \to 0$ as $m \to \infty$

[why do $f(0+)$ $f(0-)$ exist? BV functions are difference of two ~~increasing~~ increasing function and increasing functions have countable jump point.

A more general proof is that, if right limit does not exist, then we can choose two sequences that have different limit points and it leads contradiction since $f \in BV$.

We can replace $f(t)$ by $f(t+)$ and ~~then~~ we can assume f is right cts

Since f is right continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$. $|f(x) - f(0)| < \varepsilon/c$, $x < \delta$

① $\int_0^{\delta} [f(x) - f(0+)] D_m\,dx \overset{\text{Second mean}}{=} [f(\delta) - f(0+)] \Big| \underbrace{\int_{\eta}^{\delta} D_m(x)\,dx}_{\le c} \Big| < \varepsilon$

Dirichlet kernel integral is bound wrt m and integral interval $\subseteq [-\frac{1}{2}, \frac{1}{2}]$.

② $\left| \int_{\delta}^{\frac{1}{2}} [f(x) - f(0+)] D_m\,dx \right| \to 0$ by R-L lemma $\square$

$\boxed{\text{Thm 7.27}}$ If $f, g \in L^1(\mathbb{T})$ $f = g$ on an open interval I, then $S_m f - S_m g \rightrightarrows 0$ on the compact subset of I.

proof: It suffices to show that if $f = 0$ on $I = (-c, c)$ $c < \frac{1}{2}$.

~~and f~~ then $S_m f \rightrightarrows 0$ on $[-\delta, \delta] \subseteq I$

111.

$$S_m f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y) D_m(y)\, dy$$

$$D_m(y) = \frac{\sin(2m+1)\pi y}{\sin \pi y} \qquad g_{x,\pm}(y) = \frac{f(x-y)\, e^{\pm \pi i y}}{2i \sin \pi y}$$

$$f(x-y) D_m(y) = \frac{f(x-y)}{\sin \pi y} \sin(2m+1)\pi y = \frac{f(x-y)}{\sin \pi y} \frac{e^{i(2m+1)\pi y} - e^{-i(2m+1)\pi y}}{2i}$$

① $\Rightarrow S_m f = \hat{g}_{x,+}(m) - \hat{g}_{x,-}(m) \to 0$ by R-L. $\quad x_1, x_2 \in [-\delta, \delta]$. $\cancel{|x_1 - x_2| < \delta}$

② $S_m f(x_1) - S_m f(x_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin(2m+1)\pi y}{\sin \pi y} [f(x_1 - y) - f(x_2 - y)]$ $\quad |y| < \varepsilon - \delta$ $\Rightarrow f(x_1 - y) - f(x_2 - y)$

$|\cdot| \leq \frac{1}{\sin(\varepsilon\delta)} \|\tau_{x_1} f - \tau_{x_2} f\|_1$ as $x_1 - x_2 \to 0$ $\quad$ equicts

③ $|x_1 - x_2| < \eta \Rightarrow |S_m f(x_1) - S_m f(x_2)| < \varepsilon/2$

$x_1, \cdots x_k$. $\quad |x_i - x_{i+1}| < \eta$ covering $[\delta, \delta]$.

$|S_m f(x)| \leq |S_m f(x) - S_m f(x_j)| + |S_m f(x_j)| < \varepsilon \quad \frac{m > M}{\text{finite } k}$.

$\boxed{\text{Rmk}}$ The three steps are standard. The first step is ~~that~~ using R-L lemma shows ~~that~~ pointwise convergence ~~to~~ 0. The second is ~~that~~ $\{S_m f\}$ is uniformly equicontinuous. The third is that, while AA thm tells us, we only need finite points ~~to~~ understand a family of functions' information.

Fourier Analysis ~~as~~ of measures.

I think its nontrivial but cannot introduce too much here.

$$L^1(\mu) \hookrightarrow M(\mathbb{R}^n)$$

$\mu, \nu \in M(\mathbb{R}^n)$ (complex Borel measures)

$$d(\mu \times \nu)(x,y) := \frac{d\mu(x)}{d|\mu|} \frac{d\nu(y)}{d|\nu|} d(|\mu| \times |\nu|)(x,y).$$

$$\mu * \nu(E) = \iint \chi_E(x+y)\, d\mu(x)\, d\nu(y)$$

$\boxed{\text{Thm 7.28}}$ a. Convolution of measures is associative and commutative

b. for any bounded Borel measurable function h

$$\int h \, d(\mu * \nu) = \iint h(x+y) \, d\mu(x) \, d\nu(y)$$

c. $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$

d. If $d\mu = f \, dm$ and $d\nu = g \, dm$, then $d(\mu * \nu) = (f * g) \, dm$

proof a. fubini

b. linearity + Approximation

c. $h = \dfrac{d|\mu * \nu|}{d(\mu * \nu)}$ $|h| = 1$

$$\|\mu * \nu\| = \iint h \, d(\mu * \nu) \overset{\text{def}}{\leq} \iint |h| \, d|\mu| \, d|\nu| \leq \|\mu\| \cdot \|\nu\|$$

$\|\cdot\| = \|\cdot\|_{TV}$

d. $$\int h \, d(\mu * \nu) = \iint h(x+y) f(x) g(y) \, d\mu \, d\nu$$
$$= \iint h(x) f(x-y) g(y) \, d\mu \, d\nu$$
$$= \int h(x) (f * g)(x) \, d\mu.$$

Since μ, ν are both complex measures ✓ $\square$

$$f * \mu(x) = \int f(x-y) \, d\mu(y)$$

$\boxed{\text{prop 7.30}}$ $f \in L^p$, $\mu \in M(\mathbb{R}^n)$ $\Rightarrow$ $f * \mu(x)$ $\exists$ a.e. $f * \mu \in L^p$

$$\|f * \mu\|_p \leq \|f\|_p \|\mu\|$$

proof $f, \mu \geq 0$

$$\|f * \mu\|_p = \left\| \int f(x-y) \, d\mu(y) \right\|_p \leq \int \|f(\cdot - y)\|_p \, d\mu(y)$$
$$= \|f\|_p \|\mu\| \qquad \square$$

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, d\mu(x). \Rightarrow \|\hat{\mu}\|_u \leq \|\mu\|_{TV}$$

$$\Rightarrow (\mu * \nu)^\wedge = \hat{\mu} \hat{\nu}.$$

Something in PDE

$\boxed{\text{Thm 7.31}}$ A differential operator L satisfies $L(f \circ T) = (Lf) \circ T$ iff $L = P(\Delta)$.

Chapter 8 Elements of Distribution Theory.

First we shall clarify that "what are the objects we are treating".

If $U \subseteq \mathbb{R}^n$ is an open set, $C_c^\infty(U) = \bigcup_{K \subset\subset U} C_c^\infty(K)$, where

$C_c^\infty(K)$ is a Frechet space with the topology defined by norms

$\phi \longmapsto \|\partial^\alpha \phi\|_u \quad (\alpha \in \{0,1,\cdots\}^n)$ in which a sequence $\{\phi_j\}$

converges to ϕ iff $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly for all α (its completeness can be proved by similar argument as "Schwartz space".)

$C_c^\infty(U)$ is a $\boxed{\text{inductive limit topology}}$ space. $\to$ embedding maps are cts.

- $T\phi : C_c^\infty(U) \longrightarrow X$ is cts iff $\phi T\big|_{C_c^\infty(K)} : C_c^\infty(K) \to X$ is cts.

  (Universality)

- $\{\phi_j\} \subseteq C_c^\infty(U)$ converges to ϕ if $\{\phi_j\} \subseteq C_c^\infty(K)$ for some compact set $K \subset\subset U$ & $\phi_j \to \phi$ in the topology $C_c^\infty(K)$

  ~~Maybe we can consider net convergence~~
  (referred to Yosida's Functional Analysis).

- $T\big|_{C_c^\infty(K)} : \underline{\underline{C_c^\infty(K)}} \to X$ is cts

  Frechet

  iff $T\phi_j \to T\phi$ whenever $\phi_j \to \phi$ in $C_c^\infty(K)$.

- A linear map $T: C_c^\infty(U) \to C_c^\infty(U')$ is cts if for each $K \subset\subset U$ there is a $K' \subset\subset U'$ s.t. $T(C_c^\infty(K)) \subseteq C_c^\infty(K')$ and T is cts from $C_c^\infty(K)$ to $C_c^\infty(K')$.

$\boxed{\text{Def 8.1}}$ A distribution on U is a $\boxed{\text{cts linear functional on } C_c^\infty(U)}$.
Let $\mathcal{D}'(U)$ denote all distributions on U. We impose the weak*-topology on $\mathcal{D}'(U)$.

Some ~~ex~~ examples:

- $f \in L^1_{loc}(U)$: $\int_K |f| < \infty$ for $\forall K \subset\subset U$.

$$\phi \longmapsto \int f \phi$$

- Every Radon measure μ on U defines a distribution $\phi \mapsto \int \phi \, d\mu$.

- If $x_0 \in U$, α is a multi-index, the map $\phi \mapsto \partial^\alpha \phi(x_0)$ is a distribution.

$$\boxed{L^1_{loc}(U) \hookrightarrow \mathcal{D}'(U)}$$

In order to avoid confusion, we adopt a different notation for the pairing between $\underset{\phi \in}{C_c^\infty(U)}$ and $\underset{F \in}{\mathcal{D}'(U)}$ $\rightsquigarrow \langle F, \phi \rangle$.

Folland says that we can sometimes pretend that distribution F is a function and write that $\langle F, \phi \rangle = \int F(x) \phi(x) \, dx$.

Special notations : (1) $\tilde{\phi}(x) = \phi(-x)$; (2) $\langle \delta, \phi \rangle = \phi(0)$.

$\boxed{\text{prop 8.2}}$ Suppose that $f \in L^1(\mathbb{R})$ and $\int f = a$ and for $t > 0$ let $f_t = \frac{1}{t^n} f(x/t)$. Then $f_t \to a\delta$ in $\mathcal{D}'$ as $t \to 0$.

proof $\langle f_t, \phi \rangle = \int f_t \phi = f_t * \tilde{\phi}(0) \to a \tilde{\phi}(0) = a \langle \delta, \phi \rangle$ $\square$

If $\phi \in C_c^\infty$

$\boxed{\text{Rmk}}$ It does make sense to say that two distributions agree on an open set $V \subseteq U$, i.e. $\boxed{F = G \text{ on } V \text{ iff } \langle F, \phi \rangle = \langle G, \phi \rangle} \; \forall \phi \in C_c^\infty(V)$.

$\boxed{\text{prop 8.3}}$ $\{V_\alpha\}$ is a collections of open subsets of V & $V = \bigcup_\alpha V_\alpha$. If $F, G \in \mathcal{D}'(U)$ and $F = G$ on each V_α, then $F = G$.

proof: If $\phi \in C_c^\infty(V) \Rightarrow \exists \alpha_1, \cdots \alpha_m$ s.t. $\operatorname{Supp} \phi \subseteq \bigcup_{i=1}^m V_{\alpha_i}$.

pou $\Rightarrow$ $\operatorname{supp} \psi_j \subseteq V_{\alpha_j}$, $\sum \psi_j = 1$.

$\langle F, \phi \rangle = \sum \langle F, \phi \psi_j \rangle = \sum \langle G, \phi \psi_j \rangle = \langle G, \phi \rangle$. $\square$

$F \in \mathcal{D}'(U)$ $\quad U' = \{F = 0\}$ $\quad U'^{c}$ is called the support of F.

We have a general procedure for extending linear operators to ~~distributions~~ functions

Formally, we can write $\langle TF, \phi \rangle = \langle F, T'\phi \rangle$

where $T: \text{subset of } L_{loc}(U) \to L_{loc}(V)$ ("The expression in Folland's book is confusing
$T': L_{loc}(V) \to L_{loc}(U)$ but I think it's actually linear algebra.)

After some search, the existence of T' is ~~not~~ universal enough, at least the case that we are interested in.

The continuity of T': I don't know how to deal with general cases.
But in the examples below, the continuity may be easy to see.

1. Differentiation. $Tf = \partial^\alpha f$ defined on $C^{|\alpha|}(U)$. If $\phi \in C_c^\infty(U)$

$\Rightarrow \int \partial^\alpha f \phi = (-1)^{|\alpha|} \int f (\partial^\alpha \phi)$

$\Rightarrow T' = (-1)^{|\alpha|} T \big|_{C_c^\infty(U)}$

We then define the $\boxed{\text{derivative } \partial^\alpha F \in \mathcal{D}'(U)}$ of any $F \in \mathcal{D}'(U)$ by

$$\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle.$$

2. Multiplication by Smooth Functions. Given $\psi \in C^\infty(U)$, define $Tf = \psi f$.

$T' = T\big|_{C_c^\infty(U)}$ $\Rightarrow$ "product" $\psi F \in \mathcal{D}'(U)$ for $F \in \mathcal{D}'(U)$

$$\langle \psi F, \phi \rangle = \langle F, \psi \phi \rangle$$

Moreover, if $\psi \in C_c^\infty(U)$, this formula makes sense. for $\forall \phi \in C_c^\infty(\mathbb{R})$.

3. Translation Given $y \in \mathbb{R}^n$. $T = \tau_y$

$$\int f(x-y) \phi(x)\, dx = \int f(x) \phi(x+y)\, dx$$

$\Rightarrow T' = \tau_{-y}\big|_{C_c^\infty(U+y)}$. For $F \in \mathcal{D}'(U)$ $\quad \tau_y F \in \mathcal{D}'(U+y)$

$$\langle \tau_y F, \phi \rangle = \langle F, \tau_{-y} \phi \rangle.$$

point mass at y is $\tau_y \delta$.

4. Composition with Linear Maps. Given an invertible linear transform S

$V = S^{-1}(U)$ $\quad Tf = f \circ S$

$\langle Tf, \phi \rangle = \int f \circ S \cdot \phi = \int f \; \phi \circ S^{-1} \; |\det S^{-1}|$

$T'\phi = |\det S|^{-1} \phi \circ S^{-1}$

So for $F \in \mathcal{D}'(U)$, we can define $F \circ S \in \mathcal{D}'(S^{-1}(U))$.

In ~~par~~particular, $\langle \tilde{F}, \phi \rangle = \langle F, \tilde{\phi} \rangle$.

5. Convolution. (1) Given $\psi \in C_c^\infty$.

$V = \{x : x - y \in U \text{ for } y \in \operatorname{supp} \psi\}$

$f \in L^1_{loc}(U)$

$f * \psi(x) = \int f(x-y)\psi(y)\,dy = \int f(y)\psi(x-y)\,dy$

$= \int f \, \boxed{\tau_x \tilde{\psi}}$

$\boxed{\tau_x \tilde{\psi}(y) = \tilde{\psi}(y-x) = \psi(x-y).}$

the notation is confusing . . .

is well defined for $\forall x \in V$

$F \in \mathcal{D}'(U)$ $\quad F * \psi(x) = \langle F, \tau_x \tilde{\psi} \rangle$

since $\tau_x \tilde{\psi} \to \tau_{x_0} \tilde{\psi}$ in C^∞ as $x \to x_0$

$\Rightarrow F * \psi$ is C^∞.

example: $\delta * \psi(x) = \langle \delta, \tau_x \tilde{\psi} \rangle = \tau_x \tilde{\psi}(0) = \psi(x)$.

δ is the multiplicative identity for convolution.

6. convolution (2) Let $\psi, \tilde{\psi}$ $\quad V = \{x : x - y \in U, \text{ for } y \in \operatorname{supp} \psi\}$

If $f \in L^1_{loc}(U)$ $\quad \phi \in C_c^\infty(V)$

$\int (f * \psi)\phi = \int\int f(y)\psi(x-y)\phi(x)\,dy\,dx$

$= \int f \,(\tilde{\psi} * \phi)$

$Tf = f * \psi, \Rightarrow T'\phi = \tilde{\psi} * \phi$

$F \in \mathcal{D}'(U)$

$\Rightarrow \langle F * \psi, \phi \rangle = \langle F, \tilde{\psi} * \phi \rangle$.

17 $\boxed{\text{prop 8.9}}$ $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $\psi \in C_c^\infty$. $V = \{x : x - y \in U \text{ for } y \in \operatorname{supp} \psi(y)\}$

For $F \in \mathcal{D}'(U)$ $x \in V$, Let $F * \psi(x) = \langle F, \tau_x \tilde{\psi} \rangle$. Then

a. $F * \psi \in C^\infty(V)$

b. $\partial^\alpha (F * \psi) = (\partial^\alpha F) * \psi = F * (\partial^\alpha \psi)$

c. For any $\phi \in C_c^\infty(U)$, $\int (F * \psi) \phi = \langle F, \phi * \tilde{\psi} \rangle$.

proof: $x \in V$ $\exists t_0$ $|t| < t_0$ $x + t e_j \in U$

$$t^{-1} (\tau_{x + t e_j} \tilde{\psi} - \tau_x \tilde{\psi}) \rightarrow \tau_x \partial_j \tilde{\psi}$$

$$\Rightarrow \partial_j (F * \psi) = \partial_j \langle F, \tau_x \tilde{\psi} \rangle$$

$$\overset{\|}{\lim_t} \frac{\langle F, \tau_{x + t e_j} \tilde{\psi} \rangle - \langle F, \tau_x \tilde{\psi} \rangle}{t} \underset{\text{linearity}}{=} \langle F, \tau_x \partial_j \tilde{\psi} \rangle$$

$$= \langle \partial_j F, \tau_x \psi \rangle.$$

By induction. $\boxed{\partial^\alpha (F * \psi) = F * \partial^\alpha \psi}$.

$\partial^\alpha \tilde{\psi} = (-1)^\alpha \widetilde{\partial^\alpha \psi}$, $\partial^\alpha \tau_x = \tau_x \partial^\alpha$

$\boxed{(\partial^\alpha F) * \psi(x)} = \langle \partial^\alpha F, \tau_x \tilde{\psi} \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \tau_x \tilde{\psi} \rangle$

$= (-1)^\alpha \langle F, \tau_x \partial^\alpha \tilde{\psi} \rangle = \langle F, \tau_x \widetilde{\partial^\alpha \psi} \rangle = \boxed{F * (\partial^\alpha \psi)}$.

$= \boxed{\partial^\alpha (F * \psi)}$

If $\phi \in C_c^\infty(U)$. $\phi * \tilde{\psi}(x) = \int \phi(y) \psi(y - x) dy = \int \phi(y) \tau_y \tilde{\psi}(x) dx$

supported in $\underset{\subseteq U}{\boxed{\text{cpt set}}}$ approximation by Riemann Sums

$$S^m = 2^{-nm} \sum \phi(y_j^m) \tau_{y_j^m} \tilde{\psi} \rightrightarrows \phi * \tilde{\psi}$$

likewise, $\partial^\alpha S^m \rightrightarrows \partial^\alpha (\phi * \tilde{\psi})$

$$\langle F, \phi * \tilde{\psi} \rangle = \lim_m \langle F, S^m \rangle = \lim_m \langle F, 2^{-nm} \sum \phi(y_j^m) \tau_{y_j^m} \tilde{\psi} \rangle$$

$$= \lim \sum 2^{-mn} \langle F, \phi(y_j^m) \tau_{y_j^m} \tilde{\psi} \rangle$$

$$= \int \phi(y) \langle F, \tau_y \tilde{\psi} \rangle = \int \phi (F * \psi) \qquad \square$$

How should we understand distributions which can be highly ~~sig~~ singular objects?

They can be approximated in the (weak*) topology by C_c^∞ functions.

$\boxed{\text{Thm 9.5}}$ For any open set $U \subseteq \mathbb{R}^n$, $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ in the topology of $\mathcal{D}'(U)$.

proof: "Idea: approximate first by $\mathcal{D}'(U)$ supported ~~by~~ in U, then approximate by $C_c^\infty(U)$."

1. $V_1 \subseteq \underbrace{\bar{V}_1}_{\text{cpt}} \subseteq V_2 \subseteq \underbrace{\bar{V}_2}_{\text{cpt}} \cdots \subseteq U$ | Exhaustion in LCH-T.

$\bigcup_i V_i = U$

By Urysohn lemma, $\zeta_j = 1$ on $\bar{V}_j$ $\zeta_j \in C_c^\infty(U)$

$\forall \phi \in C_c^\infty(U) \Rightarrow \exists j > N.\ \operatorname{supp}\phi \subseteq V_j$

$\Rightarrow \langle F, \phi \rangle = \langle F, \zeta_j \phi \rangle = \langle \zeta_j F, \phi \rangle$

$\Rightarrow \zeta_j F \to F$ as $j \to \infty$

2. For completeness, we state the lem in Folland's book which is quite obvious: $\phi \in C_c^\infty$, $\psi \in C_c^\infty$. $\int \psi = 1$ $\psi_t = \frac{1}{t^n}\psi(\frac{x}{t})$

a. $\operatorname{supp}\phi \subsetneq U$ open $\exists t \to 0$ $\operatorname{supp} \phi * \psi_t \subseteq U$

b. $\phi * \psi_t \to \phi$

$\zeta_j F$ should be regarded as a distribution since we have noted that ζ_j is C_c^∞ function.

$\phi \in C_c^\infty$ $\quad \phi * \tilde{\psi}_t \to \phi$

$\Rightarrow \boxed{(\zeta_j F) * \psi_t \in C_c^\infty}$ by $\boxed{\text{prop 9.4}}$

$\langle (\zeta_j F) * \psi_t, \phi \rangle = \langle \zeta_j F, \phi * \tilde{\psi}_t \rangle \to \langle \zeta_j F, \phi \rangle$

$\Rightarrow \underset{\overset{\uparrow}{C_c^\infty}}{(\zeta_j F) * \psi_t} \to \zeta_j F$ in weak* - topology

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3. Finally, it suffices to show $(\varphi_j F) * \psi_t$ is of compact support.

$\mathrm{supp}(\varphi_j) \subseteq V_k$ for some k.

If $\mathrm{supp}\,\phi \cap \bar{V}_k = \phi \Rightarrow (\phi * \psi_t) \cap \bar{V}_k = \phi$ for $t \to 0$

$\Rightarrow \langle (\varphi_j F) * \psi_t, \phi \rangle = \langle F, \varphi_j(\phi * \tilde{\psi}_t) \rangle = 0$

$\Rightarrow \mathrm{supp}\left((\varphi_j F) * \psi_t\right) \subseteq \bar{V}_k \subseteq U$ $\square$

By distribution theory, we can understand some derivative of functions which do not possess classical derivative.

1. " $\chi'_{(0,\infty)} = \delta$ "

2. $\partial_j \partial_k = \partial_k \partial_j$ on $\mathcal{D}'$

---

compactly supported, tempered, periodic

Our purpose, maybe my purpose, is to extend Fourier transform to a larger domain

$C^\infty(U)$ is a Frechet space with the C^∞ topology

$V_1 \subseteq \bar{V}_1 \subseteq V_2 \subseteq \cdots \subseteq U$ $\quad \bigcup_i V_i = U$

$\|f\|_{[m,\alpha]} := \sup_{x \in \bar{V}_m} |\partial^\alpha f(x)|$ consists a family of seminorms.

$\boxed{\text{prop 3.6}}$ $C_c^\infty(U)$ is dense in $C^\infty(U)$.

proof $\psi_m = 1$ on $\bar{V}_m$

$\Rightarrow \|\psi_m \phi - \phi\|_{[m_0,\alpha]} = 0$ for $m \geq m_0$ $\Rightarrow \psi_m \phi \xrightarrow{C^\infty} \phi$ $\square$

Let $\mathcal{E}'(U)$ denote all the distributions on U whose support is a compact subset of U.

$\boxed{\text{prop } 9.7}$ $\mathcal{E}'(U)$ is the dual space of $C^\infty(U)$. More precisely: If $F \in \mathcal{E}'(U)$, 120 then F uniquely extends to a continuous linear functional on $C^\infty(U)$; if G is a continuous linear functional on $C^\infty(U)$, then $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$.

proof. If $F \in \mathcal{E}'(U)$, choose $\underset{\in C^\infty}{\psi} = 1$ on $\mathrm{supp}(F)$. then we can define a linear funtional G on $C^\infty(U)$ by $\langle G, \phi\rangle = \langle F, \psi\phi\rangle$. Since F is continuous on $C_c^\infty \mathrm{supp}(\psi) \subseteq U$, and the topology of the latter is defined by $\phi \mapsto \|\partial^\alpha\phi\|_u$ $\Rightarrow \exists N. C$ s.t. $\underset{|\langle F, \phi\psi\rangle|}{\underset{\shortparallel}{|\langle G, \phi\rangle|}} \le C \sum_{|\alpha|\le N} \|\partial^\alpha \psi\phi\|_u$ for $\phi \in C^\infty(U)$.

$\Rightarrow |\langle G, \phi\rangle| \le C' \sum_{|\alpha|\le N} \|\phi\|_{[m,\alpha]}$ m is large enough.

$\Rightarrow G$ is continuous.

Since $C_c^\infty(U)$ is dense in $C^\infty(U)$

$G|_{C_c^\infty(U)} = F$ $\Rightarrow$ Uniqueness.

On the other hand, if $G \in \mathcal{E}'(U)$

$\Rightarrow |\langle G, \phi\rangle| \le \sum_{|\alpha|\le N} \|\phi\|_{(m,\alpha)}$ for all $\phi \in C^\infty(U)$

$= \sum_{|\alpha|\le N} \|\phi\|\partial^\alpha\phi\|_u$

$\Rightarrow G$ is cts on $C_c^\infty(K)$ $K \subset\subset U$. $\Rightarrow G|_{C_c^\infty(U)} \in \mathcal{D}'(U)$.

If $\mathrm{supp}\,\phi \cap \bar{V}_m = \phi$

$\Rightarrow \langle G, \phi\rangle = 0$ $\Rightarrow \mathrm{supp}\,G \subseteq \bar{V}_m$ $\Rightarrow G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$. $\square$

If $F \in \mathcal{E}'$, $\phi \in C_c^\infty$ $\Rightarrow F * \phi \in C_c^\infty$

$\phi \in C^\infty$ $\Rightarrow F * \phi \in C^\infty$?

Check: $F * \phi(x) = \langle F, \phi(x-\cdot)\rangle = \langle F\chi^\infty_{cut}, \phi(x-\cdot)\rangle$ well-defined.

$= \langle F, \underbrace{\phi(x-\cdot)\chi^\infty_{cut}}_{C_c^\infty}\rangle$ ✓

~~$\langle \overset{C^\infty}{F*\phi}, \overset{C_c^\infty}{\psi}\rangle = \langle (F*\phi)\chi^\infty_{cut}, \psi\rangle$~~

~~$= \int (F*\phi)(x)\psi(x)\,dx = \lim \sum 2^{-nm}(F*\phi)\psi$~~

~~$= \int_K (F*\phi)(x)\psi(x)\,dx$~~

$$\underset{\mathcal{S}'}{\underset{\cap}{\langle F * \underset{C^\infty}{\phi}}}, \underset{C_c^\infty}{\psi}\rangle \overset{?}{=} \langle F, \psi * \tilde{\phi}\rangle$$

$$\| \quad \int F(x)(\psi * \tilde{\phi})(x)\,dx$$

$$= \int_K F(x)(\psi * \tilde{\phi})(x)\,dx \underset{\text{of } *}{\overset{R_s}{=}} \sum 2^{-nm} \langle F_\phi, \cancel{\psi * \tilde{\phi}}\ \psi(y_j^m)\,\tau_{y_j^m}\tilde{\phi} \rangle$$

$$= \sum 2^{-nm} \psi(y_j^m) \langle F, \tau_{y_j}\tilde{\phi}\rangle$$

$$\langle F, \psi * \tilde{\phi}\rangle = \langle F, \chi(\psi * \tilde{\phi})\rangle \qquad \int_K \psi(y)\tilde{\phi}(x-y)\,dy$$

$$= \sum \langle F, \psi(y_j^m)\,\tau_x\tilde{\phi}(y_j^m)\rangle\, 2^{-nm}$$

$$= \sum \psi(y_j^m) \langle F, \tau_x \tilde{\phi}\rangle\, 2^{-nm}$$

$$= \int \psi(y)\,(F * \phi)(y)\,dy$$

$$= \langle F * \phi, \psi\rangle.$$

$\mathcal{F} : C_c^\infty \not\to C_c^\infty$

$\mathcal{S} \quad \|\phi\|_{(N,\alpha)} = \sup_x (1+|x|)^N |\partial^\alpha \phi(x)|$ defines its topology.

$\boxed{\text{prop 9.8}}$ $\psi \in C_c^\infty$, $\psi(0) = 1$, $\psi^\varepsilon(x) = \psi(\varepsilon x)$

$\Rightarrow \forall \phi \in \mathcal{S}$, $\psi^\varepsilon \phi \to \phi$ in $\mathcal{S}$ as $\varepsilon \to 0$

proof In K $\psi^\varepsilon \rightrightarrows 1$; $(1+|x|^N)|\phi| < \infty \quad \forall x$

$$(1+|x|^N)\partial^\alpha(\psi^\varepsilon\phi - \phi) = (1+|x|^N)(\psi^\varepsilon \partial^\alpha\phi - \partial^\alpha\phi) + E_\varepsilon(x)$$

$$|\partial^\beta \psi^\varepsilon(x)| = |\varepsilon^{|\beta|}\partial^\beta\psi| \le C_\beta\, \varepsilon^{|\beta|}$$

In and Out ✓ $\square$

$\boxed{\text{Rmk}}$ $C_c^\infty \overset{\text{dense}}{\subseteq} \mathcal{S}$.

$\boxed{\text{Def 9.9}}$ A tempered distribution is a cts linear functional on $\mathcal{S}$

Notation: $\mathcal{S}'$ topology : weak* - topology

By $\boxed{\text{prop 9.7}}$ We shall identify $\mathcal{S}'$ with the set of distributions that extend ctsly from C_c^∞ to $\mathcal{S}$.

Example: • $\mathcal{E}'$

• $f \in L^1_{loc}$ $\quad \int (1+|x|)^N |f| < \infty$ for some N

$\left|\int f\phi\right| < \infty$

• $f(x) = e^{iex}$

• $f(x) = e^x \cos x$

$\left|\int f(x)\phi(x)\right| = \left|-\int \phi'(x) \sin e^x \, dx\right|$
$\leq C \|\phi\|_{(2,1)}$.

$\boxed{\text{Def 9.10}}$ A function $\psi \in C^\infty$ is called slowly increasing if ψ and all its derivatives have at most polynomial growth at infinity:
$$|\partial^\alpha \psi(x)| \leq C_\alpha (1+|x|)^{N(\alpha)} \quad \text{for all } \alpha.$$

example: $(1+|x|^2)^s$ $\quad s \in \mathbb{R}$.

$\boxed{\text{Prop 9.11}}$ If $F \in \mathcal{S}'$ & $\psi \in \mathcal{S}$ $\quad F * \psi(x) := \langle F, \tau_x \tilde{\psi} \rangle$, then we have $\langle F*\psi, \phi \rangle = \langle F, \phi * \tilde{\psi} \rangle$ for $\forall \phi \in \mathcal{S}$

proof: $\partial^\alpha (F*\psi) \overset{cts}{=} F * (\partial^\alpha \psi) \Rightarrow F*\psi \in C^\infty$

slowly increasing: $|\langle F, \phi \rangle| \overset{cts}{\leq} C \sum_{|\alpha| \leq N} \|\phi\|_{(m,\alpha)}$ $\quad \forall \phi$

$$\Rightarrow |F*\psi(x)| \leq C \sum_{|\alpha| \leq N} \sup (1+|y|)^m |\partial^\alpha \phi(x-y)|$$
$$\leq C (1+|x|)^m \sum_{\alpha \leq N} \sup_y (1+|x-y|)^m |\partial^\alpha \phi(x-y)|$$
$$\leq C (1+|x|^m) \sum_{|\alpha| \leq N}^m \|\phi\|_{(m,\alpha)}$$

So does $\partial^\alpha \psi$.

$\int (F*\psi)\phi = \cancel{\int F *} \langle F, \phi * \tilde{\psi} \rangle$ holds for $\forall \phi, \psi \in C_c^\infty$

$C_c^\infty \overset{dense}{\subseteq} \mathcal{S}$ $\quad \phi_j \to \phi \quad \psi_j \to \psi$ in $\mathcal{S}$

$\Rightarrow \phi_j * \tilde{\psi}_j \to \phi * \tilde{\psi}$ in $\mathcal{S}$

$$\left.\begin{aligned} |F*\psi_j| &\leq C(1+|x|)^m \\ |\phi_j| &\leq C(1+|x|)^{-m-n-1} \end{aligned}\right\} \Rightarrow \int (F*\psi_j)\phi_j \xrightarrow{DCT} \int (F*\psi)\phi.$$

$\square$

$\Rightarrow \langle \hat{F}, \phi \rangle := \langle F, \hat{\phi} \rangle$

Using test functions, we can show that those basic properties of Fourier transform holding for tempered distribution

$$\langle \widehat{\tau_y F}, \phi \rangle = \langle \tau_y F, \hat{\phi} \rangle = \langle F, \tau_{-y}\hat{\phi} \rangle = \langle F, \widehat{e^{-2\pi i y x}\phi(x)} \rangle$$

$$= \langle \hat{F}, e^{-2\pi i y x}\phi(x) \rangle = \langle e^{-2\pi i \xi \cdot x}\hat{F}, \phi \rangle$$

$$\tau_y \hat{F} = (e^{2\pi i y \cdot x} F)^\wedge$$

$$\partial^\alpha \hat{F} = ((-2\pi i x)^\alpha F)^\wedge \qquad \widehat{\partial^\alpha F} = (2\pi i \xi)^\alpha \hat{F}$$

$$\cancel{(f * T)^\wedge =}$$

$$(F * \psi)^\wedge = \hat{\psi}\hat{F} \quad \underline{(\psi \in \mathcal{S})}$$

If we define its inverse by $\langle \check{F}, \phi \rangle = \langle F, \check{\phi} \rangle$, then we shall see

$$\langle (\hat{F})^\vee, \phi \rangle = \cdots = \langle F, \phi \rangle$$

Thus the Fourier ~~tra~~ transform is an isomorphism on $\mathcal{S}'$.

There is an alternative way to define $\hat{F}$

$\boxed{\text{Prop 9.12}}$ If $F \in \mathcal{E}'$, then $\hat{F}$ is a slowly increasing C^∞ function and it's given by $\hat{F}(\xi) = \langle F, E_{-\xi} \rangle$ where $E_\xi(x) = e^{2\pi i \xi \cdot x}$.

proof: Let $g(\xi) = \langle F, E_{-\xi} \rangle$. since E_ξ is smooth, by standard argument, we can show $g(\xi) \in C^\infty$ and $\partial^\alpha g(\xi) = (-2\pi i)^{|\alpha|} \langle F, x^\alpha E_{-\xi} \rangle$.

$$|\partial^\alpha g(\xi)| \overset{\text{continuity}}{\leq} C \sum_{|\beta| \leq N} \sup_{\text{supp cc}\ldots} \left(\partial^\beta x^\alpha E_{-\xi}(x) \right) \lesssim (1 + |\xi|)^N.$$

It ~~suffices~~ remains ~~to~~ show $g = \hat{F}$.