最大模原理与Schwarz引理

最大模原理. 设f在区域D中全纯，不是常值函数，则$|f(z)|$不可能在Ω中取到最大值

引理(平均值) 若$f(z)$在$|z-a|<R$内全纯，则有

$$f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta \quad 0<r<R.$$

pf. $$f(a)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-a}\,d\xi = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}\cdot rie^{i\theta}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

Pf of thm. 设 $\sup\limits_{\Omega}|f|=M<\infty$，用连通性论证

令 $\Omega_1=\{|f|=M\}$ ←闭 $\quad \Omega_2=\{|f|<M\}$

下证 Ω_1 开. 设 $a\in\Omega_1\subseteq\Omega \Rightarrow \exists B(a,\delta)$ s.t. $B(a,\delta)\subseteq\Omega$

由平均值公式 $\Rightarrow |f(a)|=\left|\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta\right|=M \quad r\in(0,\delta)$

$\Rightarrow M=|f(a+re^{i\theta})| \Rightarrow B(a,\delta)\subset\Omega_1$

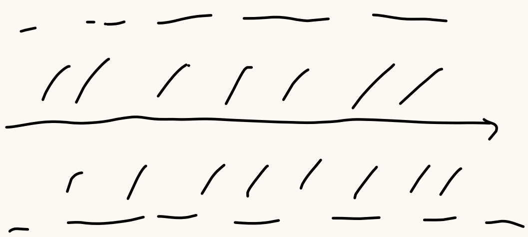
若$\Omega_1\neq\phi \Rightarrow \Omega_1=\Omega$ 矛盾 $\Rightarrow \Omega_1=\phi$ □

推论. 设Ω有界区域. $f(z)$在Ω上全纯, $\bar{\Omega}$上连续. 则

$$|f(z)|\le\max_{\partial\Omega}|f(z)| \quad \forall z\in\Omega$$

"=" 成立 iff $f(z)\equiv$ const.

若Ω无界, 例 e^{e^z} 在 $\{-\frac{\pi}{2}<\operatorname{Im} z<\frac{\pi}{2}\}$



$$\left|e^{e^z}\right|=\left|e^{e^x\cdot e^{iy}}\right|=\left|e^{e^x\cos y+ie^x\sin y}\right|=e^{e^x\cos y}$$

也没有"最小值原理", 如 $f(z)=z$, 但 若 $f\neq0$ 在Ω上, 可以处理 $\frac{1}{f(z)}$ 得到所谓的"最小值原理".

无界区域的最大模原理

设 $F(z)$ 在 $G=\{0<\operatorname{Im} z<1\}$ 上全纯,有界,且在 $\bar{G}$ 上连续,若 $\sup_{\partial G}|F|\le 1$,则 $|F(z)|\le 1$ $\forall z\in G$

Pf. 思路是造一个 F_ε,使

① $|F_\varepsilon|\le 1$ on ∂G

② $|F_\varepsilon|\to 0$ as $z\to\infty$

③ $F_\varepsilon\to F$ as $\varepsilon\to 0$

令 $F_\varepsilon(z)=F(z)e^{-\varepsilon z^2}$. 当 $\operatorname{Im} z=0$ $z=x$ $|F(z)e^{-\varepsilon x^2}|\le 1$

当 $\operatorname{Im} z=1$ $z=x+i$ $|F(z)|e^{-\varepsilon(x^2-1)}\le\begin{cases} e^{\varepsilon} & |x|<1 \\ 1 & |x|\ge 1\end{cases}$

令 ε 充分小即可

若 $0\le \operatorname{Im} z\le 1$, 令 $z=x+iy$ $x\in\mathbb{R}$ $y\in[0,1]$

$$|F_\varepsilon(z)|=|F(z)|\cdot e^{-\varepsilon(x^2-y^2)}\le Me^{-\varepsilon x^2+\varepsilon}\to 0 \quad \text{as } |x|\to\infty$$

so $\exists R>0$ 当 $|x|>R$. $|F_\varepsilon(z)|<\frac{1}{2}$

最大值点 (x,y) 落在 $[-R,R]\times[0,1]$ 的有界区域

$\Rightarrow |F_\varepsilon|\le 1$ for $\forall z\in G$, $\varepsilon>0$

$\Rightarrow |F|\le 1$ $\forall$ $z\in G$ $\square$

Rmk. 另一个可行的"辅助函数"是 $F_\varepsilon=F\dfrac{1}{1-i\varepsilon z}$.

调和函数的极值原理: $u(x,y)$ 在区域 D 中调和,且 $u\not\equiv$ const. 则 u 在 D 中的最值不在内点达到

例. f 在 $\Omega=\{0<r_1<|z|<r_2\}$ 内全纯, $\bar{\Omega}$ 上连续 令 $M=\max_{|z|=r}|f(z)|$, 则

$$\log M(r)\le\frac{\log r_2-\log r}{\log r_2-\log r_1}\log M(r_1)+\frac{\log r-\log r_1}{\log r_2-\log r_1}\log M(r_2)$$

(Hadamard 三圆定理)

Pf. $\forall\alpha\in\mathbb{R}$. 在 $\{f\ne 0\}$ 上有 $g(z)=\alpha\log|z|+\log|f|$ 调和

在 $\{f=0\}$ 上 $g=-\infty$ 达不到最大值

$\Rightarrow g \le \max\{\max\limits_{|z|=r_1} g, \max\limits_{|z|=r_2} g\} \Rightarrow g \le \max\{\alpha\log r_1 + \log M(r_1), \alpha\log r_2 + \log M(r_2)\}$

令 $\alpha\log r_1 + \log M(r_1) = \alpha\log r_2 + \log M(r_2)$ □

Schwarz引理. 设 $f: D \to D$ 全纯, $f(0)=0$ 则

1. $|f(z)| \le |z|$
2. 若对某 $z_0 \ne 0$. $|f(z_0)| = |z_0|$, 则 $f(z) = e^{i\theta}z$
3. $|f'(0)| \le 1$. 等号成立当 $f(z) = e^{i\theta}z$.

Pf. 1. $f(0)=0$, 全纯 $\Rightarrow f(z) = a_1 z + \cdots + a_n z^n + \cdots$

令 $\varphi(z) = \begin{cases} \frac{f(z)}{z} & z \ne 0 \\ a_0 & z = 0 \end{cases}$ $\varphi(z)$ 全纯

$\Rightarrow |\varphi| \le \max\limits_{|z|=r} |\varphi|$ $z \in B(0,r)$ $r \in (0,1)$

$|\varphi(z)| = \left|\frac{f(z)}{z}\right| \le \max\limits_{|z|=r} \frac{|f(z)|}{r} \le \frac{1}{r}$ $\forall z \in B(0,r)$

$\xRightarrow{r\to 1}$ $\varphi(z) \le 1$ $\forall z \in B(0,1)$ $\Rightarrow |f(z)| \le |z|$

2. $\varphi(z)$ 在内点达到最大模 $\Rightarrow$ $\varphi(z) = \text{Const}$

3. $|\varphi(0)| \le 1$. $|f'(0)| = |a_1| = |\varphi(0)| \le 1$ □

定义. ① $f: U \to V$ 是全纯双射, 则称 f 为共形映射
此时 U, V 称为共形等价/双全纯等价
② 区域 $\Omega \to \Omega$ 的共形映射 称为 Ω 的自同构
$Aut(\Omega)$ 表示所有 Ω 的自同构集合

这个定义缺少了一些结果的支撑, 不是很好的.

Riemann 曲面中只有 $\mathbb{D}$、$\mathbb{C}$、$\overline{\mathbb{C}}$ 在全纯等价意义下

定理. $Aut(\mathbb{C}) = \{az+b \mid a \ne 0\}$
$Aut(\overline{\mathbb{C}}) = \{$分式线性变换$\}$

Pf. $Aut(\mathbb{C})$: f 为 $\mathbb{C}$ 上的整函数. 考虑 ∞.

∞ 可去: f 为常数
∞ 极点: f 为多项式 $\xrightarrow{\text{单射}}$ 一次多项式
∞ 本性. $f(z_n) \to A$ $z_n = f^{-1}(f(z_n)) \to f^{-1}(A) \ne \infty$

$\mathrm{Aut}(\overline{\mathbb{C}})$ $\quad f(\infty)=\infty$. 一次多项式

$f(\infty)=a$. 令 $\varphi: a\to\infty$ 为分式线性变换

i.e. $\varphi=\frac{1}{z-a}$ $\Rightarrow$ $\overline{\mathbb{C}}\xrightarrow{f}\overline{\mathbb{C}}\xrightarrow{\varphi}\overline{\mathbb{C}}$

$\infty\to a\to\infty$

$$\Rightarrow \frac{1}{f-a}=cz+d \Rightarrow f=a+\frac{1}{cz+d}.$$

定理. $f\in\mathrm{Aut}(\mathbb{D})$, $\exists\,\theta\in\mathbb{R}$. $\alpha\in\mathbb{D}$, $f(z)=e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ $\quad$ 感觉这个之前算过.

Pf: 令 $f(0)=\alpha$. 有 $\varphi: \alpha\mapsto 0$ $\quad \frac{z-\alpha}{1-\bar{\alpha}z}$

$0\xmapsto{f}\alpha\xmapsto{\varphi}0$ $\quad g=\varphi\circ f$

$\overset{\text{Schwarz}}{\Rightarrow}|g'(0)|\leq 1$ $\quad h=g^{-1}\Rightarrow|h'(0)|\leq 1$

$\Rightarrow|g'(0)|=|h'(0)|=1 \overset{\text{Schwarz}}{\Rightarrow} g=e^{i\theta}z$

$f=\varphi^{-1}\circ g$ $\quad\cdots$ $\quad\square$

例. $\mathrm{Aut}(\mathbb{H})$ $\quad$ 思路

$$\begin{array}{ccc}\mathbb{H} & \xrightarrow{f} & \mathbb{H}\\ \varphi\downarrow & & \downarrow\varphi\\ \mathbb{D} & \xrightarrow[\tilde{f}]{} & \mathbb{D}\end{array} \qquad \varphi^{-1}\circ\tilde{f}\circ\varphi=f \qquad \mathrm{Aut}(\mathbb{H})\simeq\mathrm{Aut}(\mathbb{D})$$

不具体计算.

例. $w=f(z):\mathbb{D}\to\mathbb{D}$ 全纯, 则 $\quad$ (Hint: use Schwarz lem)

$$\begin{cases}\left|\dfrac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right|\leq\left|\dfrac{z-z_0}{1-\bar{z}_0z}\right|\\[2ex] \dfrac{|f'(z)|^2}{(1-|f(z)|^2)^2}\leq\dfrac{1}{(1-|z|^2)^2}\end{cases}$$

Pf. (1)

$$\begin{array}{ccc}z_0 & \xrightarrow{f} & f(z_0)\\ \varphi_{z_0}\downarrow & & \downarrow\varphi_{f(z_0)}\\ 0 & \xrightarrow[\tilde{f}]{} & 0\end{array} \Rightarrow f=\varphi^{-1}_{f(z_0)}\circ\tilde{f}\circ\varphi_{z_0}$$

$$\Rightarrow \begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "\varphi_{z_0}"'] & \mathbb{D} \arrow[d, "\varphi_{f(z_0)}"] \\ \mathbb{D} \arrow[r, "\bar{f}"] & \mathbb{D} \end{tikzcd}$$

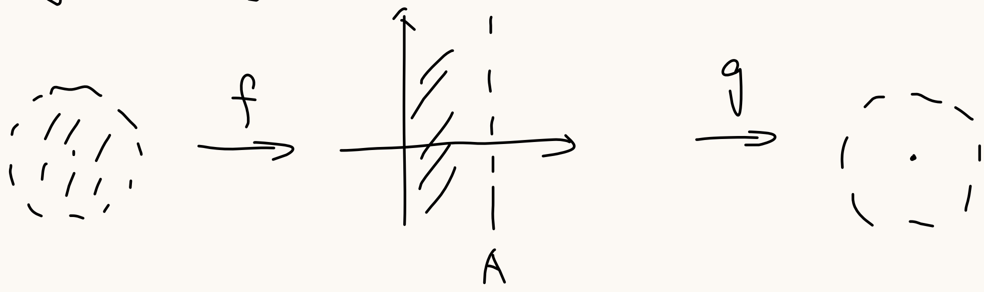
令 $\varphi_a = \dfrac{z-a}{1-\bar{a}z}$

$\Rightarrow$ 原式为 $\left|\varphi_{f(z_0)} f(z)\right| \le \left|\varphi_{z_0}(z)\right|$

那么由 $\left|\varphi_{f(z_0)} \circ f \circ \varphi_{z_0}^{-1}\right| \le |\mathrm{Id}|$ 令 $\varphi_{z_0}^{-1}(w) = z$ 即可。

(2) 显然 $\square$

例. $f(0)=0$. f在$\mathbb{D}$上全纯. $\mathrm{Re} f \le A$ $(A>0)$. 则 $|f(z)| \le \dfrac{2A|z|}{1-|z|}$ $\forall z \in \mathbb{D}$



$0 \mapsto 0$

$2A \mapsto \infty$. $\quad g = \dfrac{z}{z-2A}$

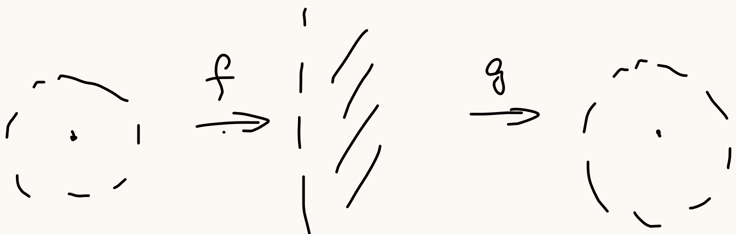
Schwarz

$\Rightarrow |g \circ f(z)| \le |z|$ i.e. $g \circ f(z) = \dfrac{f(z)}{f(z)-2A} = t$ $\quad |t| \le |z|$

$f(z) = \dfrac{2At}{1-t}$ $\Rightarrow |f(z)| \le \dfrac{2A|z|}{1-|z|}$. $\square$

例. f 在 $\mathbb{D}$ 内全纯 $f(0)=1$, $\mathrm{Re} f > 0$, 则 $\dfrac{1-|z|}{1+|z|} \le |f(z)| \le \dfrac{1+|z|}{1-|z|}$ | 有点像 Harnack...

Pf.



$g(z) = \dfrac{z-1}{z+1}$

$0 \mapsto 1$ $\quad 1 \mapsto 0$

$-1 \mapsto \infty$

$\Rightarrow g \circ f(0) = 0 \Rightarrow |g \circ f| \le |\mathrm{Id}|$ 令 $\dfrac{f(z)-1}{f(z)+1} = t$ $\quad |t| \le |z|$

$$f(z) = \frac{1+t}{1-t} \qquad |f(z)| = \left|\frac{1+t}{1-t}\right| \ge \frac{1-|t|}{1+|t|} \ge \frac{1-|z|}{1+|z|}$$

$$\le \frac{1+|t|}{1-|t|} \le \frac{1+|z|}{1-|z|} \qquad \square$$

上面基本就是利用全纯映射把函数改写为 Schwarz 引理成立的情形.

例、f 在 $|z|<R$ 上全纯，在 $|z|\le R$ 上连续、设

$M(r)=\max\limits_{|z|=r}|f|$，$A(r)=\max\limits_{|z|=r}\operatorname{Re}f$，则 $M(r)\le\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|$ $\forall r\in(0,R)$

Pf. ① f 常数 c. $|c|\le\frac{2r}{R-r}\operatorname{Re}c+\frac{R+r}{R-r}|c|$ 显然.

② $f(0)=0$ 由之前例题. $|f(z)|\le\frac{2A|z|}{1-|z|}$. $|z|<1$

$F(z)=f(Rz)\Rightarrow|F(z)|\le\frac{2A|z|}{1-|z|}$

$\Rightarrow|f(Rz)|\le\frac{2A|Rz|}{R-|Rz|}\Rightarrow f(u)\le\frac{2A|u|}{R-|u|}\le\frac{2A(R)|u|}{R-|u|}$

此即右式.

③ $g(z)=f(z)-f(0)$

$\Rightarrow\max\limits_{|z|=r}|g|\le\frac{2r}{R-r}(A(R)-f(0))$

$\quad\quad\ \ \lor|$

$\max\limits_{|z|=r}|f|-f(0)$

□

例、设 f 在 $\mathbb{C}\setminus\{0\}$ 上全纯，0 和 ∞ 都是本性奇点. 则 $\forall\ 0<r<\infty$

$\lim\limits_{r\to\infty}\frac{\log A(r)}{\log r}=\infty$ $\quad\lim\limits_{r\to 0}\frac{\log A(r)}{\log\frac{1}{r}}=\infty$

Pf. 两式等价，只证其一.

$f=\sum\limits_{-\infty}^{-1}a_nz^n+\sum\limits_{0}^{\infty}a_nz^n=g+h$

$\lim\limits_{r\to\infty}g=0$ so $|z|$ 充分大. $|g|<\varepsilon$

$A(r)\ge A_h(r)-\varepsilon$ 当 $|z|$ 充分大

由于 ∞ 是本性奇点、$M(r)\to\infty$ as $r\to\infty$ □

由 $M(r)\le\frac{2R}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|$

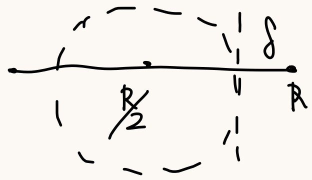
$\Rightarrow A(R)\ge\frac{R-r}{2R}M(r)-\frac{R+r}{2R}|f(0)|$ $\quad R>r>0$

令 $r=\frac{R}{2}\Rightarrow A(R)\ge\frac{1}{2}M(\frac{R}{2})-\frac{3}{2}|f(0)|>0$ 当 R 充分大

现在要估计 $M_h(\frac{R}{2})$ 的阶，注，由最大模原理. $M_h(\frac{R}{2})$ 关于 $R\nearrow$

时 ∞ 为本性奇点 $\Rightarrow\infty$ 是 $h^{(n)}$ 的本性奇点

令 $M_n(R) = \max\limits_{|z|=R} |h^{(n)}(z)|$



$$|z|=\frac{R}{2}\quad |h^{(n)}(z)| = \left|\frac{n!}{2\pi i}\int_\gamma \frac{h(\xi)}{(\xi-z)^{n+1}}\,d\xi\right|$$

$$\le \frac{n!}{2\pi} M(R-\delta)\int_\gamma \frac{1}{(\frac{R}{2}-\delta)^{n+1}}\,d\xi$$

$$= \frac{n!\,M(R-\delta)}{(R/2-\delta)^n}$$

$$\Rightarrow M_n(R/2) \le \frac{n!}{(R/2-\delta)^n} M(R-\delta) \le \frac{n!}{(R/2-\delta)^{n+1}}\left(\frac{2R}{\delta}A(R) + \frac{2R-\delta}{\delta}|f(0)|\right)$$

$$\le \frac{2R\cdot n!}{(R/2-\delta)^n\,\delta}\left(A(R)+|f(0)|\right)$$

$$\delta=\frac{R}{4} \Rightarrow M_n(R/2) \le \frac{2R\cdot n!}{(\frac{R}{4})^{n+1}}\left(A(R)+|f(0)|\right)$$

$$= \frac{2^{2n+3}\cdot n!}{R^n}\left(A(R)+|f(0)|\right)$$

$$\Rightarrow A(R) \ge \frac{R^n}{2^{2n+3}\,n!} M_n(R/2) - |f(0)|$$

$$\frac{\log A(R)}{\log R} = \frac{n\log R - (2n+3)\log 2 - \log n! + \log\left(M_n(R/2) - |f(0)|\right)}{\log R}$$

as $r\to\infty$. $\frac{\log A(R)}{\log R} \ge n$. let $n\to\infty$. □

最后的证明是对的，但是感觉有点粗糙...

另证. 只要有 $\forall n$. $\exists r_n$，当 $r>r_n$ 有 $M(r)\ge r^n$ 即可.

令 $g=\frac{f(z)}{z^n}$ 用 f 的展开.（∞ 是本性奇点）

可找 r_n（随便一点，或者一列，要连续化，这一步只要最大模）

↳ 使 $\max\limits_{r_n}|g| \ge M+1$

现在 $r>r_n$ 由最大模 $\max\limits_{r}|g| \ge M+1$ □

感觉这个证明更自然.

辐角原理.

设 0是 f 的极点或全纯, 则 $f = a_m z^m + a_{m+1} z^{m+1} + \cdots \quad m \in \mathbb{Z}$ 联想亚纯

极点 $m<0$. 全纯 $m \geq 1$

定义. 记上述 $m = \mathrm{Ord}_0 f$ 一般记 $\mathrm{Ord}_{z_0} f = m$

在 $B(0,\delta)$ 中. $f(z) = a_m z^m (1+h(z))$, 那么 $h(z)$ 在 $B(0,\delta)$ 中全纯 (δ 取小一些)

$$f'(z) = a_m m z^{m-1}(1+h(z)) + a_m z^m h'(z)$$

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \underbrace{\boxed{\frac{h'(z)}{1+h(z)}}}_{\text{全纯}}$$

$$\Rightarrow \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{f'(z)}{f(z)}\,dz = m = \mathrm{Ord}_0 f$$ 这一观察归结为以下结果

定理. 若 f 在 D 内亚纯, γ 是 D 内简单闭曲线. f 在 γ 上无零点, 无极点. 则

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z\text{在}\gamma\text{内}} \mathrm{Ord}_z f = \gamma\text{内零点重数之和} - \gamma\text{内极点阶数之和}$$

$$= N(f,\gamma) - P(f,\gamma)$$

Pf. 若 z_0 不是零点/极点 $\Rightarrow \frac{f'}{f}$ 在 z_0 处全纯.

设 γ 内的零点 极点为 $\{z_1, \cdots z_n\}$, 则 Cauchy 定理

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^n \frac{1}{2\pi i}\int_{|z-z_i|=\varepsilon} \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^n \mathrm{Ord}_{z_i} f \qquad \square$$

绕数 设 γ 为闭曲线(未必简单). 设 $\gamma = [a,b] \to \mathbb{C}$. $z(t) = \rho(t)e^{i\theta(t)}$

$z(a) = z(b)$, 则

$$\int_\gamma \frac{dz}{z} = \int_a^b \frac{1}{\rho(t)e^{i\theta(t)}}\cdot\left(\rho'(t)e^{i\theta(t)} + \rho(t)\cdot i\theta'(t)e^{i\theta(t)}\right)dt$$

$$= \int_a^b \left(\frac{\rho'(t)}{\rho(t)} + i\theta'(t)\right)dt = \log\rho(b) - \log\rho(a) + i\Delta_\gamma \mathrm{Arg}\, z$$

$$= i\Delta_\gamma \mathrm{Arg}\, z$$

then $\frac{1}{2\pi i}\int \frac{dz}{z} = \frac{1}{2\pi}\Delta \mathrm{Arg}\, z$ = 曲线 γ 绕原点的圈数

$$\Rightarrow N(f,\gamma) - P(f,\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$ 辐角原理

例. $f(z)=(z-1)(z-2)^3(z-4)$ $\gamma: |z|=3$

$N=4,\ P=0$

$\Delta_\gamma \operatorname{Arg} f(z) = \Delta_\gamma (z-1) + 3\Delta_\gamma (z-2) + \Delta_\gamma (z-4) = 8\pi$

定理(Rouché) f, g 在D内全纯，γ 可求长简单闭曲线，γ上 $|g|<|f|$

则 f 和 $f\pm g$ 在 γ 内部零点个数相同.

Pf. f, $f\pm g$ 在 γ 上无零点

$N(f,\gamma) = \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} f$

$N(f\pm g,\gamma) = \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} (f\pm g) = \frac{1}{2\pi}\left(\Delta_\gamma \operatorname{Arg} f \pm \Delta_\gamma \operatorname{Arg}\left(1+\frac{g}{f}\right)\right)$

只要证 $\Delta_\gamma \operatorname{Arg}\left(1+\frac{g}{f}\right) = 0$

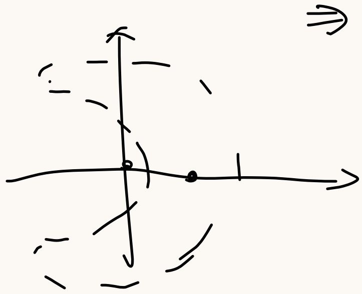
因 $\left|\frac{g}{f}\right|<1 \Rightarrow \left|\left(1+\frac{g}{f}\right)-1\right|<1$ on γ 在 0 外 $\square$

定理. $f, g \in H(D)$. γ上. $|g| < |f| + |f+g|$. 则 f, $f+g$ 在 γ内部有相同零点个数

Pf. 只要证 $\Delta_\gamma \operatorname{Arg}\left(1+\frac{g}{f}\right) = 0$

令 $w = 1+\frac{g}{f}$ $\Rightarrow$ γ上: $|(w-1)f| < |f| + |fw|$

$\Rightarrow$ γ上: $|w-1| < 1 + |w|$



到 $w \in \mathbb{C}\setminus(-\infty, 0]$

$\Rightarrow \Delta_\gamma \operatorname{Arg} w = 0$ $\square$

总结: 零点个数相同, 只要 $\begin{cases} |f\pm g| < |f| \\ |f\pm g| < |f| + |g| \end{cases}$ $\forall z\in\gamma$.

例. $z^4-6z+3=0$ 在 $|z|<1$ 与 $1<|z|<2$.

令 $f = z^4-6z+3$ $\quad |f-g| \le 4 < 6 = |g|$

$g = -6z$

$\Rightarrow$ f 与 g 在 $|z|<1$ 内 $\cdots$ $\Rightarrow$ 1个零点

$f = z^4-6z+3$ $\quad g = z^4$ $\quad |f-g| = |6z+3| \le 15 < 16 = |g|$

$\Rightarrow$ f 与 g 在 $|z|<2 \cdots$ $\quad 1<|z|<2$ 3个. $\square$

例、代数基本定理.

$p(z)=a_n z^n+\cdots+a_1 z+a_0 \quad (a_n\neq 0)$

$f=p.\quad g=a_n z^n$

$|f-g|=|a_{n-1}z^{n-1}+\cdots+a_0|<|g|$ when $|z|$ is big enough

$\Rightarrow f$ 与 g 在 $|z|<R$ 中零点个数相同.

$|f|\geq|g|-|f-g|>0$ when $|z|>R$.

$\Rightarrow n$ 个零点. $\square$

定理. 开映射定理 非常数的全纯函数是开映射

Topo version.
连续单射 $f: U\to\mathbb{R}^n$, $U\subset\mathbb{R}^n$

Pf. 引理. $f\in H(D)$ $z_0\in D$, $w_0=f(z_0)$

若 z_0 是 $f(z)-w_0$ 的 m 阶零点, 那么对充分小的 $\rho>0$ 有 $\delta>0$, 使 $\forall a\in B(w_0,\delta)$, $f(z)-a$ 在 $B(z_0,\rho)$ 中恰有 m 个零点.

这个引理可以稍微加强使 m 个零点不同 (f' 在去心邻域上 $\neq 0$.)

证. 由零点孤立性. $\exists\rho>0$. $f(z)-w_0$ 在 $\overline{B(z_0,\rho)}$ 中仅有 z_0 为零点

记 $\min\{|f(z)-w_0|: |z-z_0|=\rho\}=\delta>0$

当 $z\in\{|z-z_0|=\rho\}$ $\underbrace{|f(z)-w_0|}_{f}\geq\delta>|w_0-a|$

$|w_0-a| = |\underbrace{(f(z)-w_0)}_{f}-\underbrace{(f(z)-a)}_{g}|$

$\Rightarrow f(z)-w_0$ 与 $f(z)-a$ 在 $B(z_0,\rho)$ 中有 m 个零点 $\square$

$\Rightarrow f(B(z_0,\rho))\supseteq B(w_0,\delta)$ 内点到内点 $\square$

用开映射定理证明最大模原理.

设 f 在 Ω 中达到最大模 设为点 z_0 $\Rightarrow\exists\rho>0.\ \delta>0$

$f(B(z_0,\rho))\supseteq B(f(z_0),\delta)$ 但 $B(f(z_0),\delta)$ 中有点模大于 $f(z_0)$ 之.

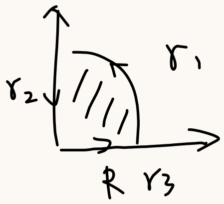
例 $z^4+2z^3-2z+10=0$ 在每一象限内恰一根.

$$\begin{cases} P(z)=(x^2-1)(x+1)^2+11>0 & z\in\mathbb{R} \\ P(z)=P(iy)=y^4+10-2iy(y^2+1)\neq 0 & z\in i\mathbb{R} \end{cases}$$

排除轴上零点

无实/纯虚根

只要证 $P(z)$ 在第一象限仅一根.



$$\gamma = \gamma_1 + \gamma_2 + \gamma_3$$

$$N = \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} P(z)$$

$$\Delta_{\gamma_3} \operatorname{Arg} P(z) = 0$$

$$\Delta_{\gamma_1} \operatorname{Arg} P(z) = \Delta_{\gamma_1} \operatorname{Arg} z^4\left(1 + \frac{2z^3 - 2z + 10}{z^4}\right)$$

$$= 2\pi + \varepsilon(R) \qquad \varepsilon \to 0 \quad \text{as} \quad R \to \infty$$

$$\Delta_{\gamma_2} \operatorname{Arg} P(z) = \operatorname{Arg} P(0) - \operatorname{Arg}\left(\left(1 - 2i\frac{R(R^2+1)}{R^4+10}\right)\right)$$

$$\to 0$$

$$\Rightarrow N = 1.$$

$\square$

例. $0 < a_0 < \cdots < a_n$. 求证: $a_0 + a_1\cos\theta + \cdots + a_n\cos n\theta$ 在 $[0, 2\pi)$ 中有 n 个不同零点

pf. 先证 $a_0 + a_1 z + \cdots + a_n z^n$ 在 $|z|<1$ 内 n 个根. 首先零点不是非负实数.

$$(1-z)(a_0 + \cdots + a_n z^n) = a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} \Rightarrow$$

设 z_0 为 $P_n(z)$ 零点. $|z_0| \geq 1$

$$\Rightarrow |a_n z_0^{n+1}| = |a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n|$$

$$z_0 \notin \mathbb{R} \quad \Rightarrow < a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z^n|$$

$$\Rightarrow a_n(|z_0|^{n+1} - |z_0|^n) + \cdots + a_0(|z| - 1) < 0 \quad \text{矛}$$

$$\Rightarrow |z_0| < 1.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Rightarrow \text{令 } z = e^{i\theta} \quad \cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2} = \frac{z^k + z^{-k}}{2}$$

$$\Rightarrow a_0 + a_1\frac{1}{2}\left(z + \frac{1}{z}\right) + \cdots + a_n\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

$$= \frac{1}{2}z^{-n}\left(a_0 z^n + a_1 z^{n-1} + \cdots + a_n + a_0 z^n + a_1 z^{n+1} + \cdots + a_n z^{2n}\right) \text{ 有 } 2n \text{ 个根}$$

$$\Rightarrow \Delta_C \operatorname{Arg} P(z) = 2\pi \cdot n \quad \Rightarrow \exists \text{ 到 } 2n \text{ 个不同的 } \theta \text{ s.t. } \operatorname{Re} P(e^{i\theta}) = 0$$

$C: |z| = 1$

$$= \frac{1}{2}\, P\left(\frac{1}{z}\right) + P(z)$$

$$\text{LHS} = \operatorname{Re} P(z) + \operatorname{Re} P\left(\frac{1}{z}\right)$$

$\square$

以两题过程理由.

单叶函数（单射）

定理. $f\in H(D)$ 单叶 $\Rightarrow$ $f'(z)\neq 0\quad \forall z\in D$

pf. 若 $f'(z_0)=0$，$z_0\in D$.（用导数造重根）z_0 是 $f(z)-f(z_0)$ 的 m 级零点，$m\geq 2$.

取充分小 $\rho>0$，在 $B(z_0,\rho)$ 上，有 $\delta>0$，使 $f'(z)$ 在附近没有别的零点

$$f(B(z_0,\rho))\supseteq B(f(z_0),\delta)\Rightarrow \text{取 } a\in B(f(z_0),\delta)$$

$\exists\, z_1,z_2\in B(z_0,\rho)$ s.t. $f(z_1)=a$，$f(z_2)=a$.

而 $f'(z_1)=f'(z_2)\neq 0$ $\Rightarrow$ 矛盾 $\square$

上述结果不是充分的，逆命题的例子是 e^z

当然，由实的反函数定理，$f'(z_0)\neq 0$ 得到局部单问题不大，如下

定理. $f\in H(D)$，若 $z_0\in D$，$f'(z_0)\neq 0$，$\Rightarrow$ f 在 z_0 的邻域单叶.

pf. $f'(z_0)\neq 0\Rightarrow z_0$ 是 $f(z)-f(z_0)$ 的一阶零点 $\Rightarrow\forall\rho$ 充分小，$\exists\,\delta>0$，

s.t. 在 $B(z_0,\rho)$ 上，$\forall\, a\in B(f(z_0),\delta)$，$f(z)-a$ 仅有一个零点

由 f 的连续性，f 在 $B(z_0,\hat{\rho})$ 上单叶.（$\hat{\rho}<\rho$）.

最后拉回 $B(f(z_0),\delta)$ 时不能用 f^{-1} 大范围拉回，不然失去了 $B(z_0,\rho)$ 对 f 的控制.

定理. 若 $w=f(z)$ 在 D 上单叶全纯，$G=f(D)$，则反函数 $z=g(w)$ 在 G 上单叶全纯，$g'(w)=\dfrac{1}{f'(z)}$

pf. 首先 g 连续，因为是开映射，也显然单叶.

$$\lim_{w\to w_0}\frac{g(w)-g(w_0)}{w-w_0}=\lim_{w\to w_0}\frac{1}{\dfrac{w-w_0}{g(w)-g(w_0)}}=\lim_{w\to w_0}\frac{1}{\dfrac{f\circ g(w)-f\circ g(w_0)}{g(w)-g(w_0)}}$$

$$\overset{\text{连续}}{=}\lim_{z\to z_0}\frac{1}{\dfrac{f(z)-f(z_0)}{z-z_0}}=\frac{1}{f'(z_0)}\qquad\square$$

留数

定义、设 f 在 $0<|z-a|<r$ 上全纯，$\mathrm{Res}(f,a):=\frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)dz$，$0<\rho<r$ 与 ρ 选取无关

由 Laurent 展开知 $\mathrm{Res}(f,a) = c_{-1}$

命题。a 是 f 的 m 阶极点，那么 $\mathrm{Res}(f,a)=\frac{1}{(m-1)!}\lim_{z\to a}\left(\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right)$

pf. $f(z)=\frac{g(z)}{(z-a)^m}$ $g(z)$ 在 a 处全纯且 $g(a)\neq 0$

$\Rightarrow \mathrm{Res}(f,a)=\frac{1}{2\pi i}\int_{|z-a|=\rho}\frac{g(z)}{(z-a)^m}dz \quad \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}g\Big|_{z=a} = \cdots$ □

推论。对一阶极点 $\mathrm{Res}(f,a)=\lim_{z\to a}(z-a)f(z)$

命题。$f=\frac{g}{h}$，$g(a)\neq 0$，$h(a)=0$ $h'(a)\neq 0$ g,h 都有 a 处全纯，则

$\mathrm{Res}(f,a)=\frac{g(a)}{h'(a)}$.

pf. $\mathrm{Res}(f,a) = \frac{1}{2\pi i}\int_{|z-\rho|=a}\frac{g(z)}{h(z)}dz = \lim_{z\to a}(z-a)\frac{g(z)}{h(z)}$

$= \lim_{z\to a}\frac{g(z)}{\frac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$ □

定理。(留数定理) 设 $f(z)$ 在 γ 内部 $D\setminus\{z_1,\cdots z_n\}$ 全纯且在 $\bar{D}\setminus\{z_1,\cdots z_n\}$ 上连续

则 $\int_\gamma f(z)dz = \sum_{k=1}^{n}\mathrm{Res}(f,z_k)$

这是 Cauchy 定理 ...

∞ 远点留数 $\mathrm{Res}(f,\infty) = -\frac{1}{2\pi i}\int_{|z|=\rho} f(z)dz \quad (\rho>R) = -c_{-1}$

定理。设 f 在 $\mathbb{C}\setminus\{z_1,\cdots z_n\}$ 上全纯，则 f 所有孤立奇点的留数之和为 0

$\sum_{k=1}^{n}\mathrm{Res}(f,z_k) = \mathrm{Res}(f,\infty)$

还是 Cauchy 定理。

例 $I=\int_{|z|=2}\frac{z^5}{1+z^6}dz = -2\pi i\,\mathrm{Res}(f,\infty) = 2\pi i\, c_{-1} = 2\pi i$

$\frac{z^5}{1+z^6} = \frac{z^5}{z^6}\left(1-\frac{1}{z^6}+\cdots\right) = \frac{1}{z} \cdots$

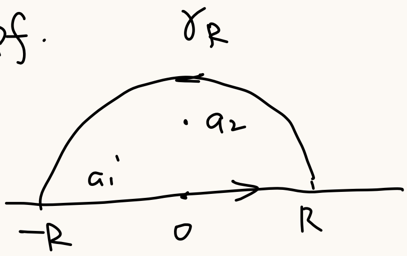
实积分的计算

$\int_{-\infty}^{\infty} f(x)\,dx$ 型.

定理　设 f 在 $\mathbb{H}\setminus\{a_1,\cdots a_n\}$ 上全纯，$\overline{\mathbb{H}}\setminus\{a_1,\cdots a_n\}$ 上连续. 若

$\lim\limits_{z\to\infty} z f(z) = 0$　那么　$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum\limits_{k=1}^{n} \mathrm{Res}(f, a_k)$.

Pf.



$$\int_{-R}^{R} f(x)\,dx + \int_{\gamma_R} f(z)\,dz \underset{\text{残数定理}}{=} 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k)$$

记 $M(R) = \max\limits_{z\in\gamma_R} |f(z)|$

$$\Rightarrow \quad RM(R) \to 0 \quad \text{as } R\to\infty$$

$$\Rightarrow \left|\int_{\gamma_R} f(z)\,dz\right| = \left|\int_0^{\pi} f(Re^{i\theta})\cdot Rie^{i\theta}\,d\theta\right|$$

$$\le \int_0^{\pi} R\left|f(Re^{i\theta})\right| d\theta \le \pi R M(R) \to 0 \quad \text{as } R\to\infty$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \sum_{k=1}^{n} \mathrm{Res}(f, a_k).$$

□

推论. $f(x) = \dfrac{P(x)}{Q(x)}$　$\deg Q - \deg P \geq 2$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(\frac{P(z)}{Q(z)}, a_k\right)$$　a_k 是 $Q(z)$ 在 $\mathbb{H}$ 上的根

例　$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{n+1}} = 2\pi i\, \mathrm{Res}\left(\frac{1}{(z^2+1)^{n+1}}, i\right)$$

$$\mathrm{Res}\left(\frac{1}{(z^2+1)^{n+1}}, i\right) = \frac{1}{n!}\left(\frac{(z-i)^{n+1}}{(z^2+1)^{n+1}}\right)^{(n)}\Bigg|_{z=i}$$

$$= \frac{1}{n!}\left((z+i)^{-(n+1)}\right)^{(n)}\Bigg|_{z=i}$$

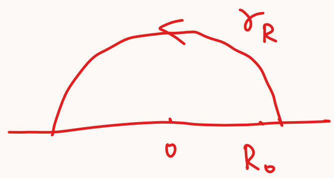
$$= \frac{(2n)!}{(n!)^2}(z+i)^{-2n+1}\Bigg|_{z=i}.$$

□

$\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$ 型

引理. f 在 $\{R_0 \le |z| < \infty,\ \operatorname{Im} z \ge 0\}$ 上连续, $\lim\limits_{\substack{R\to\infty \\ \operatorname{Im} z \ge 0}} f(z) = 0$.

$\forall \alpha > 0.$ $\lim\limits_{R\to\infty} \int_{\gamma_R} e^{i\alpha z} f(z)\,dz = 0$



Pf 直接计算.

$$\left|\int_{\gamma_R} e^{i\alpha z} f(z)\,dz\right| = \left|\int_0^{\pi} e^{i\alpha(R\cos\theta + iR\sin\theta)} f(Re^{i\theta})\, Rie^{i\theta}\,d\theta\right|$$

$$\le \int_0^{\pi} R\,M(R)\cdot e^{-\alpha R\sin\theta}\,d\theta \le 2R\,M(R)\int_0^{\pi/2} e^{-\alpha R\cdot\frac{2}{\pi}\theta}\,d\theta$$

$$= \frac{\pi}{2} M(R)\left(1 - e^{-\alpha R}\right) \to 0 \quad \text{as } R\to\infty. \qquad \square$$

那么. $\int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,dx = 2\pi i \sum\limits_{i=1}^{\infty} \operatorname{Res}\left(e^{i\alpha z} f(z),\ a_k\right)$

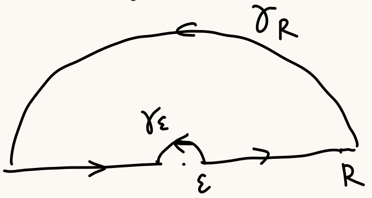
例 $\int_{-\infty}^{\infty} \frac{\cos\alpha x}{b^2+x^2}\,dx = \operatorname{Re}\left(2\pi i \operatorname{Res}\left(\frac{e^{i\alpha z}}{b^2+z^2},\ ib\right)\right)$

$\operatorname{Res}\left(\frac{e^{i\alpha z}}{b^2+z^2}, ib\right) = \frac{e^{-\alpha b}}{2ib}$

$\text{LHS} = 2\pi \frac{e^{-\alpha b}}{2b} = \frac{\pi}{b} e^{-\alpha b}.$ $\square$

$\mathbb{R}$ 上有奇点

例 $\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$



$$\left(\int_{\varepsilon}^{R} + \int_{\gamma_R} + \int_{-R}^{-\varepsilon} - \int_{\gamma_\varepsilon}\right)\frac{e^{iz}}{z}\,dz = 0$$

由之前引理 $\lim\limits_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z} = 0$

现在算 $\lim\limits_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}$

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz = \int_0^{\pi} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}\cdot \varepsilon i e^{i\theta}\,d\theta = i\int_0^{\pi} e^{i\varepsilon e^{i\theta}}\,d\theta \to \pi i \quad \text{as } \varepsilon\to 0$$

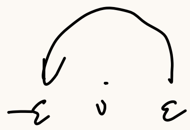
$\Rightarrow \varepsilon\to 0, R\to\infty \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$

例 $\int_{-\infty}^{\infty}\frac{\sin^3 x}{x^3}dx = \int_{-\infty}^{\infty}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3\cdot\frac{1}{x^3}dx$

$$= \int_{-\infty}^{\infty}\frac{3\sin x-\sin 3x}{4x^3}dx$$

类似选择围道. 但是 $\frac{1}{4x^3}$ 难以处理. 以下.

$\lim\limits_{\varepsilon\to 0}\int_{\gamma_\varepsilon} z^k = ?$ $\quad k\in\mathbb{Z}$. $\qquad \gamma_\varepsilon$



令 $z=\varepsilon e^{i\theta} \Rightarrow \int_0^\pi \varepsilon^k e^{ik\theta}\cdot \varepsilon i e^{i\theta}\,d\theta$

$$=\varepsilon^{k+1}\int_0^\pi e^{i(k+1)\theta}\,d\theta$$

$$=\varepsilon^{k+1}\frac{(-1)^{k+1}-1}{k+1}\to\begin{cases}0 & k\geq 0 \text{ 或 } k\leq -2 \text{ 奇}\\ \text{不存在} & k\leq -2 \text{ 偶}\\ \pi i & k=-1.\end{cases}$$

将 $3e^{iz}-e^{3iz}$ 展开

$$3\left(1+iz+\frac{-z^2}{2}+\frac{-iz^3}{6}\cdots\right)-\left(1+3iz+\frac{-9z^2}{2}+\frac{-27iz^3}{6}\cdots\right)$$

$$\int_{\gamma_\varepsilon}\frac{2+3z^2+4iz^3}{4z^3}\to\frac{3}{4}\pi i$$

有理三角式

$\int_0^{2\pi}R(\cos\theta,\sin\theta)\,d\theta \qquad z=e^{i\theta} \quad \frac{dz}{iz}=d\theta$

$$\int_0^{2\pi}\frac{d\theta}{3+\cos\theta+2\sin\theta}=\int_{|z|=1}\frac{1}{3+\frac{1}{2}(z+\frac{1}{z})+\frac{1}{i}(z-\frac{1}{z})}\frac{dz}{iz}$$

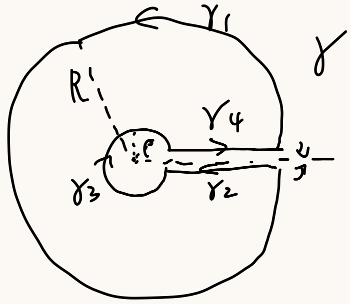
$=\int_{|z|=1}\frac{dz}{3iz+\frac{i}{2}z^2+\frac{i}{2}+z^2-1}=\int_{|z|=1}\frac{dz}{(1+\frac{i}{2})z^2+3iz+(\frac{i}{2}-1)}$ 然后计算 Res

多值相关

例 $\int_0^\infty\frac{dx}{(1+x)x^\alpha} \qquad 0<\alpha<1$

若令 $f(z)=\frac{1}{(1+z)z^\alpha} \qquad z^\alpha$ 支点为 0 和 ∞

构造如下围道



在围道上积分有

$$\int_\gamma f(z)\,dz = 2\pi i \,\mathrm{Res}(f,-1)$$

$$\gamma_1:\ \left|\int_{|z|=R} f\,dz\right| \le \frac{2\pi R}{(R-1)R^\alpha} \sim R^{-\alpha} \to 0 \quad R\to\infty$$

$$\gamma_2:\ \int_R^\rho \frac{1}{(1+x)\,x^\alpha\cdot e^{i2\pi\alpha}}\,dx$$

$$\gamma_3\ \ \left|\int_{|z|=\rho} f\,dz\right| \le \frac{2\pi\rho}{(1-\rho)\,\rho^\alpha} \sim 2\pi\rho^{1-\alpha} \to 0 \quad \rho\to 0$$

$$\gamma_4\ \ \int_\rho^R \frac{1}{(1+x)\,x^\alpha}\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{(1+x)x^\alpha} = \frac{2\pi i}{1-e^{-i\alpha\cdot 2\pi}}\,\mathrm{Res}(f,-1)$$

$$\mathrm{Res}(f,-1) = \left.\frac{1}{z^\alpha}\right|_{z=-1} = \frac{1}{|z|^\alpha e^{\pi\alpha i}} = e^{-\pi\alpha i}$$

$$\Rightarrow \int_0^\infty \frac{1}{(1+x)\,x^\alpha} = \frac{\pi}{\sin\pi\alpha} \qquad 0<\alpha<1.$$

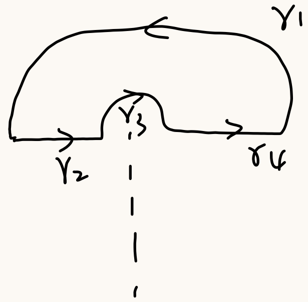
$\square$

再算几个经典积分结束.

1. $I = \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

令 $f(z) = \frac{\log z}{(1+z^2)^2}$

若 keyhole contour 注意到值在分子,会抵消.



$$\int_{\gamma_1} f\,dz \to 0$$

$$\int_{\gamma_2} f\,dz = \int_{-R}^{-\rho} \frac{\log|x| + i\pi}{(1+x^2)^2}\,dx \to I + i\pi\int_0^\infty \frac{1}{(1+x^2)^2}\,dx$$

$$\left|\int_{\gamma_3} f\,dz\right| \lesssim \log\rho\cdot\pi\rho \to 0 \quad \text{as } \rho\to 0$$

$$\int_{\gamma_4} f\,dz = \int_\rho^R \frac{\log x}{(1+x^2)^2}\,dx \to I$$

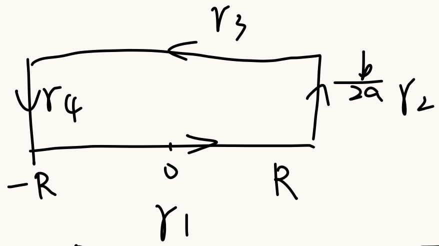
$$\Rightarrow 2I + \pi i\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = 2\pi i\,\mathrm{Res}(f,i).$$

$$2I + \pi i \cdot \frac{\pi}{4} = 2\pi i \Rightarrow I = -\frac{\pi}{4}. \qquad \square$$

2. poisson积分和 Fresnel 积分

1. $I = \int_0^\infty e^{-ax^2} \cos bx \, dx$

令 $f(z) = e^{-az^2}$ 取围道



$\int_\gamma f \, dz = 0 \qquad \int_{\gamma_1} f(z) = \int_{-R}^{R} e^{-ax^2} dx \to \sqrt{\frac{\pi}{a}}$

$\int_{\gamma_2} f(z) dz \to 0$

$\int_{\gamma_4} f(z) dz \to 0$

$\int_{\gamma_3} f(z) dz = \int^{-R} e^{-a(x+\frac{b}{2a}i)^2} dx$

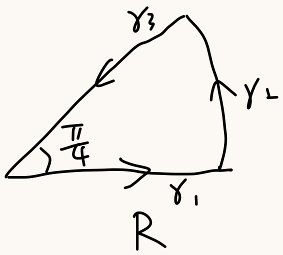
$= \int_R^{-R} e^{-a(x^2 + \frac{b}{a}xi - \frac{b^2}{4a^2})} dx$

$= -\int_{-R}^{R} e^{-ax^2 + bxi + \frac{b^2}{4a^2}} dx$

$= -e^{\frac{b^2}{4a^2}} \int_{-R}^{R} e^{-ax^2} \cdot e^{ibx} dx$

$\Rightarrow e^{-\frac{b^2}{4a^2}} \int_{-\infty}^{\infty} e^{-ax^2} \cos bx \, dx = \sqrt{\frac{\pi}{a}} \qquad \square$

2. $\int_0^\infty \cos x^2 \, dx \qquad \int_0^\infty \sin x^2 \, dz$



$f(z) = e^{iz^2}$

$\int_\gamma f(z) dz = 0 \qquad \int_{\gamma_1} f \, dz = \int_0^R e^{ix^2} dx$

$$\left| \int_{\gamma_2} f \, dz \right| = \left| \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} \cdot iRe^{i\theta} d\theta \right| \leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} d\theta \leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \cdot \frac{2}{\pi} \cdot 2\theta} d\theta$$

$= \frac{\pi}{4R}(1 - e^{-R^2}) \to 0 \quad$ as $R \to \infty$

$\int_{\gamma_3} f(z) dz = \int_R^0 e^{i(re^{i\frac{\pi}{4}})^2} e^{i\frac{\pi}{4}} dr = -e^{i\frac{\pi}{4}} \int_0^r e^{-r^2} d \to -e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$

$\Rightarrow \int_0^\infty e^{ix^2} dx = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \frac{\sqrt{\pi}}{2} \Rightarrow \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \int_0^\infty \sin x^2 dx$

$\square$

全纯开拓　　就是全纯的延拓，名字而已…

$f\in H(D)$，$\hat f\in H(G)$　$G\supsetneq D$，都是区域．$\hat f|_D=f$

例　$f(z)=\sum_{n=0}^{\infty}z^n$　$D=\{|z|<1\}$

$\hat f=\frac{1}{1-z}$　　$G=\mathbb{C}\setminus\{1\}$

NOTE. 1. 不是所有全纯函数都能开拓．e.g. $\sum_{n=0}^{\infty}z^{n!}$

2. 由唯一性定理．$\underset{f}{D}\longrightarrow\underset{\hat f}{G}$ 的开拓唯一．

定理（Painlevé 定理）区域Ω被γ分为两区域Ω_1,Ω_2，f在Ω_1,Ω_2上全纯，Ω上连续

$\Rightarrow f\in H(\Omega)$

Pf. Morera 定理．之前证过．

NOTE. 就是某种 paste lem.

cor. $f_1\in H(D_1)\cap C(D_1\cup \ell)$，$f_2\in H(D_2)\cap C(D_2\cap\ell)$

$f_1|_\ell=f_2|_\ell\Rightarrow F=f_1\cup f_2\in H(D)$

定理（Schwarz 对称原理）D 关于实轴对称，若 f 满足

(1) $f\in H(D\cap\mathbb{H})$，

(2) $f\in C(D\cap\overline{\mathbb{H}})$，

(3) f 在 $D\cap\partial\mathbb{H}$ 上取实值

$$\Rightarrow F(z)=\begin{cases}f(z), & z\in D\cap\overline{\mathbb{H}}\\ \overline{f(\bar z)} & z\in D\cap(\overline{\mathbb{H}})^c\end{cases}$$

Pf. $F\in H(D\cap\mathbb{H})$ 是显然的

$\forall z\in D\cap\overline{\mathbb{H}}^c$，$\frac{\partial F}{\partial\bar z}=\frac{\partial}{\partial\bar z}\overline{f(\bar z)}=\overline{\left(\frac{\partial f}{\partial z}(\bar z)\right)}=0$

$z\in\overline{\mathbb{H}}^c$：$\lim_{z\to x_0}F(z)=\lim_{\bar z\to x_0}\overline{f(\bar z)}=\lim_{z\to x_0}\overline{f(z)}=f(x_0)=F(x_0)$．□

定理．（推广的 Schwarz 定理）

设 ① Ω 与 Ω' 关于 $|z-a|=r$ 对称

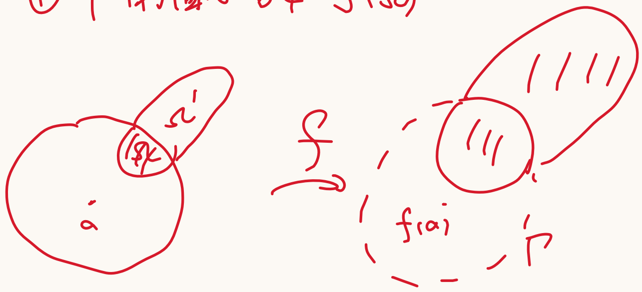
② $f(z)$ 在 Ω 内全纯，在 $\Omega\cup S$ 上连续

③ $f(S)$ 为一段圆弧

④ Γ 的圆心 $b\notin f(\Omega)$

对称指 $z\in\Omega \Longleftrightarrow z^*\in\Omega'$

$$z^* = a + \frac{R^2}{\bar z-\bar a}$$



对称开拓的思路

$$\begin{array}{ccc} z^* & \xrightarrow{\checkmark} & f^*(z) \\ \downarrow & & \uparrow \\ z & \longrightarrow & f(z) \end{array}$$

Pf. 令 $F(z)=\begin{cases} f(z) & z\in\Omega\cup S \\ (f(z^*))^* & z\in\Omega'\cup S\end{cases}$

直接计算 $(f(z^*))^* = b+\dfrac{R^2}{\overline{f(z^*)-b}} = b+\dfrac{R^2}{\overline{f\left(a+\frac{r^2}{\bar z-\bar r}\right)-b}}$ ④

$$\frac{\partial F}{\partial \bar z} = R^2\,\frac{-1}{\left(\bar f\left(a+\frac{r^2}{\bar z-\bar r}\right)-\bar b\right)^2}\cdot\frac{\partial}{\partial\bar z}\bar f(z^*) = \frac{-R^2}{(\bar f-\bar b)^2}\,\overline{\frac{\partial}{\partial z}f(z^*)}$$

$$\frac{\partial f}{\partial z}(z^*) = f'(z^*)\cdot\frac{\partial}{\partial z}z^* = f'(z^*)\,\frac{\partial}{\partial z}\left(a+\frac{R^2}{\bar z-\bar a}\right)=0$$

(求导时 $z,\bar z$ 是无关的，由 $\frac{\partial}{\partial z}$ $\frac{\partial}{\partial\bar z}$ 的定义决定)

连续性 拆 $*$ 算即可. 由 Painlevé ⋯ □

注意 Γ 是直线也没关系. 因此我们实际得到了 4 种 Schwarz 原理.

幂级数全纯开拓

设 $f(z)=\sum_{n=0}^{\infty} a_n z^n$ 收敛半径为 R. f 在 $D=\{|z|<R\}$ 上全纯

定义. ($\partial B(0,R)$ 上)

正则点. $\xi_0\in\partial B$, $\exists\, B(\xi_0,\delta)$ 上全纯函数 $g(z)$ 在 $B(\xi_0,\delta)\cap D$ 上 $g=f$

奇点 $\xi_0\in\partial B$, 不是正则点

$f\in H(D_1)$ $g\in H(D_2)$ $D_1\cap D_2\neq\phi$ $g|_{D_1\cap D_2}=f|_{D_1\cap D_2}\Rightarrow (f,D_1)\sim(g,D_2)$

定理. 幂级数的收敛圆周上必有奇点

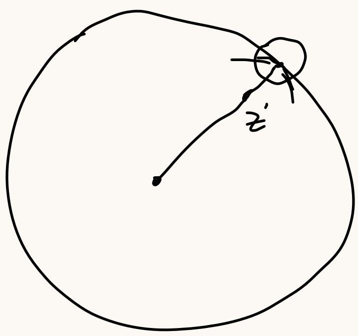
Pf. 若全部正则. 选出一个开覆盖. 由 ∂B 的紧性.

设 $\{B(\xi_1,r_1),\cdots,B(\xi_n,r_n)\}$ 令 $D=B(0,R)\cup(\bigcup_{i=1}^{n}B_i)$

令 $g(z)=\begin{cases} f(z) & z\in B(0,R) \\ f_i(z) & z\in B(\xi_i,r_i)\end{cases}$ 由唯一性定理. g 良定. 在 D 上全纯

$\Rightarrow g$ 在 $\bar{B}\supsetneq B$ 上可展开, 由唯一性 $\Rightarrow f$ 在 $\bar{B}$ 上收敛 矛盾 □

如何判断奇异点? $z_0\in\partial B(0,R)$



① $\lim\limits_{z\to z_0}|f(z)|=+\infty$. 奇

② $\lim\limits_{z\to z_0}|f(z)|\neq+\infty$

取 $z'\in Oz_0$ z'处展开 $\rho\geq R-|z'|$

if $\rho>R-|z'|$ 正则

if $\rho=R-|z'|$ 奇点 (因为其他点都不是奇点)

③ f 与 f' 有相同的奇点/正则点

④ f 在 $|z|=R$ 上的收敛/发散与正则/奇异无必然关系.

例. 1. $\sum\limits_{n=0}^{\infty}z^n$ 在 $z=-1$ 处发散, 但 $z=-1$ 正则点 $\frac{1}{1-z}$

2. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$ 在 $z=1$ 处收敛, 但 $z=1$ 正则 $\log z$

3. $\sum\limits_{n=0}^{\infty}z^n$ 在 $z=1$ 处发散, 但 $z=1$ 奇异 $\frac{1}{1-z}$

4. $\sum\limits_{n=2}^{\infty}\frac{z^n}{n(n-1)}$ 在 $z=1$处收敛. 但 $z=1$ 奇异 求导

例 $\sum\limits_{n=1}^{\infty}z^{2^n}$ 在 $|z|=1$ 上处处奇异 (technical)

(1) $z=1$ 奇异

(2) $z^2=1$ 的根 $f(z)=z^2+f(z^2)$ $\Rightarrow$ $z^2=1$的根也是奇异点

(3) $z^{2^n}=1$ 的根

(4) 奇异点稠密 $\Rightarrow$ 放不下正则点.

例、$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad 0<R<\infty$ 若某项起 $a_n \geqslant 0$.

$\Rightarrow z=R$ 是奇异点

Pf. 若 $z=R$ 正则 $\Rightarrow \frac{R}{2}$ 处展开, $\rho>\frac{R}{2}$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\frac{R}{2})}{n!}\left(z-\frac{R}{2}\right)^n$$

$$\Rightarrow \limsup_{n\to\infty}\left(\left|\frac{f^{(n)}(R/2)}{n!}\right|\right)^{\frac{1}{n}}=\frac{1}{\rho}<\frac{2}{R}$$

计算 $f^{(n)}(R/2)$: $\quad f^{(n)}=a_n\cdot n!+\cdots$

$$=\sum_{m\geqslant n} a_m m(m-1)\cdots(m-n+1)\left(\frac{R}{2}\right)^{m-n}\geqslant 0$$

$$\left|f^{(n)}\left(\tfrac{1}{2}Re^{i\theta}\right)\right|\leqslant\left|f^{(n)}(R/2)\right| \quad\Rightarrow \rho_\theta\geqslant\rho>R/2. \text{ 全部正则! 矛盾 } \square$$

黎曼映照

Q: 开集 Ω 何时使存在共形映射 $f: \Omega \to D$

由 liouville 定理，$\Omega \neq \mathbb{C}$ （因此此时 f 只能为常数）

定理(Hurwitz) $f_n \in H(D)$，f_n 内闭一致收敛到 $f(z)$，$f(z) \not\equiv 0$.

若 $\gamma \subseteq D$ 不经过 f 的零点，则 $\exists N>0$，当 $n>N$，f_n 与 f 在 γ 内的零点个数相同

设 $\varepsilon = \min\{|f(z)| : z \in \gamma\} > 0$ $\gamma \subset\subset D$

$\Rightarrow \exists N$，$|f_n - f| < \varepsilon$ if $n > N$

$\Rightarrow$ $|f_n - f| < \varepsilon \leq |f|$ □

因为我们最后想要做的结论是全纯等价意义下的，我们到要处理单叶函数

定理. ① f_n 在 D 中全纯单叶.

② f_n 内闭一致收敛到 f，$f \not\equiv \text{const}$

则 f 也是全纯单叶

Pf. f 当然全纯 (Weierstrass) 反证. 有 $z_1 \neq z_2$，$f(z_1) = f(z_2)$ $z_1, z_2 \in D$

令 $F(z) = f(z) - f(z_1)$ z_1, z_2 是孤立的

设 $\varepsilon > 0$，$B(z_1, \varepsilon) \cap B(z_2, \varepsilon) = \emptyset$ 且 F 在 $B(z_1, \varepsilon)$，$B(z_2, \varepsilon)$ 上无其他零点

令 $F_n = f_n - f(z_1) \overset{内闭}{\rightrightarrows} F$ 由 Hurwitz. 选出 z_1'，z_2' 两个零点，矛 □

我们会处理到一族具有某性质的函数，为此引入概念.

定义. $\mathcal{F}$ 为区域 D 上的函数族.

① $K \subset\subset D$，$\forall f \in \mathcal{F}$，$\exists M = M(K) > 0$，$|f| \leq M$ on K

则称 $\mathcal{F}$ 内闭一致有界

② $\forall \{f_n\} \subseteq \mathcal{F}$，$\{f_n\}$ 有 D 上内闭一致收敛子列，则称 $\mathcal{F}$ 是 D 上的一个正规族

接下来用完整版的 AA 定理证明 Montel 定理.

定理.(AA) $(X,\mathcal{T}),(Y,d)$, $\mathcal{F}\subseteq \mathcal{C}(X,Y)$, $\mathcal{T}_{c.c.}$)

(1) if $\mathcal{F}$ 等度连续，逐点预紧 $\Rightarrow$ $\mathcal{F}$ 在 $(\mathcal{C}(X,Y),\mathcal{T}_{c.c})$ 中的闭包紧

(2) if X is LCH，逆命题对.

定理.(Montel) $\mathcal{F}$ 是 D 上的全纯函数族，那么 $\mathcal{F}$ 正规族 $\Leftrightarrow$ $\mathcal{F}$ 在 D 上内闭一致有界

Pf. ($\Rightarrow$) 这个容易，因为若不然，$\exists K$ $\sup\{|f| : z\in K\}=\infty$

$\Rightarrow \{f_n\}$ $\sup\{|f_n|,\ z\in K\}\geq n$ $f_{n_k}\rightrightarrows f$ on K

f on K 有界 ↯

($\Leftarrow$) D 的拓扑性质够好. LCH + σ-cpt $\{f_n\}$ 在 $K\subset\subset D$ 上有界 M_K

$$|f_n(z_1)-f_n(z_2)| = \left|\frac{1}{2\pi i}\int_\gamma \left(\frac{f_n(\xi)}{\xi-z_1}-\frac{f_n(\xi)}{\xi-z_2}\right)d\xi\right|$$

$$= \frac{|z_1-z_2|}{2\pi}\left|\int_\gamma \frac{f_n(\xi)\,d\xi}{(\xi-z_1)(\xi-z_2)}\right|$$

$$\leq \frac{M\cdot 2\pi r\,|z_1-z_2|}{r^2} \leq \frac{C}{r}|z_1-z_2|$$

只要让 $\forall z\in K$. $B(z,3r)\subseteq D$ 即可

那么用 AA + 对角线 Argument 即可 □

定理 (Riemann映照定理) 设 $\Omega \subsetneq \mathbb{C}$，且单连通，则 Ω 与 $\mathbb{D}$ 全纯同构. i.e.

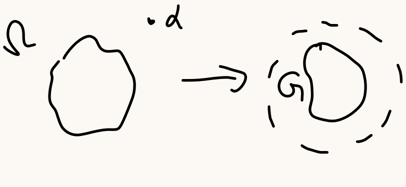
$\forall z_0\in\Omega$ $\exists!$ 共形映射 $f:\Omega\to\mathbb{D}$ s.t. $f(z_0)=0$, $f'(z_0)>0$

Pf. Idea: ① $\mathcal{F}=\{f\in H(\Omega): f:\Omega\to\mathbb{D},\ f(z_0)=0,\ f\text{ 单射}\}$ $\mathcal{F}$ 非空，一致有界

② $\exists \tilde{f}\in\mathcal{F}$ $|f'(z_0)|$ 达到最大. 应是我们要的

1. Ω 单连通 $\Omega\subsetneq\mathbb{C}$, $\exists\ G\subseteq D$. $0\in G$, $G\sim\Omega$ $z_0\mapsto 0$

设 $\alpha\in\mathbb{C}$, $\alpha\notin\Omega$ 令 $g(z)=\log(z-\alpha)$，g 在 Ω 上有单值分支



① g 单射. 因 $g(z_1)=g(z_2)\Rightarrow e^{g(z_1)}=e^{g(z_2)}\Rightarrow z_1-\alpha=z_2-\alpha$

② $w_0=g(z_0)$ 若 $\tilde{w}_0=w_0+2\pi i\in g(\Omega)$

$\Rightarrow g(\tilde{z}_0)=w_0+2\pi i$ $e^{g(\tilde{z}_0)}=e^{g(z_0)}\Rightarrow \tilde{z}_0=z_0$ ↯

③ $g(\Omega)$ 与 $\widetilde{w_0}$ 是"严格分离"的。否则 $g(z_n) \to w_0+2\pi i$

$z_n-\alpha = e^{g(z_n)} \to e^{\widetilde{w_0}} = e^{w_0} = z_0-\alpha \Rightarrow z_n \to z_0$ 但 $g(z_n) \to g(z_0) = w_0 \neq \widetilde{w_0}$

因此有 $g(\Omega) \cap B(\widetilde{w_0}, \delta) = \phi$

④ 令 $f(z) = \dfrac{1}{g(z)-\widetilde{w_0}}$ 良定 on Ω 在 Ω 上有界

令 $\tilde{f}(z) = f(z) - f(z_0) \Rightarrow \underset{|\tilde{f}| \leq C}{\underline{\tilde{f} \text{ 有界}}}$ 则 $\tilde{f}(z_0)=0$

令 $h(z) = \dfrac{1}{2C}\tilde{f}$ 即可

2.① 因此 可设 $\Omega \subseteq \mathbb{D}$ 现在 $\mathcal{F}$ 非空(Id) 且一致有界($\mathbb{D}$)

$0 \in \Omega \Rightarrow \exists \varepsilon>0.\ B(0,\varepsilon) \subseteq \Omega$

$|f'(0)| \underset{\underset{\text{Cauchy}}{\uparrow}}{\leq} \dfrac{1}{\varepsilon} \quad \forall f \in \mathcal{F}$

令 $\sup\{|f'(0)|,\ f \in \mathcal{F}\} = A$. $\exists \{f_n\} \subseteq \mathcal{F}.\ f_n'(0) \to A$

由 Montel, 有 $\{f_{k_n}\}$ 内闭一致收敛到 f_∞ 由 Hurwitz 推论

f_∞ 全纯单叶 $\left(|f_\infty'(0)|>0,\ f_\infty(0)=0\right)$

且 $|f_\infty(z)|<1 \quad \forall z \in \Omega$ (最大模)

$\Rightarrow f_\infty \in \mathcal{F}$

② $f_\infty(G) = \mathbb{D}$、 if not. $f_\infty(G) \subsetneq \mathbb{D}$

$\Rightarrow \exists \alpha \in \mathbb{D} \setminus f_\infty(G)$

令 $\varphi_\alpha(z) = \dfrac{\alpha-z}{1-\bar{\alpha}z} \quad \mathbb{D} \to \mathbb{D}$

$\Rightarrow$ ① $\varphi_\alpha \circ f_\infty$ 单连通 (全纯同构)

② $\varphi_\alpha \circ f_\infty$ 不含 0

$\Rightarrow R(z) = (\varphi_\alpha \circ f_\infty)^{\frac{1}{2}}$ on G 有单值分支, 单叶

令 $\beta = \sqrt{\alpha}$, $\varphi_\beta = \dfrac{\beta-z}{1-\bar{\beta}z} \quad \mathbb{D} \to \mathbb{D}.\ \beta \to 0$

$G \xrightarrow{R} \mathbb{D} \xrightarrow{\varphi_\beta} \mathbb{D}$

$0 \longrightarrow \beta \longrightarrow 0$

$F = \underbrace{\varphi_\beta \circ S \circ \varphi_\alpha}_{T^{-1}} \circ f_\infty \qquad S = \sqrt{w}$

$\Rightarrow f_\infty = \underbrace{T\circ F}_{\text{不是转 on } \mathbb{D}}$ $\quad T(0)=0$ $\Rightarrow T$ 不是 $\mathbb{D}$ 上旋转 $\Rightarrow |T'(0)|<1$

$$\Rightarrow |f_\infty'(0)| \le |T'(0)||F'(0)| < |F'(0)| \quad 矛盾!$$

最后加个旋转即可 $f'(0)>0$

③ 一般地 若 f, g 都是满足的映射

令 $h = f\circ g^{-1}$ $\quad h(B(0,1)) = B(0,1) \Rightarrow h\in \mathrm{Aut}(\mathbb{D})$

又 $h(0)=0 \Rightarrow h(z) = e^{i\theta}z \qquad h'(0) = \dfrac{f'(z_0)}{g(z_0)} > 0$

而 $h(z)=e^{i\theta}z \Rightarrow \theta=0 \Rightarrow f\equiv g$ on $\mathbb{D} \Rightarrow f\equiv g$ $\square$

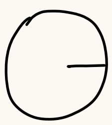
NOTE. 单连通不能被连续映射保持. 但我们用的是双全纯

边界对应

定理. G 是简单闭曲线 Γ 所围成的区域. 若 $w=f(z)$ 把 G 双全纯地映为 $B(0,1)$, 那么 f 的定义可以扩充到 Γ 上, 使 $f\in C(\bar G)$, 且 f 把 Γ 一一地映成 $|w|=1$, Γ 关于 G 的定向对应于 $f(\Gamma)$ 关于 $B(0,1)$ 的正向

G

这时边界为



. 不能想当然定义中间的部分

Pf. 1. 定义 $f|_{\bar G}$ $\quad$ 先证 $\forall \zeta\in\partial G$. $f(\zeta) := \lim\limits_{\substack{z\to\zeta\\ z\in G}} f(z)$ 存在

设 $\{z_n\}, \{z_n'\} \to \zeta$

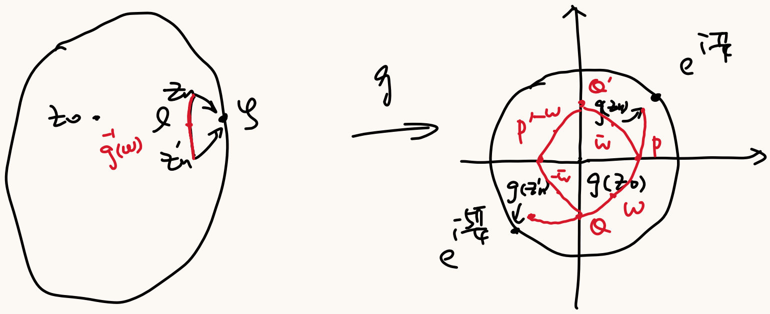
$\lim\limits_{n\to\infty} f(z_n) = a$, $\lim\limits_{n\to\infty} f(z_n') = b$ $\quad \boxed{\text{if } a\ne b}$. denote $w_n = f(z_n)$

首先 $a, b\in\partial\mathbb{D}$

$\zeta = \lim\limits_{n\to\infty} f^{-1}(w_n) = \lim\limits_{w_n\to a} f^{-1}(w_n) = f^{-1}(a) \in G$

取 $T\in \mathrm{Aut}(\mathbb{D})$ s.t. $T(a) = e^{i\pi}$ $\quad T(b) = e^{i\frac{\pi}{2}}$

$g = T\circ f \qquad\qquad g(z_0) = 0$



适当分离, 有 ℓ 连 z_n, z_n'

$L = g(\ell)$

控制 ℓ 与 ζ 的距离, 先设

$\ell \in B(\zeta,\delta) \cap G.\quad \delta > 0$

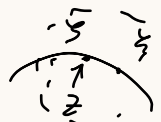
「 按理红线在左端收为一点, 右端的对径点应使红线的像收缩至 0, 需要用F 刻画使一收缩过程 $\perp$

令 $F(\omega) = (g^{-1}(\omega) - \zeta)\overline{(g^{-1}(\overline{\omega}) - \zeta)}\,(g^{-1}(-\omega) - \zeta)\overline{(g^{-1}(-\overline{\omega}) - \zeta)}$

$M = \sup\limits_{z \in \partial G} |\zeta - z| \quad \Rightarrow |F(\omega)| \le \delta M^3 \xRightarrow{\text{最大模}} |F(0)| \le \delta M^3$

$\Rightarrow |g^{-1}(0) - \zeta|^4 \le \delta M^3 \quad \delta \to 0 \Rightarrow g^{-1}(0) = \zeta$ 矛

那么定义 $\zeta \in \partial G.\ f(\zeta) = \lim\limits_{\substack{z \to \zeta \\ z \in G}} f(z)$, 用三角不等式可证



$\lim\limits_{\substack{\xi \to \zeta \\ \xi \in G}} f(\xi) = f(\zeta)$

$\forall \varepsilon > 0.\ \exists \delta > 0\ |f(z) - f(\zeta)| < \varepsilon/2$ if $|z - \zeta| < \delta$

$\xi \in B(\zeta,\delta) \cap \partial G.\ \delta_1 > 0\ |f(\xi) - f(z)| < \varepsilon/2$, if $|\xi - z| < \delta_1$

两圆相交 $\Rightarrow |f(\xi) - f(\zeta)| < \varepsilon$

2. 证明 一一. 现在 $f|_{\partial G} : \partial G \hookrightarrow \partial D$.

(1) 单射. 若 $f(\zeta) = f(\zeta')$, 有 $B(\zeta,\varepsilon) \cap B(\zeta',\varepsilon) = \phi$

f 在 G 上单射 $\Rightarrow f(B(\zeta,\varepsilon) \cap G) \cap f(B(\zeta',\varepsilon) \cap G) = \phi$

$\Rightarrow f(\zeta) \ne f(\zeta')$

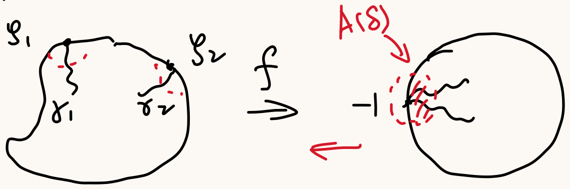
感觉不太对

显然中间可能不知道去哪

满 $f(\overline{G}) \underset{\text{紧}}{=} \overline{f(G)} \supseteq f(G) = \overline{D}$

$\Rightarrow f$ 同胚 (f^{-1} 连续)

「单射另证, 计算办法. 若 $\zeta_1, \zeta_2 \in \partial G,\ f(\zeta_1) = f(\zeta_2) = -1$



设曲线 $\gamma_1, \gamma_2 \quad \gamma_1(1) = \zeta_1,\ \gamma_2(1) = \zeta_2$

$\exists\, c \in [0,1]$ s.t. $|\gamma_1(t_1) - \gamma_2(t_2)| \ge \frac{1}{2}|\zeta_1 - \zeta_2|$

$\forall\, t_1 \in [c,1]$

NOTE. 这两根曲线的构造: $z_n \to \zeta_1$, γ_1 经过 $z_n \cdots$ $\gamma_i(t) \in G$ $t \in [0,1)$

$\Rightarrow$ for $([0,c])$ 紧 $\exists \delta > 0$ s.t $\left(B(-1,\delta) \cap B(0,1)\right) \cap \left(\gamma_1([0,c]) \cup \gamma_2([0,c])\right) = \phi$

记 相交部分为 $A(\delta)$, $A(\delta)$ 中的点以极坐标表示为 $0 < r < \delta$.

$-\varphi(r) < \theta < \varphi(r)$. $\varphi(r) = \arccos \frac{r}{2} \leq \frac{\pi}{2}$

$$\text{Area}\left(f^{-1}(A(\delta))\right) = \iint_{A(\delta)} \left|(f^{-1})'(w)\right|^2 dw = \underbrace{\int_0^\delta \int_{-\varphi(r)}^{\varphi(r)} \left|(f^{-1})'(-1+re^{-i\theta})\right|^2 r\,d\theta\,dr}_{\text{估计这个积分}}$$

$\forall r \in (0,\delta)$. γ_1, γ_2 与 $|w+1| = r$ 交于 α_1, α_2

$$\Rightarrow \frac{1}{2}|\zeta_1 - \zeta_2| \leq |f^{-1}(\alpha_1) - f^{-1}(\alpha_2)| \leq \int_{\overset{\frown}{\alpha_1\alpha_2}} \left|(f^{-1})'\right|(w)\,dw \leq \int_{-\varphi(r)}^{\varphi(r)} \left|(f^{-1})'\right|(w)\,dw$$

Cauchy

$$\Rightarrow \frac{1}{4}|\zeta_1 - \zeta_2|^2 \leq \left(\int_{-\varphi(r)}^{\varphi(r)} \left|(f^{-1})'(w)\right| dw\right)^2 \leq \left(\int_{-\varphi(r)}^{\varphi(r)} \left|(f^{-1})'(-1+re^{i\theta})\right|^2 r^2 d\theta\right) \int_{-\varphi(r)}^{\varphi(r)} 1\, d\theta$$

$$\leq I \cdot \pi r$$

$$\Rightarrow I \geq \frac{|\zeta_1 - \zeta_2|^2}{4\pi r}$$ 两边从 0 到 δ 积分导出矛盾!

Appendix: $\text{Area}(f(u)) = \iint_u |f'(z)|^2 dx\,dy$

$f: (x,y) \mapsto (u,v)$

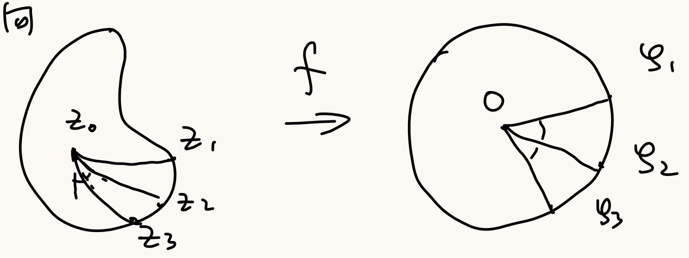
$$\text{Area}(f(u)) = \iint_{f(u)} du\,dv = \iint \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \overset{C-R}{=\!=} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$\left|\frac{\partial f}{\partial z}\right|^2 = \left|\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)\right|^2 = \frac{1}{4}\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right|$$

$$= \left|\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right|^2 = \frac{\partial(u,v)}{\partial(x,y)}$$

3. 保夹角



保角.

□

Fourier Transform

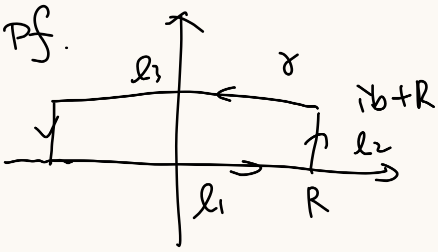
$$\mathfrak{F}_a = \left\{ f : f \text{ is holomorphic on } |\operatorname{Im} z| < a,\ |f(x+iy)| \le \frac{A}{1+x^2} \right\}$$

$$\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \qquad \xi \in \mathbb{R}$$

Thm. if $f \in \mathfrak{F}_a$, $a>0$, then $|\hat{f}(\xi)| \le B e^{-2\pi b|\xi|}$ for $\forall\, b \in [0,a)$

Pf.



$$\int_\gamma f(z) e^{-2\pi i z \xi} dz = 0$$

$$\int_{l_1} * = \int_{-R}^{R} f(x) e^{-2\pi i x \xi} dx$$

$$\int_{l_2} * = \int_0^b f(R+iy)\, e^{-2\pi i (R+iy)\xi}\, i\, dy$$

$$|I_2| \le \int_0^b \frac{A}{1+R^2} e^{2\pi y \xi} dy = \frac{A}{1+R^2} \cdot \frac{e^{2\pi b}-1}{2\pi \xi} \to 0 \quad \text{as } R \to \infty$$

$$\int_{l_3} * = \int_R^{-R} f(x+ib)\, e^{-2\pi i (x+ib)\xi} dx \qquad (*) \quad \text{appears later}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx = e^{2\pi b \xi} \int_{-\infty}^{\infty} f(x+ib)\, e^{-2\pi i x \xi} dx$$

$$\le e^{2\pi b \xi} \cdot \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx = B e^{-2\pi b |\xi|}. \quad (\xi < 0)$$

if we shift the real line down by $-b$ ... then we can get the case $\xi > 0$. $\square$

Thm. if $f \in \mathfrak{F}$ then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad x \in \mathbb{R}$.

Pf. $$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i (u-ib)\xi} du \qquad (\xi > 0)$$

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} = \underbrace{\int_0^{+\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi}_{I_1} + \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$I_1 = \int_0^{+\infty} \int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i (u-ib)\xi}\, e^{2\pi i x \xi} du\, d\xi \quad \text{(obviously converge)}$$

$$\overset{\text{fubini}}{=} \int_{-\infty}^{\infty} f(u-ib)\,du \int_0^{+\infty} e^{-2\pi i(u-ib-x)\xi}\,d\xi$$

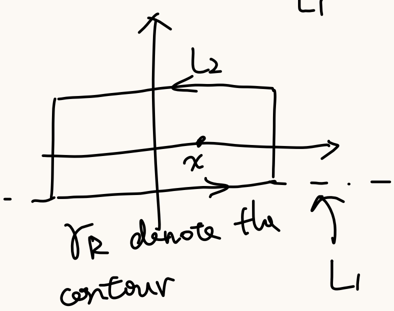
NOTE. $\underbrace{\int_0^{+\infty} e^{-2\pi i(u-ib-x)\xi}\,d\xi}_{\text{converge}} = \lim\limits_{R\to\infty} \int_0^R e^{-2\pi i(u-ib-x)\xi}\,d\xi$

$$= \lim_{R\to\infty} \frac{e^{-2\pi i(u-ib)R}-1}{-2\pi i(u-x-ib)} = \frac{1}{2\pi i(u-x-ib)}$$

$$\Rightarrow I_1 = \int_{-\infty}^{\infty} \frac{f(u-ib)}{2\pi i(u-x-ib)}\,du = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(u-ib)}{u-ib-x}\,du$$

$$= \frac{1}{2\pi i}\int_{L_1} \frac{f(\zeta)}{\zeta - x}\,d\zeta$$

for $\xi<0$, $\int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = \frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta-x}\,dx$



γ_R denote the contour

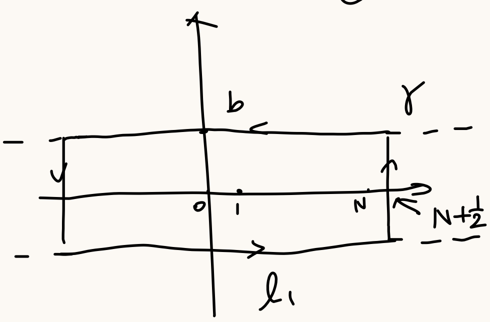
$$f(x) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)}{\zeta - x}\,d\zeta = \int_{L_1} + \int_{L_2} \cdots$$

as $R\to\infty$, the integral over the vertical sides goes to 0. $\square$

我们唯一强的条件（相比于 moderate decrease）就是 $\mathbb{R}$"附近"（一致的 sense）的解析条件，由于 f 的光滑性 $\Rightarrow$ $\hat{f}$ 的衰减速度. 可以预见 $\hat{f}$ 的衰减.

Thm. if $f \in \mathfrak{F}$, then $\sum\limits_{n\in\mathbb{Z}} f(n) = \sum\limits_{n\in\mathbb{Z}} \hat{f}(n)$

Pf consider $\dfrac{f(z)}{e^{2\pi i z}-1}$ $\quad \mathrm{Res}\left(\dfrac{f(z)}{e^{2\pi i z}-1}, n\right) = \dfrac{f(n)}{2\pi i}$



$$\sum_{|n|\le N} \frac{f(n)}{2\pi i} = \sum_{n\le N} \mathrm{Res}\left(\frac{f(z)}{e^{2\pi i z}-1}, n\right) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

on the vertical sides $|e^{2\pi i z}-1| \geq |e^{2\pi(N+\frac{1}{2})}-1| \to \infty$

so the integral goes to 0 as $N\to\infty$

$$\int_{\ell_1 \cup \ell_3} * = \int_{-\infty}^{+\infty} \frac{f(x-ib)}{e^{2\pi i(x-ib)}-1}\,dx - \int_{-\infty}^{+\infty} \frac{f(x+ib)}{e^{2\pi i(x+ib)}-1}\,dx$$

since $e^{2\pi b} > 1$

$$\Rightarrow \frac{1}{e^{2\pi i z}-1} = \frac{1}{e^{2\pi i z}} \sum_{n=0}^{\infty} \left(e^{-2\pi i z}\right)^n$$

$$\Rightarrow \int_{l_1} * = \int_{l_1} f(z) \sum_{n=0}^{\infty} e^{-2\pi i (n+1) z} dz \overset{\text{order}}{=\!=} \sum_{n=0}^{\infty} \int_{l_1} f(z) e^{-2\pi i (n+1) z} dz$$

$$= \sum_{n=1}^{\infty} \hat{f}(n)$$

$$\int_{l_3} * = \int_{l_3} f(z) \sum_{n=0}^{\infty} \left(e^{2\pi i z}\right)^n dz = \sum_{n=-\infty}^{0} \hat{f}(n)$$

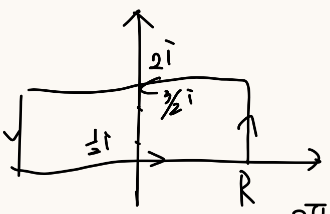
$$\Rightarrow \sum_{\mathbb{Z}} f(n) = \sum_{\mathbb{Z}} \hat{f}(n) \qquad \square$$

「Calculate two integrals

1. $$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2} \quad (\text{poisson 积分})$$

2. $$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx = \int_{-\infty}^{\infty} \frac{2 e^{-2\pi i x \xi}}{e^{-\pi x} + e^{\pi x}} dx$$

$e^{-\pi z} + e^{\pi z} = 0 \Rightarrow e^{2\pi z} + 1 = 0 \quad 2z = (2k+1) i \quad z = (k+\frac{1}{2}) i$



$$\left| \int_0^2 \frac{e^{-2\pi i (R+iy)\xi}}{e^{-\pi(R+iy)} + e^{\pi(R+iy)}} i\, dy \right| \le \int_0^2 \frac{e^{2\pi y \xi} dy}{e^{\pi R} - e^{-\pi R}} \to 0 \text{ as } R \to \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh x} dx - \int_{-\infty}^{\infty} \frac{e^{-2\pi i (x+2i)\xi}}{e^{-\pi(x+2i)} + e^{\pi(x+2i)}} dx = 2\pi i \left(\mathrm{Res}(\tfrac{1}{2} i) + \mathrm{Res}(\tfrac{3}{2} i) \right)$$

$$(1 - e^{4\pi \xi}) I = 2\pi i \left(\frac{e^{\pi \xi}}{\pi i} - \frac{e^{3\pi \xi}}{\pi i} \right)$$

$$\Rightarrow I = \frac{2(e^{\pi \xi} - e^{3\pi \xi})}{1 - e^{4\pi \xi}} = \frac{2 e^{\pi \xi}}{1 + e^{2\pi \xi}} = \frac{1}{\cosh \pi \xi}. \quad \dashv$$

Paley-Wiener

thm. $|\hat{f}(\xi)| \le A e^{-2\pi a |\xi|}$, $A, a > 0 \Rightarrow f$ can be extended on $S_b = \{ z : |\mathrm{Im}\, z| < b \}$ and the extension on S_b is holomorphic. $\forall b \in (0, a)$

pf. since $|\hat{f}(\xi)| \le A e^{-2\pi a |\xi|} \Rightarrow f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$

define $f_n(z) = \int_{-n}^{n} \hat{f}(\xi) e^{2\pi i z \xi} d\xi \overset{\text{cpt}}{\rightrightarrows} f(z) := \underbrace{\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} dz}_{\text{converge}}$

f_n holomorphic

So f is holomorphic $\square$

Thm. f is cts and $|f(x)| \le \frac{A}{1+x^2}$ on $\mathbb{R}$, then f has an extension to $\mathbb{C}$ that is entire with $|f(z)| \le Ae^{2\pi M|z|}$ for some $A>0$ iff and only if $\hat{f}$ is supported on $[-M, M]$

Pf. if $\hat{f}$ is cpt supp. let $a \to 0$ in lem "$\Leftarrow$" ✓

($\Rightarrow$) 1. first we assume f has stronger bound.

$$|f(x+iy)| \le A' \frac{e^{2\pi M|y|}}{1+x^2}$$, we plan to prove $\hat{f}(\xi) = 0$ ($|\xi|>M$)

when $\xi > M$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx \overset{\text{contour}}{=\!=} \int_{-\infty}^{\infty} f(x-iy) e^{-2\pi i (x-iy)\xi} dx$ ($y>0$)

$$\Rightarrow |\hat{f}(\xi)| \le \int_{-\infty}^{\infty} \frac{A' e^{2\pi M|y|}}{1+x^2} \cdot e^{-2\pi y \xi} dx \lesssim e^{-2\pi y(\xi - M)}$$

$\hat{f}(\xi) = 0$ as $y \to \infty$

2. $|f(x+iy)| \le e^{2\pi M|y|}$

$\ulcorner$ $\xi > M$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = \int_{-\infty}^{\infty} f(x-iy) e^{-2\pi i (x-iy)\xi} dx$

$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} e^{2\pi(\xi-M)|y|} dx$ we can deal with it as this $\lrcorner$

i.e. we need an assistant part to make the integral at least converged

let $f_\varepsilon(z) = \frac{f(z)}{(1+\varepsilon i z)^2}$ ($\operatorname{Im} z \le 0$, since we gonna use $\cdots$)

$$|f_\varepsilon(z)| \le \frac{Ae^{2\pi M|y|}}{|1-\varepsilon y + \varepsilon i x|^2} \le \frac{Ae^{2\pi M|y|}}{\varepsilon^2(1+x^2)} \Rightarrow \hat{f}_\varepsilon(\xi) = 0 \text{ as } \xi > M$$

and $|\hat{f}_\varepsilon(\xi) - \hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \left|\frac{1}{(1+i\varepsilon x)^2} - 1\right| dx$ using DCT $\cdots$

3. $|f(x)| \le 1$. $|f(z)| \le e^{2\pi M|z|}$

[We prove Phragham & Lindelöf Thm to use the inequalities above

$S = \{z : \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4})\}$, $|f(z)| \le 1$ on ∂S. $|f(z)| \le Ae^{c|z|}$ on S

$\Rightarrow |f(z)| \le 1$ on S.

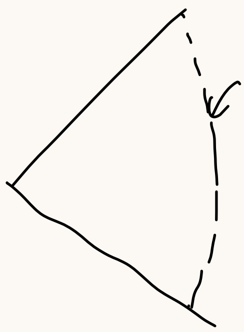
pf. "notice that e^{z^2} is a func not satisfies the conclusion since its increase speed is too high."

let $F(z) = f(z)e^{-\varepsilon z^{\frac{3}{2}}}$ there $z^{\frac{3}{2}} = e^{\frac{3}{2}\log z} = |z| e^{\frac{3}{2} i\theta}$ $\theta = \arg z$

$\Rightarrow |e^{-\varepsilon z^{\frac{3}{2}}}| = e^{-\varepsilon |z|^{\frac{3}{2}} \cos(\frac{3}{2}\theta)}$ $|F(z)| \le Ae^{cz - \varepsilon z^{\frac{3}{2}}} \to 0$

$-\frac{\pi}{2} < \frac{3}{2}\theta < \frac{3\pi}{8} < \frac{\pi}{2}$

$\Rightarrow$ when $|z|$ is large enough ($|z| \ge R$, $F(z)$



$|F(z)| \le 1$

$\Rightarrow |F(z)| \le 1$ on S

$\Rightarrow |f(z)| \le e^{\varepsilon |z|^{\frac{3}{2}} \cos(\frac{3}{2}\theta)}$ $\varepsilon \to 0$

$\Rightarrow |f(z)| \le 1$ on S $\square$

so, using this thm. let $g(z) = f(z)e^{-2\pi M y}$ for $y > 0$

$\Rightarrow |g(z)| \le 1$ on $x = 0$ or $y = 0$

$|g(z)| \le e^{2\pi M(|z| - y)} \le e^{2\pi M|z|}$

$\Rightarrow |g(z)| \le 1$ on $x > 0, y > 0$ $\Rightarrow |f(z)| \le e^{2\pi M y}$

In total. $|f(z)| \le e^{2\pi M|y|}$ . . . $\square$

Entire funcs   WLOG. the funcs in this chapter satisfies $f \not\equiv 0$.

Thm. $\overline{D_R} \subseteq \Omega$. $f \in H(\Omega)$, $f(0) \neq 0$. $f(z) \neq 0$ for $z \in C_R$, $z_1, \cdots z_N$ denote the zeros of f inside D_R (counted with multiplicities), then

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta \quad (*)$$

Idea. We have known $\log|f|$ is harmonic, then the formula above is mean value formula

if $f = z - z_k$. $z_k \in D_R \setminus \{0\}$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\log|f(Re^{i\theta})| - \log|f(0)|\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\log|R| + \log\left|e^{i\theta} - \frac{z_k}{R}\right| - \log|z_k|\right) d\theta$$

So we need to prove " $\frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - \frac{z_k}{R}\right| d\theta$ "

Pf. ① if f_1, f_2 satisfy Jensen formula $\Rightarrow f_1 f_2$ does so.

② $\forall f$. suppose $z_1, \cdots z_N$ are all the zeroes of f (they may denote many the same points)

$$\Rightarrow g(z) := \frac{f(z)}{(z-z_1)\cdots(z-z_N)}$$ it's a holomorphic func (each z_j is removable)

hence we write $f(z) = g(z)\prod_{i=1}^{N}(z - z_i)$, $g(z)$ vanishes nowhere on D_R

Obviously. g satisfies $(*)$

Now prove $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - a|\, d\theta = 0$ for $\forall\, |a| < 1$

$$LHS = \frac{1}{2\pi}\int_0^{2\pi} \log|1 - ae^{-i\theta}|\, d\theta$$ and nowhere vanishing.

$1 - az$ is holomorphic on $\overline{B(0,1)}$ □

let $n(r)$ denote the number of zeroes of f (counted with multiplicities)

lem if f satisfies the condition of last theorem. then

$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta - \log|f(0)|.$$

Pf. what we need to prove is the formula below.

$$\int_0^R n(r)\frac{dr}{r} = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|.$$

consider characteristic func $\eta_k(r) = \begin{cases} 0 & r \le |z_k| \\ 1 & r > |z_k| \end{cases}$

$$\Rightarrow n(r) = \sum_{k=1}^N \eta_k(r)$$

$$\Rightarrow \int_0^R n(r)\frac{dr}{r} = \int_0^R \sum_{k=1}^N \eta_k(r) = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r} = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right| \qquad \square$$

How does a function behave at ∞?

let f be an entire function, if there exists $\rho > 0$. A, B constant > 0, s.t.

$$|f(z)| \le A e^{B|z|^\rho} \quad \text{for all } z \in \mathbb{C}$$

then we say f has an order of growth $\le \rho$. define the order of growth of f as $\rho_f = \inf \rho$.

Thm oog of $f \le \rho$

(i) $n(r) \le C r^\rho$ for some $C > 0$. $\forall r$ large sufficiently

(ii) if $z_1, \cdots$ denote zeros of f with $z_k \ne 0$ then all $s > \rho$

$$\sum_{k=1}^\infty \frac{1}{|z_k|^s} < \infty$$

Pf. (i) $\log|f| \le \log A + B|z|^\rho \lesssim |z|^\rho$

NOTE the $F(z) := \frac{f(z)}{z^\ell}$ $\quad (\ell = \text{order of } 0 \text{ if } f(0) = 0)$

then $n_F(z)$ & $n_f(z)$ differ by a constant

hence, WLOG, $f(0) \ne 0$.

$$\int_0^R n(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

$$\Rightarrow \int_r^{2r} n(x)\frac{dx}{x} \le \frac{1}{2\pi}\int_0^{2\pi}\log|f(2re^{i\theta})|\,d\theta - \log|f(0)|$$

$$\int_r^{2r} n(x)\frac{dx}{x} \geq n(r)\int_r^{2r}\frac{dx}{x} = n(r)\log 2$$

$$\Rightarrow n(r)\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi}\log\left|Ae^{B|2r|^\rho}\right| d\theta - \log|f(0)|$$

$$\lesssim c\, r^\rho$$

(ii) in $B(0,1)$ at most contains finite zeroes of f

$$\sum_{|z_k|\geq 1}|z_k|^{-s} = \sum_{j=0}^{\infty}\left(\sum_{2^j\leq|z_k|<2^{j+1}}|z_k|^{-s}\right) \qquad (\rho-s)<0$$

$$\leq \sum_{j=0}^{\infty} 2^{-js} n(2^{j+1}) \leq c\sum_{j=0}^{\infty} 2^{-js}\, 2^{(j+1)\rho} < \infty$$

NOTE. $s>\rho$ can't improve

$$f(z) = \sin\pi z = \frac{e^{i\pi z}-e^{-i\pi z}}{2} \qquad |f(z)|\leq e^{\pi|z|} \qquad \rho=1 \quad (z=ix)$$

$$\sum_{n\neq 0}\frac{1}{|n|^s} < \infty \quad \text{if} \quad s<1$$

fixed $\{a_k\}$ want to construct an entire function f s.t $f=0 \Leftrightarrow z=a_k$.

WLOG. $a_k\neq 0$. $\{a_k\}$ has no limit point

Idea $\quad f = \prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{g_k}$

define $\prod_{n=1}^{\infty}(1+a_n) := \lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$ (allow the limit $=0$)

prop if $\sum|a_n|<\infty$. then $\prod_{n=1}^{\infty}(1+a_n)$ converge & the limit $=0$ iff $a_n=0$

(AC)

Pf. $\sum|a_n|<\infty \Rightarrow |a_n|<\frac{1}{2}$ for n large enough

$b_n = \log(1+a_n) \quad B_N = \sum_{n=1}^{N} b_n$

$$\left|\log(1+z)\right| = \left|z-\frac{z^2}{2}+\frac{z^3}{3}\cdots\right| \leq 2|z| \text{ if } |z|<\frac{1}{2}$$

Hence $|b_n|\leq 2|a_n| \Rightarrow \sum b_n$ converge $\Rightarrow \lim_{N\to\infty} B_N = B$ $\quad e^B$ $\quad\square$

prop. $\{F_n\}$ is a sequence of holomorphic functions on Ω. if $\exists\, C_n>0$ s.t

$\sum C_n<\infty$. $|F_n(z)-1|\leq C_n$ on Ω

then ① $\prod_{n=1}^{\infty} F_n(z) \rightrightarrows F(z)$ on Ω

② if $F_n(z) \neq 0$ then $\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$

Pf. ① $\prod_{n=1}^{\infty}\left((F_n(z)-1)+1\right)$ . . .

② $\forall K \subset\subset \Omega$

$G_n = \prod_{k=1}^{n} F_k \rightrightarrows F \Rightarrow G_n' \rightrightarrows F' \Rightarrow \frac{G_n'(z)}{G_n(z)} \to \frac{F'(z)}{F(z)}$ on K

$LHS = \sum_{k=1}^{n} \frac{F_k'(z)}{F_k(z)} \to \frac{F'(z)}{F(z)}$ on K

lem. if f is an entire function, $f \neq 0$ then $\exists$ an entire function g s.t.

$f = e^g$

Pf. $\frac{f'}{f}$ is entire $\quad g(z) := \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)}\, d\xi + C_0 \quad e^{C_0} = f(z_0)$

$g(z)$ is well-defined & $g'(z) = \frac{f'(z)}{f(z)}$

$\Rightarrow (fe^{-g})' = f'e^{-g} - fe^{-g}g' = (f' - fg')e^{-g} = 0$

$\Rightarrow fe^{-g} = f(z_0)e^{-g(z_0)} = e^{C_0}e^{-C_0} = 1$

$\Rightarrow f = e^g$ □

Thm. given any $\{a_n\} \subseteq \mathbb{C}$ with $|a_n| \to \infty$ as $n \to \infty$. there exists an entire function exactly vanishes at those point. and all these functions has the form $f(z)e^{g(z)}$ (g is entire)

Pf. ①. f_1, f_2 vanish exactly at $\{a_k\}$ $\Rightarrow \frac{f_1}{f_2}$ is entire and vanishing nowhere

$\Rightarrow \frac{f_1}{f_2} = e^g \Rightarrow f_1 = f_2 e^g$

② construct f let $E_0(z) = 1 - z \quad E_k(z) = (1-z)e^{\left(z + \cdots + \frac{z^k}{k}\right)}$ (Canonical factor)

lem. if $|z| \le \frac{1}{2}$ then $|1 - E_k(z)| \le c|z|^{k+1}$

$E_k = e^{\log(1-z) + z + \cdots + z^k/k} = e^w$

NOTE $\log(1-z) = -\left(z + \cdots + \frac{z^k}{k} + \cdots\right) \Rightarrow w = -\left(\sum_{n=k+1}^{\infty} \frac{z^n}{n}\right)$

$$|w| \le |z|^{k+1} \sum_{n=k+1}^{\infty} |z|^{n-k-1}/n \le |z|^{k+1} \sum_{j=0}^{\infty} 2^{-j} \le 2|z|^{k+1} \le 1$$

$$\Rightarrow |1-e^{w}| \le c'|w| \le c|z|^{k+1} \quad \left(\text{more precisely. } |1-e^{w}| \le e|w| \le 2e|z|^{k+1} \right)$$

define $f(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n)$ $\quad a_n \ne 0 \quad$ m is the order of 0

suppose $z \in \mathbb{C}$. $|z| < R$ when $|a_n| > 2R \Rightarrow \left|\frac{z}{a_n}\right| < \frac{1}{2}$

$\Rightarrow \left|1 - E_n \frac{z}{a_n}\right| \le c|z|^{k+1} \Rightarrow f(z)$ converge uniformly in $|z| < R$

$\Rightarrow f$ is entire

$\square$

Thm (Hadamard) f is entire and has growth order ρ_0, let k be the integer that $k \le \rho_0 < k+1$, if $a_1, a_2, \ldots$ denote the (non-zero) zeroes of f, then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n) \qquad \deg p \le k$$

Pf. WLOG. $f(0) \ne 0$ $\quad \left(f(z) \overset{?}{=} e^{p(z)} \prod_{n=1}^{\infty} E_k(z/a_n), \quad E_k = (1-z)e^{z + \cdots + z^k/k} \right)$

$\dfrac{f}{\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})}$ has no zeroes $\Rightarrow \dfrac{f}{\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})} = e^{p}$ $\quad$ i.e suffices to prove p is a polynomial

estimate ~~its~~ increase order upper bound of e^p
i.e the lower bound of $\prod_{n=1}^{\infty} E_n(\frac{z}{a_n})$

① $E_k(z) \ge e^{-c|z|^{k+1}}$ $\qquad |z| \le \frac{1}{2}$

$$E_k(z) = e^{-\sum_{n=k+1}^{\infty} z^n/n} = e^{w} \Rightarrow |E_k(z)| \ge e^{-|w|} \ge e^{-c|z|^{k+1}}$$

② $E_k(z) \ge |1-z| e^{-c'|z|^k}$ $\qquad |z| \ge \frac{1}{2}$

$$|E_k(z)| = |1-z| e^{-|z + \cdots + z^k/k|} \ge |1-z| e^{-c'|z|^k}$$

the proof has two steps below

step 1. $\exists \{r_n\}$ $r_n \to \infty$ s.t. $\prod_{n=1}^{\infty} E_n(\frac{z}{a_n}) \ge e^{-B|z|^s}$, $s > \rho_0$ holds on $|z| = r_i$

then $|f| \dfrac{1}{\left|\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})\right|} = e^{\operatorname{Re} p} \le e^{c|z|^s} \Rightarrow \operatorname{Re} p \le c|z|^s$ $(*)$

Step 2. $P \in H(\mathbb{C})$ satisfies $(*)$ on $|z| = r_n$ must be polynomial & $\deg P \le s$

the proof of step 2. : $P(z)$ is entire

$\Rightarrow P(z) = \sum_{n=0}^{\infty} b_n z^n$ use cauchy integral formula

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} p(re^{i\theta}) e^{-in\theta} d\theta = \begin{cases} b_n r^n, & n \ge 0 \\ 0 & n < 0 \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \overline{p(re^{i\theta})} e^{-in\theta} d\theta = \begin{cases} \bar{b}_n r^n & n \le 0 \\ 0 & n > 0 \end{cases}$$

$$b_n = \frac{1}{\pi r^n}\int_0^{2\pi} \operatorname{Re} P(re^{i\theta}) e^{-in\theta} d\theta$$

$$= \frac{1}{\pi r^n}\int_0^{2\pi} [\operatorname{Re} P(re^{i\theta}) - c|r|^s] e^{-in\theta} d\theta$$

$$|b_n| \le \frac{-1}{\pi r^n}\int_0^{2\pi} (\operatorname{Re} P(re^{i\theta}) - cr^s) d\theta$$

$$= 2r^{s-n} - 2\operatorname{Re} P(0) \cdot r^{-n} \qquad n > s, \ r \to \infty \Rightarrow b_n = 0$$

the proof of step 1.

for Any $\rho_0 < s < k+1$. we have $\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \ge e^{-c|z|^s}$ on $\mathbb{C} \setminus \overline{\bigcup_{n=1}^{\infty} B(a_n, |a_n|^{-k-1})}$

$$\prod_{n=1}^{\infty} E_k(z/a_n) = \prod_{|a_n| \le 2|z|} \cdot \prod_{|a_n| > |z|}$$

$$\left|\prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right)\right| \ge \prod_{|a_n| > 2|z|} e^{-c|z/a_n|^{k+1}} = e^{-c|z|^{k+1}\sum_{|a_n|>2|z|} |a_n|^{-(k+1)}}$$

$$|a_n|^{-k-1} = |a_n|^{-s} |a_n|^{s-k-1} = \underbrace{C|a_n|^{-s}}_{} \cdot |z|^{s-k-1}$$

$\sum |a_n|^{-s}$ converge

$$\Rightarrow \left|\prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right)\right| \ge e^{-c|z|^s}$$

$$\left|\prod_{|a_n| \le 2|z|} E_k\left(\frac{z}{a_n}\right)\right| \ge \left|\prod_{|a_n| \le 2|z|}^{\infty} \left(1 - \frac{z}{a_n}\right)\right| \left|\prod_{|a_n| \le 2|z|} e^{-c'|z/a_n|^k}\right| \ge \left|\prod_{|a_n| \le 2|z|} \left(1 - \frac{z}{a_n}\right)\right| \cdot e^{-c'|z|^s}$$

$$\prod_{|a_n| \le 2|z|} e^{-c'|z/a_n|^k} = e^{-c'|z|^k \sum_{|a_n| \le 2|z|} |a_n|^{-k}}$$

$$|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \le C|a_n|^{-s}|z|^{s-k}$$

NOTE than $z \notin \bigcup_{n=1}^{\infty} \overline{B(a_n, |a_n|^{-k-1})} \Rightarrow |z-a_n| > |a_n|^{-k-1}$

$$\Rightarrow \left| \prod_{|a_n| \le 2|z|} \left(1-\frac{z}{a_n}\right) \right| = \prod_{|a_n| \le 2|z|} \frac{|a_n - z|}{|a_n|} > \prod_{|a_n| \le 2|z|} \frac{|a_n|^{-k-1}}{|a_n|} = \prod_{|a_n| < 2|z|} |a_n|^{-(k+2)}$$

$$\sum_{|a_n| \le 2|z|} (k+2) \log |a_n| \le (k+2)\, n(2|z|) \log(2|z|)$$

$$\overset{(s>\rho_0)}{\le} (k+2) |z|^s \log(2|z|)$$

$$\le C |z|^{s'}$$

in total, the lemma holds ~~true~~.

since $\sum \frac{1}{|a_n|^{k+1}}$ converge, we can take a sequence of $\{r_i\}$ $r_i \to \infty$. $\square$