特殊模型

1. 可分、伯努利、里卡蒂 —解不能解?

2. 解的延拓 —存在性说了吗?

3. 较难的证明. SL问题、线性近似 稳定性

4. 全轴相图 极限环问题

5. 非自治 线性 稳 & 不稳

自治 线性 on lyapunov

一阶偏微分方程.

$$\begin{cases} \frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = f(x,t) & -\infty < x < +\infty \\ u(x,0) = \phi(x) \end{cases}$$

特征线法
$$\begin{cases} \frac{dx}{dt} = a(x(t), t) \\ x(0) = c \end{cases}$$

$$U(t) = u(x(t), t)$$

$$\frac{dU}{dt} = \partial_x u \frac{dx}{dt} + \partial_t u$$
$$= \left(a(x(t), t)\partial_x u\right)(x(t), t) + \partial_t u$$
$$= -b(\ )U(t) + f \cdots$$

例
$$\begin{cases} \partial_t u - a\partial_x u = f \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} \frac{dx(t)}{dt} = -a \\ x(0) = c \end{cases}$$

$$x = -at + c$$

$$U(t) = u(c-at, t)$$

$$\frac{dU}{dt} = \left(-a\partial_x u + \partial_t u\right)(c-at, t)$$
$$= f(c-at, t)$$

$$U(0) = u(c, 0) = \phi(c)$$

$$U(t) - \phi(c) = \int_0^t f(c-as, s)\,ds$$
$$u(c-at, t) = \phi(c) + \int_0^t f(c-as, s)\,ds$$
$$\Rightarrow u(\underbrace{c-at}_{x}, t)$$
$$u(x,t) = \phi(x+at) + \int_0^t f(x+a(t-s), s)\,ds$$

例2. $\begin{cases} \frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} + u = x. \\ u(x,0) = x \end{cases}$

$\begin{cases} \frac{dx}{dt} = x(t) + t \\ x(0) = c \end{cases}$ $x = ce^t + e^t - t - 1.$

$U(t) = u(ce^t + e^t - t - 1, t)$

$\cancel{\frac{dU}{dt} = ((ce^t+e^t-1)u + \partial_t u)(t)}$

$\Rightarrow U(t) = -t + \frac{1}{2}(e^t - e^{-t}) + \frac{c}{2}(e^t - e^{-t})$

$\parallel$

$u(x(t), t)$

$ce^t + e^t - t - 1 = x$

$c = \cdots$

波动方程 $\partial_t^2 u - Au = f(x,t)$ $t \in \mathbb{R}$ $x \in \Omega \subseteq \mathbb{R}^n$

初值 $u(x,0) = \phi(x)$, $\partial_t u(x,0) = \psi(x)$

边值 Dirichlet 边值 $u(x,t) = g(x,t)$. $\forall x \in \partial\Omega, t \in I.$

Neumann 边值 $\frac{\partial u}{\partial n}(x,t) = g(x,t)$ $\forall x \in \partial\Omega$ $t \in I$

求解 Robin 边值 $\frac{\partial u}{\partial n} + \alpha(x,t) u = g(x,t)$ $\forall x \in \partial\Omega$. $t \in I$.

4.1. $\tilde{u}_2 = M_\psi(x,t)$ 令 $\tilde{u} = M_\psi(x,t)$

$\cancel{\frac{\partial}{\partial t} M_\psi = \psi}$

$\partial_t^2 \tilde{u} - \Delta \tilde{u} = 0$

$\tilde{u}|_0 = 0$ $\partial_t u|_0 = \psi$

令 $V = \partial_t \tilde{u}$

$\partial_t V = \partial_t^2 \tilde{u} = 0$

$\partial_t^2 V = 0$

$\tilde{u} = M_{f_\tau}(x,t)$

$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = 0 \\ \tilde{u}(x,0) = 0 \quad \partial_t \tilde{u} = f_\tau(x) \end{cases}$

$V(x,t) = \int_0^t M_{f_\tau}(x, t-\tau)\, d\tau$

$\partial_t V = \int_0^t \partial_t \tilde{u}(t-\tau)\, d\tau$

$\partial_t^2 V = \int_0^t \partial_t^2 \tilde{u}(t-\tau)\, d\tau + f$

$AV = \int_0^t \Delta \tilde{u}\, d\tau$

$\Rightarrow \partial_t^2 V - AV = f$

$\Delta u = \frac{1}{r^2}\partial_r(r^2\partial_r u) + \frac{1}{r^2}\Delta_{S^2} u$

$n=3$

$$\begin{cases} \partial_t^2 u - \Delta_{\mathbb{R}^3} u = 0 \\ u(x,0) = 0 \\ \partial_t u(x,0) = \psi(x) \end{cases}$$

令 $\bar{u} = \frac{1}{4\pi}\int_{S^2} u(x, r\omega)\, dS(\omega)$

$$\Rightarrow \begin{cases} \partial_t^2 \bar{u} - \partial_r^2 \bar{u} - \frac{2}{r}\partial_r \bar{u} = 0 \\ \bar{u}(r,0) = 0 \\ \partial_t \bar{u}(r,0) = \bar{\psi}(r) = \frac{1}{4\pi}\int_{S^2} \psi(r\omega)\, dS(\omega) \end{cases}$$

令 $V(t,r) = r\bar{u}(r,t)$

$\Rightarrow \partial_t^2 V - \partial_r^2 V = 0$ $r>0$

$V(r,0)$, $\partial_t V(r,0) = r\bar{\psi}(r)$

偶延拓

Step 1.

$u(0,t) = \bar{u}(0,t) = \partial_r(r\bar{u})\Big|_{r=0}$

代入 V 的 D'A 公式

$\Rightarrow u(0,t) = t\bar{\psi}(t)$.

Step 2. 平移不变性 $\tilde{u} = u(x + x_0, t)$

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta_{\mathbb{R}^3} \tilde{u} = 0 \\ \tilde{u}(x,0) = 0 \quad \partial_t \tilde{u}(x,0) = \psi(x+x_0) \triangleq \tilde{\psi}(x) \end{cases}$$

$t\bar{\tilde{\psi}}(t) = t\frac{1}{4\pi}\int_{S^2} \tilde{\psi}(t\omega)\, dS(\omega)$

$\tilde{u}(0,t) = u(x_0,t)$. $\longrightarrow \frac{1}{4\pi t}\int_{|y-x_0|=t} \psi(y)\, dS(y) = \frac{t}{4\pi}\int_{|\omega|=1} \psi(x+t\omega)\, dS(\omega)$

$dS(y) = t^2\, dS(\omega)$

即 $u(x_0,t)=\frac{1}{4\pi t}\int_{|y-x_0|=t}\psi(y)\,dS(y)$ $\boxed{\forall x_0}$

$$\Rightarrow u(x,t)=\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y).$$

由之前定理 $\begin{cases}\partial_t^2 u-\Delta_{\mathbb{R}^3}u=f(x,t)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x).\end{cases}$

$$u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|x-y|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$

$$+\int_0^t\frac{1}{4\pi(t-\tau)}\int_{|y-x|=t-\tau}f(y,\tau)\,dS(y)\,d\tau.\qquad \#$$

$n=2$ 升维.

$u(x_1,x_2,t)$: $\begin{cases}\partial_t^2 u-\Delta_{\mathbb{R}^2}u=f(x,t)\\ u(x_0,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x).\end{cases}$

令 $\tilde{u}=\tilde{u}(x_1,x_2,x_3,t)=u(x_1,x_2,t)$

$\rightsquigarrow\begin{cases}\partial_t^2\tilde{u}-\Delta_{\mathbb{R}^3}\tilde{u}=0\\ \tilde{u}(\tilde{x},0)=\tilde{\varphi}(\tilde{x}),\ \partial_t\tilde{u}(\tilde{x},0)=\tilde{\psi}(\tilde{x}).\end{cases}$ $\tilde{x}=(x_1,x_2,x_3)$

先设 $\tilde{f}(x,t)$, $\tilde{\varphi}(\tilde{x})=0$ 由 Kirchoff 公式.

$\tilde{u}(x,t)=\frac{1}{4\pi t}\int_{|y-x|=t}\tilde{\psi}(\tilde{y})\,dS(\tilde{y})$. 由 $\tilde{u}$ 与 x_3 无关 令 $x_3=0$.

令 $x_1,x_2=0\Rightarrow\tilde{u}(0,t)=u(0,t)$

$$\frac{1}{4\pi t}\int_{|y|=t}\tilde{\psi}(y)\,dS(\tilde{y}).=\frac{2}{4\pi t}\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}}\tilde{\psi}(y_1,y_2)\,dS(\tilde{y}).$$

$$\sqrt{1+\left(\frac{\partial y_3}{\partial y_1}\right)^2+\left(\frac{\partial y_3}{\partial y_2}\right)^2}=\frac{t}{\sqrt{t^2-y_1^2-y_2^2}}$$

$$=\frac{2}{4\pi t}\int_{y_1^2+y_2^2\le t}\frac{\psi(y_1,y_2)\,t}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1\,dy_2$$

$$=\frac{1}{2\pi}\int_{|y|\le t}\frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1\,dy_2.$$

$V(x,t) = u(x+x_0, t)$ 则

$$\begin{cases} \partial_t^2 V - \Delta_{\mathbb{R}^2} V = 0 \\ V(x,0) = 0 \quad \partial_t V(x,0) = \psi(x+x_0) \end{cases}$$

$$u(x_0,t) = V(0,t) = \frac{1}{2\pi}\int_{|y|\le t} \frac{\psi(x_0+y)}{\sqrt{t^2-|y|^2}}\,dy_1\,dy_2$$

$$= \frac{1}{2\pi}\int_{|y-x_0|\le t} \frac{\psi(y)}{\sqrt{t^2-|y-x_0|^2}}\,dy_1\,dy_2$$

$$V(x,t) = \frac{1}{2\pi}\int_{|y-x|\le t} \frac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\,dy_1\,dy_2$$

一般形式 . . .

P179. 例) 代入 2

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) \\ u(x,0) = \varphi(x) \\ \partial_t u(x,0) = \psi(x) & 0 \le x \le l \\ u(0,t) = g_1(t) & t \ge 0 \\ u(l,t) = g_2(t) \end{cases}$$

① $g_1 = g_2 = f \equiv 0$

代入为 $\sum_{n=1}^{\infty} T_n''(t) X_n(x) - \sum_{n=1}^{\infty} T_n(t) X_n''(x) = 0$

$$\sum_{n=1}^{\infty} T_n(0) X_n(x) = \varphi(x)$$

$$\sum_{n=1}^{\infty} T_n'(0) X_n(x) = \psi(x)$$

边值已自动满足

又 $\boxed{-X_n''(x) = \lambda X_n(x)}$

$$\Rightarrow \sum_{n=1}^{\infty} T_n''(t) X_n(x) + \sum_{n=1}^{\infty} T_n(t)\,\lambda_n X_n(x) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(T_n''(t) X_n(x) + T_n(t)\,\lambda_n X_n(x)\right) = 0$$

做 L^2 内积 $\Rightarrow T_{n_0}''(t)(X_{n_0}, X_{n_0}) + T_n(t)\,\lambda_n (X_{n_0}, X_{n_0}) = 0$

$$\Rightarrow \boxed{T_{n_0}''(t) + \lambda_{n_0} T_{n_0}(t) = 0.}$$

$$T_n(t)(X_n, X_n) = (\varphi, X_n)$$

$$\Rightarrow \begin{cases} T_n(0) = \dfrac{(\varphi, X_n)}{(X_n, X_n)} \\ T_n'(0) = \dfrac{(\psi, X_n)}{(X_n, X_n)} \end{cases}$$

可以解…

设 $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \begin{cases} T''(t)\,X(x) - T(t)\,X''(x) = 0 \\ T(t)\,X(0) = 0 \quad T(t)\,X(L) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda. \\ X(0) = 0 \quad X(L) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos\frac{n\pi}{L}t + \frac{L}{n\pi}\psi_n \sin\frac{n\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

这表明考虑 SL 是自伴的.

若 $\lambda < 0$. $\begin{cases} X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ C_1 + C_2 = 0 \end{cases} \Rightarrow X(x) = 0$

$\lambda = 0 \quad X(x) = C_1 x + C_2 \Rightarrow X(x) = 0$

$\lambda > 0. \quad X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$C_1 = 0$

$X(L) = C_2 \sin(\sqrt{\lambda}L) = 0$

$\sqrt{\lambda}L = k\pi \quad \Rightarrow \lambda_n = \left(\dfrac{n\pi}{l}\right)^2.$

$$\boxed{X_n = \sin\frac{n\pi}{l}x. \quad n = \cdots}$$ 特征函数

$\|X_n\| = \dfrac{L}{2}. \quad T_n(0) = \dfrac{2}{L}\displaystyle\int_0^L \varphi \sin\left(\frac{n\pi}{l}x\right)dx = \varphi_n$

$T_n'(0) = \psi_n.$

$$\Rightarrow \begin{cases} T_n''(x) + \lambda T_n(x) = 0 \\ T_n(0) = \varphi_n \quad T_n'(0) = \psi_n \end{cases} \Rightarrow T_n = \varphi_n \cos\frac{n\pi}{L}t + \frac{\psi_n L}{n\pi}\sin\frac{n\pi}{L}t.$$

② $f(x,t) \not\equiv 0 \quad g_1 = g_2 = 0$

$\Rightarrow (T_n'' + \lambda T_n) X_n = f$

$$\begin{cases} T_n'' + \lambda T_n = \dfrac{(f, X_n)}{(X_n, X_n)} \\ T_n(0) = \varphi_n \quad T_n'(0) = \psi_n . \end{cases}$$

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③ 带非0边值

令 $V = u(x,t) - \left(\frac{l-x}{l} g_1(t) + \frac{x}{l} g_2(t) \right)$ 为零边值.

$\partial_t^2 V - \partial_x^2 V = -\left(\frac{l-x}{l} g_1''(t) + \frac{x}{l} g_2''(t) \right) + f$

$V(x,0) = \varphi(x) - \left(\frac{l-x}{l} g_1(0) + \frac{x}{l} g_2(0) \right)$

$\partial_t V(x,0) = \psi(x) - \left(\frac{l-x}{l} g_1'(0) + \frac{x}{l} g_2'(0) \right).$

例. $\begin{cases} \partial_t u - \partial_x^2 u = 0 & 0 \le x \le l \quad t \ge 0 \\ u(x,0) = \varphi(x) \\ u(0,t) = 0 & u_x(l,t) + h u(l,t) = 0 \end{cases} \qquad h > 0$

解 Step 1 求特征函数.

设 $u(x,t) = X(x) T(t)$

$\Rightarrow \begin{cases} T'(t) X(x) - T(t) X''(x) = 0 \\ T(t) X(0) = 0 \quad X'(l) T(t) + h\, T(t) X(l) = 0 \end{cases}$

$\Rightarrow \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$

$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(l) + h X(l) = 0 \end{cases}$

若 $\lambda < 0$. $\quad X(x) = 0$

$\lambda = 0 \quad X(x) = 0$

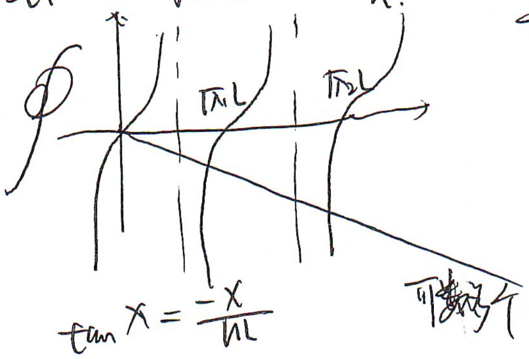
$\lambda > 0 \quad X(x) = C_1 \cos\sqrt{\lambda} x + C_2 \sin\sqrt{\lambda} x$

$\tan(\sqrt{\lambda}\, l) = -\dfrac{\sqrt{\lambda} l}{hl}$

$\Rightarrow X(x) = \sin(\sqrt{\lambda} x)$

$\Rightarrow \exists\ 0 < \lambda_1 < \lambda_2 < \cdots$

st. $\tan\sqrt{\lambda}L = -\frac{\sqrt{\lambda_n}}{h}$. 取 $X_n(x) = \sin\sqrt{\lambda_n}x$.



Step 2. $\frac{T_n'(t)}{T_n(t)} = -\lambda_n$

$\sum T_n(0)X_n(x) = \varphi(x)$

$\cancel{\sum T_n(0)}$

$$T_n(0) = \frac{(\varphi_n(x), X_n)}{(X_n, X_n)} \triangleq \varphi_n$$

$T_n(t) = \varphi_n e^{-\lambda_n t}$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda t}\varphi_n \sin\sqrt{\lambda_n}x.$$

例 $B = \{(x,y) \mid x^2+y^2<1\}$. B上的方程.

$$\begin{cases}\Delta u = 0 & \text{in } B \\ u = \varphi & \text{on } \partial B\end{cases}$$

极坐标$(r,\theta) \Rightarrow \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0$

令 $u(r,\theta) = R(r)\Theta(\theta)$.

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\left(R''(r) + \frac{1}{r}R'(r)\right)\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) = 0$$

选 Θ是因为有边值.

$$\Rightarrow \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} \triangleq \lambda. \quad \boxed{\lambda_k = k^2}$$

$$\Rightarrow \begin{cases}\Theta''(\theta) + \lambda\Theta(\theta) \\ \Theta(\theta+2\pi) = \Theta(\theta)\end{cases}$$

$\lambda > 0$. $\Theta(\theta) = 0$

$\lambda = 0$ $\Theta(\theta) = 1$

$\lambda < 0$. $\Theta_k(\theta) = C_1\cos k\theta + C_2\sin k\theta$

解 Euler 方程

$\partial_r^2 R - k^2 R = 0$

$k \neq 0 \quad R(r) = D_1 r^k + \underbrace{D_2 r^{-k}}_{舍}$

$k = 0 \quad R(r) = \underbrace{D \ln r}_{舍} + D_2 \quad \Rightarrow R_0(r) = D_2$

设 $u(x,\theta) = D_2 + \sum_{k=1}^{\infty} r^k (C_k \cos k\theta + D_k \sin k\theta)$.

由边值条件 $u(x,\theta)\big|_{\partial B} = D_2 + \sum_{k=1}^{\infty}(C_k \cos k\theta + D_k \sin k\theta) = \varphi(\cos\theta, \sin\theta) \triangleq \tilde{\varphi}(\theta)$

$$D = \int_0^{2\pi} \tilde{\varphi}(\theta)\,d\theta$$

$$C_k = \frac{\int_0^{2\pi} \tilde{\varphi}(\theta)\cos k\theta\,d\theta}{\int_0^{2\pi} \cos^2 k\theta\,d\theta} \qquad D_k = \frac{\int_0^{2\pi} \tilde{\varphi}(\theta)\sin k\theta\,d\theta}{\int_0^{2\pi} \sin^2 k\theta\,d\theta}.$$

令
$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & 0<x<l \quad t>0 \\ u(x,0) = \varphi(x), \ \partial_t u(x,0) = \psi(x) & 0 \le x \le l \\ -u_x + \alpha u \big|_{x=0} = g_1(t) & \\ u_x + \beta u \big|_{x=l} = g_2(t) & \alpha, \beta > 0 \end{cases}$$

令 $v(x) = u(x) - (C_1 x^2 g_1(t) + C_2 (l-x)^2 g_2(t))$

$\Rightarrow$ 令 $C_1 = \frac{1}{\alpha l^2 - 2l} \quad C_2 = \frac{1}{\beta l^2 + 2l} \qquad v(x) = u(x) - \left(\frac{1}{\alpha l^2 - 2l} x^2 g(t) + \frac{(l-x)^2}{\beta l^2 + 2l} g_2(t)\right)$

$$\begin{cases} v_{tt} - v_{xx} = F(x,t) \\ v(x,0) = \Phi(x) \quad \partial_t v(x,0) = \Psi(x) \\ -v_x + \alpha v \big|_{x=0} = 0 \\ v_x + \beta v \big|_{x=l} = 0 \end{cases}$$

先抽出 SL 问题, 并取 $F = 0$

暴力计算得 $X_n(x) = C_1 \cos(\sqrt{\lambda_n} x) + C_2 \sin(\sqrt{\lambda_n} x)$

$\tan(\sqrt{\lambda} l) = \frac{(\alpha+\beta)\sqrt{\lambda}}{\lambda - \alpha\beta}$

若 $\lambda < 0$ ——

$\lambda = 0$ ——

令 $u(x,t) = \sum T_n(t) X_n(x)$, 其中 $X_n'' = -\lambda X_n$

$$\begin{aligned} u_{tt} - u_{xx} &= \sum T_n'' X_n - \sum T_n X'' \\ &= \sum (T_n'' + \lambda T_n) X_n = F \end{aligned}$$

~~$T_n(0) = \Phi$~~ 内积 $T_n''(t) + \lambda T_n(t) = F_n(t) = \frac{(F, X_n)}{(X_n, X_n)}$

$T_n(0) = \Phi_n = \frac{(\Phi, X_n)}{(X_n, X_n)} \qquad T_n'(0) = \Psi_n = \frac{(\Psi, X_n)}{(X_n, X_n)}.$

$$\Rightarrow T_n = \Phi_n \cos\sqrt{\lambda_n} t + \Psi_n \frac{\sin\sqrt{\lambda_n} t}{\sqrt{\lambda_n}} + \int_0^t \frac{\sin((t-s)\sqrt{\lambda_n})}{\sqrt{\lambda_n}} F_n(s)\,ds$$

能量估计

$$\begin{cases} u_{tt} - \Delta u = \cancel{f(x,t)}\ 0 & t > 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

$\frac{\partial}{\partial x_i} u_t u_{x_i}$

$\underline{u_{tx_i} u_{x_i}} + u_t u_{x_i x_i}$

$$\partial(u_t u_{tt} - \Delta u \cdot u_t) = 0$$

$$\Rightarrow \partial_t\left(\frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2\right) - \operatorname{div}(u_t \nabla u) = 0$$

$\cancel{u_{t}u_{tt}}$

$$\Rightarrow \partial_t\Big(\underbrace{\frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2}_{e(t)\ 能量密度}\Big) = \operatorname{div}(u_t \nabla u)$$

$$\cancel{E(t)=}\int_{\mathbb{R}^n} \partial_t e(t)\, dx = \int_{\mathbb{R}^n} \operatorname{div}(u_t \nabla u) \overset{散度定理}{=\!=} \int_{\partial \mathbb{R}^n} u_t \nabla u \cdot \vec{n}\, ds = 0$$

$$\partial_t \int_{\mathbb{R}^n} e(t)\, dx \qquad 令\ E(t) = \int_{\mathbb{R}^n} e(t)\, dx. = \int_{\mathbb{R}^n}\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) dx$$

现在
$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(x,0) = \varphi(x)\ \ u_t(x,0) = \psi(x) \\ u|_{\partial\Omega} = 0 \end{cases} \quad \Omega \subseteq \mathbb{R}^n 有界$$

$$\Rightarrow \partial_t \overline{\left(\int_{\Omega} \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right)} = \int_{\Omega} \operatorname{div} u_t \nabla u\, dx = \int_{\partial\Omega} \boxed{u_t} \cdot \nabla u\, \vec{n}\, ds = 0$$

$u_t \Downarrow 0$

$$\Rightarrow E(t) = E(0) = \int_{\Omega}\left(\frac{1}{2}\psi^2 + \frac{1}{2}|\nabla\varphi|^2\right) dx$$

$$\begin{cases} u_{tt} - \Delta u = f \\ u(x,0) = \varphi(x)\ \ u_t(x,t) = \psi(x) \\ u|_{\partial\Omega} = 0 \end{cases} \qquad \partial_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) = \operatorname{div}(u_t \nabla u) + u_t f$$

Ω 上积分：
$$\partial_t \int_{\Omega}\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) dx = \int_{\Omega} u_t f(x,t)\, dx$$
$$\le \int_{\Omega}\left(\frac{1}{2}u_t^2 + \frac{1}{2}f^2\right) dx$$
$$\le \int_{\Omega}\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) dx + \int_{\Omega} \frac{1}{2} f^2\, dx$$

即 $\frac{dE}{dt} \le E(t) + \frac{1}{2}\int_{\Omega} f^2\, dx$

$$\Rightarrow \frac{d}{dt}(e^{-t} E) \le \frac{1}{2} e^{-t} \int_{\Omega} f^2\, dx$$

$$e^{-t} E \le E(0) + \frac{1}{2}\int_0^t e^{-s} \int_{\Omega} f^2(x,s)\, dx\, ds$$

$$E(t) \le e^t\left(E(0) + \frac{1}{2}\int_0^t \int_{\Omega} f^2(x,s)\, dx\, ds\right) \le e^T E(0) + \int_0^T\int_{\Omega} f^2(x,t)\, dx\, dt$$
$$= C_T\left(E(0) + \int_0^T\int_{\Omega} f^2(x,t)\, dx\, dt\right)$$

$$E(t) \le C_T\Big(\underbrace{E(0)}_{\frac{1}{2}\int_\Omega |\nabla\varphi|^2+\psi^2\,dx} + \frac{1}{2}\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt\Big)$$

令 $E_0(t) = \int_\Omega |u(x,t)|^2\,dx$

$$\frac{d}{dt}E_0(t) = \int_\Omega 2u(x,t)\cdot \cancel{\quad} u_t\,dx$$
$$\le \int_\Omega u^2\,dx + \int_\Omega u_t^2\,dx$$
$$= E_0(t) + \int_\Omega u_t^2\,dx$$

Growall $\Rightarrow$ $E_0(t) \le C_T\Big(\cancel{\quad}E_0(0) + \frac{1}{2}\int_0^T\underbrace{\int_\Omega u_t^2\,dx\,dt}\Big)$

$$\le C_T\Big(E_0(0)\Big) + C_T\frac{T}{2}\Big(C_T\big(E(0) + \int_0^T\int_\Omega f^2(x,t)\,dx\,dt\big)\Big)$$
$$\le \tilde{C}_T\Big(E_0(0) + E(0) + \int_0^T\int_\Omega f^2\,dx\,dt\Big) \quad \forall\, 0\le t\le T$$

推论：
$$\begin{cases} u_{tt} - \Delta u = f \\ u|_{t=0} = \varphi(x) \quad u_t|_{t=0} = \psi(x) \\ u|_{\partial\Omega} = 0 \end{cases}$$
古典解唯一

设有两解 u_1, u_2 $\quad V = u_1 - u_2$

$$\Rightarrow \begin{cases} V_{tt} - \Delta V = 0 \\ V|_{t=0} = \cancel{\varphi(x)}\,0, \quad V_t|_{t=0} = 0 \\ V|_{\partial\Omega} = 0 \end{cases}$$

由能量估计

$$0 \le \frac{1}{2}\int_\Omega V_t^2 + |\nabla V|^2\,dx \le 0$$

$\Rightarrow$ V 为常数

在由 E_0 的估计 $\Rightarrow V\equiv 0$ ~~on~~ on Ω

可以类似用能量方法得到"稳定性"

用能量方法解释有限传播速度.

$$\partial_t\left(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2\right)=\mathrm{div}\,(u_t\nabla u)$$

$$\int_{\Delta}\Big(\underbrace{\partial_t\, e(t)-\mathrm{div}\,(u_t\nabla u)}_{\mathrm{div}_{t,x}\left(e(t),\ -u_t\nabla u\right)}\Big)\,dx\,dt=0$$

$$\Rightarrow\int_{\partial\Delta}\left(e(t),\ -\partial_t u\,\nabla u\right)\cdot\vec{n}\cdot ds=0$$

K 侧面

$$\{(x,t)\mid |x-x_0|\le R-t\quad 0\le t\le T\}$$

$$\vec{n}=\frac{1}{\sqrt{2}}\left(1,\ \frac{x-x_0}{|x-x_0|}\right)$$

$$\int_{B} e(t)\,dx=\int_T e(t)\,dx+\frac{1}{2\sqrt{2}}\int_K\left(\partial_t u^2+|\nabla u|^2-2\,\partial_t u\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right)ds$$

$$\partial_t u^2+|\nabla u|^2-2\,\partial_t u\frac{x-x_0}{|x-x_0|}\cdot\nabla u=\underbrace{\left|\partial_t u-\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2+\underbrace{|\nabla u|^2-\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2}}_{\ge 0}$$

$$\Downarrow$$

Flux (0,t)

位势方程

$-\Delta u = f(x) \quad x\in\Omega\subseteq\mathbb{R}^n \quad n=2,3$

$u:\Omega\to\mathbb{R}$ 未知 $\quad f(x):\Omega\to\mathbb{R}$ 已知

$f\equiv 0$ Laplace 方程

$u|_{\partial\Omega}=\varphi(x)$ Dirichlet边值

$\frac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega}=\varphi(x)$ Neumann边值

$\left(\frac{\partial u}{\partial \vec{n}}+\sigma u\right)\Big|_{\partial\Omega}=\varphi(x)$ Robin边值

if $u\in C^2(\Omega)$ $\Delta u=0 \Rightarrow u$ 为调和函数

极坐标 $\int_{\mathbb{R}} f = \int_0^{+\infty}\int_{\partial B(0,r)} f(y)\,dS(y)$

$\cancel{=\int_0^{+\infty}\int_{\partial B}}$

$\frac{d}{dr}\int_{B(x_0,r)} f(x)dx = \int_{\partial B(x_0,r)} f(y)dS(y)$

调和函数的性质 $\quad$ 平移、伸缩、旋转不变

(1) 平均值性质 $\forall x\in B(x,r)\subseteq\Omega \quad u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u(y)\,dy$

(2) 第二平均 $\forall x\in B_r(x)\subseteq\Omega \quad u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)\,dy$

claim (1) $\Leftrightarrow$ (2)

(1) $\Rightarrow$ (2) $\quad u(x)\frac{4}{3}\pi r^3=\int_{B_r(x)}u(y)\,dy$

$\downarrow$ dr

$4\pi r^2 u(x) = \int_{\partial B_r(x)}u(y)\,dy$

(2) $\Rightarrow$ (1)

积分

$\Omega\subset\mathbb{R}^n$ 有界 $\partial\Omega$ 光滑

调和 $\Rightarrow$ 平均值性质

$\Delta u=0$ on Ω

$$0=\int_{B_r(y)}\Delta u\,dy=\int_{B_r(y)}\operatorname{div}(\nabla u)\,dy=\int\nabla u\cdot\vec{n}\,dS$$

$$=\int_{\partial B}\nabla u(y)\frac{y-x}{r}\,dS(y)=\int_{\partial B}\nabla u(rw+x)\frac{rw}{r}\,r^2\,dS(w)$$

$$=r^2\int_{|w|=1}\underbrace{\nabla u(rw+x)\cdot w}_{\frac{d}{dr}u(x+rw)}\,dS(w)$$

$\nabla u(x+rw)$

$$=r^2\frac{d}{dr}\left(\int_{|w|=1}u(rw+x)\,dS(w)\right)$$

$$\Rightarrow u(x)=\frac{1}{4\pi}\int_{|w|=1}u(x+rw)\,dS(w)$$

$$=\frac{1}{4\pi r^2}\int_{\partial B}u(y)\,dS(y)$$

$$=\frac{1}{|\partial B|}\int_{\partial B}u(y)\,dS(y)$$

$\square$

反过来. $u \in C^2(\Omega)$. $\forall B(x,r) \subseteq \Omega$. $u(x) = \frac{1}{|\partial B|}\int_{\partial B} u(y)\, d\cancel{\sigma}s(y) \Rightarrow u$ 调和

$$u(x) = \frac{1}{4\pi r^2}\int_{|y-x|=r} u(y)\, ds(y)$$

$$= \frac{1}{4\pi}\int_{|w|=1} u(x+rw)\, ds(w)$$

$$\int_{B_r(x)} \Delta u(y)\, dy = \int_{B_r(x)} \mathrm{div}(\nabla u(y))\, dy = \int_{\partial B} \nabla u(y)\, \vec{n}\, ds(y)$$

$$= \int_{\partial B} \nabla u \cdot \frac{y-x}{r}\, ds(y) = r^2 \frac{d}{dr}\int u = r^2\frac{d}{dr} u(x) = 0$$

$\Rightarrow \Delta u$ 在任一球上积分为0. $\xRightarrow{\text{反证}} \Delta u = 0$

$\boxed{\text{磨光}}$ 条件减弱为 $u \in C(\Omega)$. $\mathrm{supp}\,\varphi \subseteq B_1(0)$

令 $\varphi \in C_0^\infty(B_1^\circ)$ $\left(\varphi \text{ 取法 } \int_{\mathbb{R}^n}\varphi = 1,\ \varphi \geq 0,\ \varphi \text{ 径向}\right)$ 取 bump.

$$1 = \int_{\mathbb{R}^n}\varphi = \int_0^\infty \int_{|w|=1} \varphi(rw) r^2\, ds(w)\, dr \overset{\text{径向}}{=} \int_0^\infty \int_{|w|=1} \varphi(r) r^2\, ds(w)$$

$$= \underbrace{4\pi \int_0^\infty \varphi(r) r^2\, dr}$$

令 $\varphi_\varepsilon = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \rightsquigarrow \mathrm{supp}\,\varphi_\varepsilon \subseteq B_\varepsilon(0)$.

$$\int_{\mathbb{R}}\varphi_\varepsilon = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\varphi\left(\frac{x}{\varepsilon}\right)dx = \int_{\mathbb{R}^n}\varphi\left(\frac{x}{\varepsilon}\right)d\frac{x}{\varepsilon} = 1$$

Claim $u(x) = (u * \varphi_\varepsilon)(x)$ ε 充分小

$$\int_{\mathbb{R}^n}\varphi_\varepsilon(x-y)\, u(y)\, dy = \int_{B_\varepsilon}\varphi_\varepsilon(x-y)\, u(y)\, dy = \int_{|z|\leq 1}\varphi(z)\, u(x+\varepsilon z)\, dz$$

$\frac{y-x}{\varepsilon} = z \leq 1$

$$\overset{z=rw}{=} \int_0^1 \int_{|w|=1} \varphi(r)\, u(x+\varepsilon r w)\, r^2\, ds(w)\, dr$$

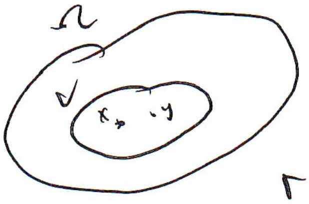
$$= \int_0^1 r^2\varphi(r)\, dr \underbrace{\int_{|w|=1} u(x+\varepsilon rw)\, ds(w)}$$

$$= \int_0^1 r^2\varphi(r)\, [4\pi u(x)]\, dr$$

$$\overset{\text{平均值}}{=} \int_0^1 r^2\varphi(r)\,[4\pi u(x)]\, dr$$

$= u(x)$. #

Harnack.



$r < \frac{1}{4} d(V, \partial\Omega)$

$r = |x-y|$

$$u(y) = \frac{1}{|B_{2r}|}\int_{B_{2r}(y)} u(z)\,dz \geq \frac{1}{|B_{2r}|}\int_{B_r(x)} u(z)\,dz \geq \frac{|B_r|}{|B_{2r}|} u(x)$$

—串有限覆盖即可

2.7. 梯度估计 $u \in C(\bar{B}_R)$ 是调和的 $B_R = B_R(x_0)$

则 $|Du(x_0)| \le \frac{n}{R}\max_{\partial B_R} u$

u调和 & 满足平均值性质 ⇒ u光滑

$$\Rightarrow \Delta \partial_{x_i} u = 0 \Rightarrow \partial_{x_i} u(x) = \frac{1}{|B_R|}\int_{B_R}\partial_{x_i} u(y)\,dy = \frac{1}{|B_R|}\int_{B_R} \mathrm{div}(0\cdots u \cdots 0)\,dy$$

$$= \frac{1}{|B_R|}\int_{\partial B_R} (0, u, 0)\cdot\vec{n}\, dS(y)$$

$$\Rightarrow |\partial_{x_i} u| \le \frac{1}{\frac{4}{3}\pi R^3}\int_{\partial B_R} |u|\, dS(y)$$

$$\le \frac{1}{\frac{4}{3}\pi R^3}\max_{\partial B_R}|u| \cdot 4\pi R^2 = \frac{3}{R}\max_{\partial B_R} u$$

Liouville $\mathbb{R}^n$上的有界调和函数是常数

用梯度估计…

$\mathbb{R}^n$ $\qquad \Delta u = f$

基本解 $\Delta\Gamma = \delta$

$$f = f*\delta = f*\Delta\Gamma = \Delta(\underline{f*\Gamma})$$

⇒ 解了

Step 1. $\Delta u = f$ Claim f 径向 $\Rightarrow u$ 径向

只要旋转不变

$\forall O \in SO(n)$ $\quad \Delta u(Ox) \underset{\uparrow}{=} (\Delta u)(Ox) = f(Ox) \underset{\uparrow}{=} f(x)$

Δ 旋转不变 $\qquad$ 径向

由"解的唯一性" $u(Ox) = u(x)$ $\forall O \in SO(n)$ $\Rightarrow u$ 为 radical

$$\boxed{\Delta(u(Ox)) = \sum \frac{\partial^2 u}{\partial x_i^2}(Ox) = \sum O_{ij}^2 \frac{\partial^2 u}{\partial x_i^2}(Ox) = (\Delta u)(Ox)}$$

$\Rightarrow \Delta \Gamma = \delta \Rightarrow \Gamma$ 径向

用极坐标 (r,θ) $\Delta_{\mathbb{R}^n} \Gamma = \partial_r^2 \Gamma + \frac{n-1}{r} \partial_r \Gamma + \frac{1}{r^2} \Delta_{S_n} \Gamma$

$\Rightarrow \partial_r^2 \Gamma + \frac{n-1}{r} \partial_r \Gamma = 0$ $\quad (r>0)$

$v = \partial_r \Gamma$

$\Rightarrow \partial_r v + \frac{n-1}{r} v = 0$ 可分离变量.

~~$n=2$:~~

$$\Rightarrow \Gamma = \begin{cases} C_1 \ln r + C_2 & n=2 \\ \frac{C_1}{2-n} r^{-(n-2)} + C_2 & n \geq 3 \end{cases}$$

令 $\Gamma = \begin{cases} \frac{1}{2\pi} \ln r & n=2 \\ -\frac{1}{4\pi} \frac{1}{r} & n=3 \end{cases} \longrightarrow \Delta \Gamma = \delta$

Green 公式 首先 $u \partial_{x_i}^2 v = \partial_{x_i}(u \partial_{x_i} v) - \partial_{x_i} u \, \partial_{x_i} v$

对 i 求和

$u \Delta v = \mathrm{div}(u \nabla v) - \nabla u \cdot \nabla v$

若 $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ $\Rightarrow \int_\Omega u \Delta v = \int_\Omega \underline{\mathrm{div}}(u \nabla v) - \int_\Omega \nabla u \cdot \nabla v$

$\int_\Omega u \Delta v = \int_{\partial\Omega} u \nabla v \cdot \vec{n} \, ds - \int_\Omega \nabla u \cdot \nabla v$

$\Rightarrow \int_\Omega (u \Delta v - v \Delta u) \overset{对称}{=} \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) ds$

Claim, $n=3$:

若 $\Delta u = 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ $\overset{x \in \Omega}{\Rightarrow} u(x) = \int_{\partial\Omega} \left[-\frac{u}{4\pi} \frac{\partial}{\partial \vec{n}} \frac{1}{|y-x|} + \frac{1}{4\pi |y-x|} \frac{\partial u}{\partial \vec{n}}\right] ds$

若 $x_0=0\in\tilde{\Omega}$ 则 $u(0)=\int_{\partial\Omega}\left(-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}\right)ds$

denote

$\tilde{\Omega}=\Omega-x_0$

transition pa

$$\int_{\partial\tilde{\Omega}}\left(u\frac{\partial}{\partial n}\Gamma-\Gamma\frac{\partial u}{\partial n}\right)ds$$

$\Omega_\varepsilon:=\Omega\setminus\overline{B_\varepsilon(0)}$ $\quad\Delta V=\delta\Rightarrow\Delta V\big|_{\partial\Omega_\varepsilon}=0$

$$\int_{\partial\Omega_\varepsilon}(\quad)\,ds\ \underset{\text{Green}}{=}\ 0$$

$$\text{lhs}=\int_{\partial\Omega}\Big(u\underset{①}{\frac{\partial}{\partial n}\Gamma}-\underset{②}{\Gamma\frac{\partial u}{\partial n}}\Big)ds$$

$①=\int_{\partial B_\varepsilon}u\frac{1}{4\pi\varepsilon^2}ds$

$=-\int_{\partial B_\varepsilon}\big((u(x)-u(0))+u(0)\big)*ds$

$\Rightarrow -u(0).$

$②=\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon}\frac{\partial u}{\partial r}ds\to 0$ $\quad$ or $\int_{\partial B}\frac{\partial u}{\partial r}ds=\int\nabla\cdot\frac{\partial u}{\partial r}=\int\Delta u=0$

$\Rightarrow\int_{\partial\Omega}+\underbrace{\int_{|x|=\varepsilon}}_{-u(0)}=\int_{\partial\Omega_\varepsilon}=0$

$\Rightarrow u(0)=\int_{\partial\Omega}\cdots$

若 $\Delta u=f$

$$\Rightarrow u(x_0)=\int_\Omega-\frac{1}{4\pi|x-x_0|}f\,dx+\int_{\partial\Omega}\left[-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right]ds \quad (A)$$

无法同时知道 $\frac{\partial u}{\partial n}$ 和 u 在 $\partial\Omega$ 上的值

若 g 都调和 且 $g|_{\partial\Omega}=\frac{1}{4\pi|x-x_0|}\Big|_{\partial\Omega}$

$$\int_\Omega gf\,dx=\int_{\partial\Omega}\left(u\frac{\partial g}{\partial n}-g\frac{\partial u}{\partial n}\right)ds$$

$$=\int_{\partial\Omega}\left(u\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-x_0|}\right)-\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right)ds \quad (B)$$

$$A+B\Rightarrow u(x_0)=\int_\Omega-\frac{1}{4\pi|x-x_0|}f\,ds+\int_\Omega fg\,dx+\int_{\partial\Omega}u\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|x-x_0|}+g\right)ds$$

$\boxed{\text{消掉了}\frac{\partial u}{\partial n}\ !}$

重写为 $u(x) = \int_\Omega f(y) g^x(y)\,dy + \int_\Omega -\frac{1}{4\pi|x-x_0|} f(y)\,dy + \int_{\partial\Omega} \varphi \frac{\partial}{\partial n}\Big(\underbrace{-\frac{1}{4\pi|x-y|} + g^x(y)}_{\text{格林函数 } G(x,y)}\Big)\,ds(y)$

$G(x,x_0)$: ① $x \neq x_0$ 调和 on $\Omega \setminus \{x_0\}$

② $G(x,x_0)|_{\partial\Omega} = 0$

③ $G(x,x_0) + \frac{1}{4\pi|x-x_0|}$ 调和 on Ω

以上为 $n=3$

$n=2$. $G(x,x_0) = \frac{1}{2\pi}\ln|x-x_0| + g^{x_0}(x)$. 称 $G(x,x_0)$ 为 Ω 上一对 x_0 的 Green 函数

性质 $G(x,x_1) = G(x_0,x)$. $\forall\, x_0, x \in \Omega$

令 $u(x) = G(x,a)$ $v(x) = G(x,b)$

要证 $u(b) = v(a)$

挖两点 $\Omega_\varepsilon = \Omega \setminus \overline{(B_\varepsilon(a) \cup B_\varepsilon(b))}$

$\Delta u = 0$

$u|_{\partial\Omega} = 0$

$u + \frac{1}{4\pi|x-x_0|}$ 调和 $\setminus \{a\}$

$0 = \int_{\Omega_\varepsilon} (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$

$= \left(\int_{\partial\Omega} + \int_{-\partial B_\varepsilon(a)} + \int_{-\partial B_\varepsilon(b)}\right) +$

$= \int_{\partial\Omega}$ 被积 $= 0$

$$\boxed{\begin{aligned}&\int_{|x-a|=\varepsilon}\left[\left(u + \frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\left(u + \frac{1}{4\pi|x-a|}\right)\right] ds \quad (a)\\ &- \int_{|x-a|=\varepsilon}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n} \quad (b) + \int_{|x-a|=\varepsilon} v\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right) ds \quad (c)\end{aligned}} + \cdots$$

改法向.

(a): $\int_{|x-a|=\varepsilon} = -\int_{|x-a|=\varepsilon}\left[u + \frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\left(u + \frac{1}{4\pi|x-a|}\right)\right] ds$

$= -\int_{B_\varepsilon} (u\Delta v - v\Delta u)\,dx = 0.$

(b) $= \frac{1}{4\pi\varepsilon}\int_{|x-a|=\varepsilon} \frac{\partial v}{\partial n}\,ds.$

$= \frac{1}{4\pi\varepsilon}\int_{|x-a|=\varepsilon} \nabla v \cdot \vec{n}\,ds = \frac{1}{4\pi\varepsilon}\int_{B_\varepsilon} \Delta v\,dx = 0$

(c) $\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-x_0|}\right) = -\frac{x-x_0}{|x-x_0|}\nabla\left(\frac{1}{4\pi|x-x_0|}\right) = \frac{x-x_0}{|x-x_0|}\frac{x-x_0}{4\pi|x-x_0|^3} = \frac{1}{4\pi|x-x_0|^2}$

(c) $= \frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon} v(x)\,ds = \frac{1}{4\pi\varepsilon^2}\left(\int_{|x-a|=\varepsilon}(v(x)-v(a))\,ds + \int v(a)\,ds\right) \to v(a)$

$\quad\quad \| $

$\quad\quad \nabla v\,|x-a|$

$\quad\quad \|$

$\quad\quad \max|\nabla v|\,\varepsilon \to 0$

$\Rightarrow 0 \quad 0 = v(a) - u(b)$

……同理即可

半空间的Green函数

$R_3^+ = \{(x_1, x_2, x_3) \mid x_3 > 0\}$

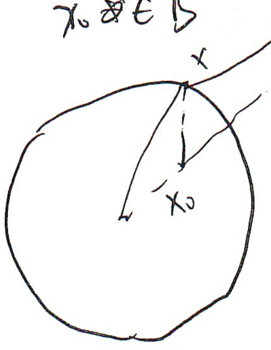
"电相法"

$x_0 \in R^+$

$$G(x, x_0) = \frac{-1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|} \qquad x_0^* = (x_1^0, x_2^0, -x_3^0)$$

球

$x_0^* \in B$



$\boxed{相似}$

$$G(x, x_0) = \frac{-1}{4\pi|x-x_0|} + \frac{c}{4\pi|x-x_0^*|}$$

$$\downarrow$$

$$G(x, x_0) = \frac{-1}{4\pi|x-x_0|} + \frac{R}{|x_0|}\frac{1}{4\pi|x-x_0^*|} \qquad x_0^* = \frac{R^2}{|x_0|^2} x$$

$x_0 \in B_R(0)$. 由Poisson公式

$$u(x_0) = \int_\Omega G(x, x_0) f(x)\,dx + \int_{\partial\Omega} u(x) \frac{\partial G}{\partial n}(x, x_0)\,ds(x)$$

$$\nabla G(x, x_0) = \cdots = \frac{1}{4\pi|x-x_0|^3}\frac{R^2-|x_0|^2}{R^2} x \qquad x \in \partial B_R(0)$$

$$\frac{\partial G}{\partial n} = \frac{x}{|x|}\cdot\nabla G = \frac{R^2-|x_0|^2}{4\pi R|x-x_0|^3} \qquad (|x|=R)$$

$$\begin{cases} \Delta u = f & x \in B_R(0) \\ u|_{\partial B_R(0)} = \varphi \end{cases}$$

$$u(x) = \int_B G(y, x) f(y)\,dy + \int_{\partial B} \varphi(y) \frac{R^2-|x|^2}{4\pi R|y-x|^3}\,ds(y)$$

$$= \int_B G(y, x) f(y)\,dy + \frac{R^2-|x|^2}{4\pi R}\int_{\partial B} \frac{\varphi(y)}{|y-x|^3}\,ds(y).$$

Harnack 不等式： u 在 $B_R(x_0)$ 内调和. $u > 0$. 则

$$\frac{R}{R+r}\frac{R-r}{R+r} u(x_0) \le u(x) \le u(x_0)\frac{R}{R-r}\frac{R+r}{R-r} \qquad r = |x-x_0| \le R$$

$|y| = R$

proof. 不妨 $x_0 = 0$

由Poisson公式

$$\begin{cases} \Delta u = 0 \\ u \quad \partial B \end{cases}$$

$$u(x) = \frac{R^2-|x|^2}{4\pi R}\int_{\partial B} \frac{u(y)}{|x-y|^3}\,ds(y)$$

$$\Rightarrow \frac{R^2-r^2}{4\pi R(R+r)^3}\int_{\partial B} u \le u(x) \le \frac{R^2-r^2}{4\pi R(R-r)^3}\int_{\partial B} u(y)\,ds(y)$$

$$= \frac{R}{R+r}\frac{R-r}{R+r} u(0). \qquad = \frac{R+r}{4\pi R(R-r)^2}\int_{\partial B} u(y)\,ds(y)$$

$$\overset{平均}{=} \frac{R}{R-r}\frac{R+r}{R-r} u(0)$$

Liouville 定理 设 u $\mathbb{R}^n$ 上有上界(或下界)的调和函数 $\Rightarrow u \equiv \text{Const.}$

Pf: u 有上界 ~~$|u| \le M$~~. $u \le M$ $\underbrace{V(x)}_{\text{调和}} = M - u(x) \ge 0$

$$\frac{R(R-r)}{(R+r)^2} V(x_0) \le V(x) \le \frac{R(R+r)}{R-r} V(x_0) \quad r \text{ 不动}$$

$R \to +\infty$ $\Rightarrow V(x) \equiv V(x_0)$

极值原理与最大模估计

$-\Delta u + C(x) u = f \quad x \in \Omega \subseteq \mathbb{R}^n$ (2.38) 若 $u \in C^2(\Omega) \cap C(\bar\Omega)$ 满足方程 (2.38) 则 $u(x)$ 不能在 Ω 上达到 $\bar\Omega$ 上

$C(x) \ge 0$ $f < 0$

的非负最大值.

$$\begin{cases} \nabla u(x_0) = 0 \\ \Delta u(x_0) \le 0 \\ u(x_0) \ge 0 \end{cases}$$ 矛盾.

最大值

$c(x) \ge 0$ $f \le 0$ 时: $L := -\Delta + C(x)$.

结论: 若 u 在 $\bar\Omega$ 存在正的最大值, 则必在 $\partial\Omega$ 上达到它在 $\bar\Omega$ 上的非负最大值.

且 $\max\limits_{x \in \bar\Omega} u(x) \le \max\limits_{\partial\Omega} \underbrace{u^+(x)}_{\max(u,0)}$

令 $w(x) = u(x) + \varepsilon V(x)$

不妨设 $0 \in \Omega$, $d = \operatorname{diam} \Omega$

$\forall x \in \Omega$, $|x| \le d$.

令 $V = |x|^2 - d^2 \le 0$

$LV = -\Delta V + c(x) V \le -2n \le 0$

$Lw = \underset{\le 0}{f} + \varepsilon \underset{< 0}{LV} < 0$

$\boxed{\varepsilon > 0}$ $Lw = \underbrace{Lu}_{\substack{f \\ \le 0}} + \underline{\varepsilon L V} \overset{?}{<} 0$ ...

$\max\limits_{\bar\Omega} u - \varepsilon \max\limits_{\bar\Omega} V \le \max\limits_{\bar\Omega} u + \varepsilon V$

$\Delta(|x|^2) = 2n$

$\Rightarrow \max\limits_{\bar\Omega} w \le \max\limits_{\partial\Omega} w^+$

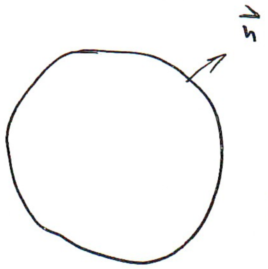
$$\max_{\bar\Omega} u - \varepsilon d^2 \le \overset{\shortparallel}{\max}(u + \underset{\le 0}{\varepsilon V}) \le \max_{\partial\Omega} w^+ \le \max_{\partial\Omega} u^+ \quad \forall \varepsilon.$$

$\Rightarrow \max\limits_{\bar\Omega} u \le \max\limits_{\partial\Omega} u^+$

Hopf Lemma. Note $c(x)$ 非负有界

$$u \in C^2(B_R) \cap C^1(\bar{B}_R)$$

$$\mathcal{L} = -\Delta + c(x)$$



(1) $\mathcal{L}u \le 0$ $x \in B_R$

(2) $x_0 \in \partial B$ u在x_0处达到"严格"非负最大值

$u(x_0) = \max_{\Omega} u \ge 0$

$u(x_0) > u(x)$ $x \in B_R$

$w(x) = u(x)$ 则 $\frac{\partial u}{\partial \nu}\Big|_{x=x_0} > 0$

Idea: 设 $w(x) = u(x) + \varepsilon v(x)$ v待定

1° $\mathcal{L}w = \underbrace{\mathcal{L}u}_{\le 0} + \varepsilon \mathcal{L}v \le 0$ $\boxed{\mathcal{L}v \le 0}$

2° 仍希望 $w(x_0)$ 达到最大值 $\boxed{\frac{\partial v}{\partial \nu} < 0}$

$\rightsquigarrow \frac{\partial w}{\partial \nu}\Big|_{x=x_0} \ge 0$

现在仍不妨设 $B_R = B(0,R)$

如果考虑 $v = R^2 - |x|^2$ $\nabla v =$ ~~$-2|x|$~~

$\frac{\partial v}{\partial n} < 0$ 这做不到

令 $\boxed{v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}}$ $\boxed{\alpha > 0}$

$$\nabla v(x) = e^{-\alpha|x|^2} \cdot (-2\alpha x)$$

$$\frac{\partial v}{\partial n} = -2\alpha|x| e^{-\alpha|x|^2} < 0$$

$$\Delta v = \sum \partial_{x_i}^2 v = \sum \partial_{x_i}\left(-2\alpha x_i e^{-\alpha|x|^2}\right)$$

$$= \sum\left(-2\alpha e^{-\alpha|x|^2} + 4\alpha^2 x_i^2 e^{-\alpha|x|^2}\right)$$

$$= -2\alpha n e^{-\alpha|x|^2} + 4\alpha^2|x|^2 e^{-\alpha|x|^2}$$

$$= \left(4\alpha^2|x|^2 - 2\alpha n\right) e^{-\alpha|x|^2}$$

$$\mathcal{L}v = -\Delta v + c(x)$$

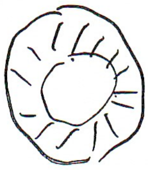
$$= \left[4\alpha^2|x|^2 + 2\alpha n + c(x)\right] e^{-\alpha|x|^2}$$

$$\le \left(-4\alpha^2|x|^2 + 2\alpha n + c\right) e^{-\alpha|x|^2}$$

α充分大即可 ≤ 0 当 $|x| \ne 0$

令 $B_R^* = B_R(0)\setminus\overline{B_{\frac{R}{2}}(0)}$ Then $Lv \le (-R^2\alpha^2+2n\alpha+C)e^{-\alpha|x|^2} \le 0$ in B_R^*

Now. $w(x) = u(x) - u(x_0) + \varepsilon v(x)$ 在 x_0 处 $v(x)=0$ $w(x_0)=0$. 非负.



由极值原理 $\max_{\overline{B_R^*}} w = \max_{\partial B_R^*} w^+$

当 $|x|=\frac{R}{2}$. $w(x) = \underbrace{u(x)-u(x_0)}_{<0} + \varepsilon v(x_0)$

$\le \max_{|x|=\frac{R}{2}} u - u(x_0) + \varepsilon\left(e^{-\alpha\frac{R^2}{4}} - e^{-\alpha R^2}\right)$

ε 充分小 <0

但最大值要 ≥ 0. ⟹ 事实上 $w(x)$ 的最大值仍在 x_0 处

因为 $|x|=R$ 时 $w(x)=u(x)-u(x_0)+\varepsilon \underset{0}{\underset{\parallel}{v(x)}}$

$= u(x)-u(x_0)\le 0$

$\Rightarrow \frac{\partial w}{\partial n}\Big|_{x_0} \ge 0 = \frac{\partial u}{\partial n}(x_0) + \varepsilon \underbrace{\frac{\partial v}{\partial n}}_{<0} \ge 0$

$\frac{\partial u}{\partial n}(x_0) > 0$

Hopf ⟹ 强极值原理 Ω 连有界连通开   Hopf's condition   若 u 在 Ω 内达到其在 $\bar\Omega$ 上的最大值,
则 u 在 $\bar\Omega$ 上是常数.

proof. 考虑集合 $O=\{x\in\Omega \mid u(x)=M\}$. note $M=\max_{\bar\Omega} u(x)$.

$u(x_0)=M$ $x_0\in\Omega$ ⟹ O 非空.

若 $x_n\in O$. $x_n\to\bar x$ 由于 u 连续 ⟹ $\bar x\in O$ ⟹ O 闭

$O\subseteq\Omega$ $\forall x_0\in\Omega$. $\exists r>0$ $B(x_0,2r)\subseteq\Omega$

若 $O\ne\Omega$ ⟹ $\Omega\setminus O$ 非空开集

令 $x_0\in\partial O$ $\exists \bar x \in B(x_0,r)$ $|x_0-\bar x|<r$

令 $d=\mathrm{dist}(\bar x, O)$ $d\le r$

则 $B(\bar x,d)\subset B(x_0,2r)\subseteq\Omega$

记一个"相切"的点为 y_0 ⟹ "$\exists y_0\in\partial O\cap\partial B(\bar x,d)$"

$u(y_0)=M>u(y)$ $\forall y\in B(\bar x,d)\subseteq\Omega\setminus O$

由Hopf lemma. $\frac{\partial u}{\partial n}(y_0)>0$ 但 $\nabla u(y_0)=0$

$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$

是 Dirichlet 边值问题 $\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases} \Rightarrow \max_{\bar\Omega} |u| \leq G + CF$

$\max_{\bar\Omega} |f| = F \quad \max_{\partial\Omega} |g| = G \quad C = C(d, n)$

命题 $v = u - z$

$\begin{cases} \Delta v = f - \Delta z \geq 0 & \rightsquigarrow -\Delta z \geq F \\ v|_{\partial\Omega} = g - z \leq 0 & \longrightarrow -z|_{\partial\Omega} \leq -G \Leftrightarrow z|_{\partial\Omega} \geq G \end{cases}$

不妨 $0 \in \Omega$ $\quad \Delta |x|^2 = 2n$

$\Rightarrow -\frac{F}{2n} \Delta |x|^2 = F$

令 $z(x) = -\frac{F}{2n}\left(|x|^2 - d^2\right) + G$

现在 v 满足 $\begin{cases} \Delta v \geq 0 \\ v|_{\partial\Omega} \leq 0 \end{cases}$ 由极值原理 $\max_{\bar\Omega} v \leq \max_{\partial\Omega} v^+ = 0$

$\Rightarrow \max \left(u(x) - \frac{F}{2n}(d^2 - |x|^2) - G \right) \leq 0$

$v \geq u(x) - \frac{F}{2n} d^2 - G$

$\Rightarrow \max_{\bar\Omega} u(x) \leq G + \frac{F}{2n} d^2$

$-u$ 也是一样的

$\Rightarrow \max_{\bar\Omega} |u(x)| \leq G + CF \quad \square$

稳定性 & 唯一性

$u_i \in C^2(\Omega) \cap C^1(\bar\Omega) \quad i = 1, 2 \quad$ satisfy

$\begin{cases} \Delta u_i = f_i & \Omega & i = 1, 2. \\ u_i = g_i & \partial\Omega \end{cases}$

$\max_{\bar\Omega} |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2| + C \max_{\bar\Omega} |f_1 - f_2|$

In particular, if $f_1 = f_2$, $g_1 = g_2$

$\Rightarrow$ "唯一性定理"

$v = u_1 - u_2$

$\Rightarrow \begin{cases} \Delta v = f_1 - f_2 \\ v|_{\partial\Omega} = g_1 - g_2 \end{cases} \xrightarrow{\text{最大核}} \cdots \quad \square$

现在考虑 $\begin{cases} -\Delta u(x) + c(x)u = f(x) & \Omega \\ \dfrac{\partial u}{\partial n} + \alpha u = g(x) & \partial\Omega \end{cases}$ $\qquad c(x) \geq 0,\ \alpha(x) \geq \alpha_0 > 0$

若 $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ 是解

Thm. $F = \max\limits_{\Omega} f \quad G = \max\limits_{\partial\Omega} g$ 则 $\max\limits_{\bar{\Omega}} u \leq C(G+F) \qquad C = C(n,d),\ d = \operatorname{diam}\Omega$

令 $w = u - z$

希望 $\begin{cases} (-\Delta + c)w \leq 0 \\ \left(\dfrac{\partial}{\partial n} + \alpha\right) w \leq 0 \end{cases}$

$f(x) - (-\Delta z + c(x) z) \leq 0 \rightsquigarrow$ 让 $-\Delta z + c z \geq F$

边界上 $\dfrac{\partial z}{\partial n} + \alpha z \geq G$

有 $-\Delta\left(\dfrac{F}{2n}(d^2 - |x|^2)\right) = F$

$\dfrac{\partial}{\partial n}\left(\dfrac{F}{2n}(d^2 - |x|^2)\right) = \dfrac{F}{2n}(-2x)\cdot \vec{n}$

$\underbrace{\left(-\dfrac{F}{n}\vec{x}\right)\cdot \vec{n}}_{|\cdot| \leq \frac{F}{n} d} + \alpha \dfrac{F}{2n}(d^2 - |x|^2) \leq \alpha \dfrac{F}{2n} d^2$

加个常数项 C s.t. $\alpha_0 C \geq \dfrac{Fd}{n} + G$

$z = \dfrac{F}{2n}(d^2 - |x|^2) + \dfrac{Fd}{n\alpha_0} + \dfrac{G}{\alpha_0}$ 即可

此时用极大值原理，若 w 有非负最大值，必在 $\partial\Omega$ 上取到

$\Rightarrow \underbrace{\dfrac{\partial}{\partial n} w}_{\geq 0} + \underbrace{\alpha w}_{\geq 0} \leq 0 \quad \Rightarrow \quad \boxed{w \leq 0}$ on $\bar{\Omega}$

$u(x) \leq z(x)$ on $\bar{\Omega}$

$u(x) \leq \dfrac{F}{2n} d^2 + \dfrac{Fd}{n\alpha_0} + \dfrac{G}{\alpha_0}$

$\leq C(F+G)$

类似可证 $-u \leq C(F+G)$

$\Rightarrow \max|u| \leq C(F+G)$ □ $\Longrightarrow$ 唯一性

热方程

$$\begin{cases} \partial_t u - \Delta u = f & x \in \Omega,\ t > 0 \\ u(x,0) = \varphi(x) \\ \text{边值} \to \end{cases}$$

Dirichlet $u(x,t) = g(x,t)$ $\partial\Omega$

Neumann $\frac{\partial}{\partial n} u(x,t) = g(x,t)$

Robin $\cancel{\frac{\partial u(x,t)}{\partial n} + g(x)}$

$\left(\frac{\partial u}{\partial n} + \partial u\right) u(x,t) = g(x,t)$ $\partial\Omega$

Fourier 变换 $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$

Schwarz $\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty \,\middle|\, \langle x \rangle^N \left| \partial^\alpha f \right| < \infty \quad \forall\, \alpha, N \geq 0 \right\}$

$\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$

$(\tau_{x_0} f)(x) := f(x - x_0)$

① $\widehat{\tau_{x_0} f}(\xi) = e^{-2\pi i x_0 \cdot \xi} \hat{f}(\xi)$

$\int_{\mathbb{R}^n} f(x - x_0) e^{-2\pi i \xi \cdot ((x - x_0) + x_0)} dx = e^{-2\pi i \xi \cdot x_0} \hat{f}(\xi)$

② $S_\lambda f(\xi) := f(\lambda \xi)$

$\widehat{S_\lambda f}(\xi) = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} f(\lambda x) e^{-2\pi i \lambda x \cdot \xi/\lambda} d\lambda x$

$= \frac{1}{\lambda^n} \hat{f}(\xi/\lambda) \quad \left(\lambda^{-n} \hat{f}(\lambda^{-1} \xi) \right)$

③ $\alpha := (\alpha_1, \cdots \alpha_n)$ $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

prove by induction: $\widehat{\partial_{x_1} f}(\xi) = \int_{\mathbb{R}^n} \partial_{x_1} f(x) e^{-2\pi i x \cdot \xi} dx$

$= \int_{\mathbb{R}^{n-1}} d\hat{x} \int e^{-2\pi i x \cdot \xi} \partial_{x_1} f$

$= \int_{\mathbb{R}^{n-1}} d\hat{x} \int_{\mathbb{R}} -2\pi i \xi_1 f e^{-2\pi i x \cdot \xi} dx$

$= -2\pi i \xi_1 \hat{f}(\xi)$

④ $\widehat{(-2\pi i x)^\alpha f}(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$

设 $\alpha = (1, 0, \cdots 0)$

$$\int_{\mathbb{R}^n} -2\pi i x_1 f(x) \cdot e^{-2\pi i x\cdot\xi} dx$$

$$= \int_{\mathbb{R}^n} f(x) \partial_{\xi_1} e^{-2\pi i x\cdot\xi} dx$$

$$= \partial_{\xi_1} \hat{f}(\xi)$$

⑤ $f * g := \int_{\mathbb{R}^n} f(x-y) g(y) dy \underset{\text{Fubini}}{\Rightarrow} \widehat{f*g} = \hat{f}\hat{g}$

$$\widehat{fg}(\xi) = \int_{\mathbb{R}^n} f(x) g(x) e^{-2\pi i x\cdot\xi} dx \quad = \hat{f} * \hat{g}$$

$$\boxed{\hat{f} * \hat{g}(\xi)} = \int_{\mathbb{R}^n} \hat{f}(\xi - y) \hat{g}(y) dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x_1) e^{-2\pi i x_1\cdot(\xi-y)} dx_1 \right) \left(\int_{\mathbb{R}^n} g(x_2) e^{-2\pi i x_2\cdot y} dx_2 \right) dy$$

$$= \iiint_{(\mathbb{R}^n)^3} f(x_1) e^{-2\pi i x_1\cdot(\xi-y)} \boxed{g(x_2)} \boxed{e^{-2\pi i x_2\cdot y}} dx_1 \boxed{dx_2} dy$$

$$\cancel{= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x_1) f(x_2) e^{-2\pi i x_1\cdot\xi} \int_{\mathbb{R}^n} e^{-2\pi i (x_2 - x_1) y} dy}$$

$$\cancel{= \iint_{(\mathbb{R}^n)^2} f(x_1) \hat{g}(y) e^{-2\pi i x_1\cdot(\xi-y)}} dx_1 dy$$

逆变换 $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi \quad =$

$\check{\hat{f}}(x) = f(x)$

变换后方程 $\begin{cases} \partial_t \hat{u}(\xi,t) + 4\pi^2 \xi^2 \hat{u}(\xi,t) = \hat{f}(\xi,t) \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi) \end{cases}$

$$\hat{u}(\xi,t) = e^{-4\pi^2|\xi|^2} \hat{\varphi}(\xi) + \int_0^t e^{-4\pi^2|\xi|^2(t-s)} \hat{f}(\xi,s) ds$$

若 $f=0$. $u(x,t) = \left(e^{-4\pi^2|\xi|^2}\right)^\vee * \varphi$

例 e^{-x^2} $x\in\mathbb{R}$.

$$F(\xi)=\int_{\mathbb{R}} e^{-x^2} e^{-2\pi i x\cdot\xi}dx$$

$$F'(\xi)=\int_{\mathbb{R}} e^{-x^2}(-2\pi i x)e^{-2\pi i x\xi}dx$$

$$=\pi i\int_{\mathbb{R}} \partial_x(e^{-x^2})e^{-2\pi i x\xi}dx$$

$$=-\pi i\int_{\mathbb{R}} e^{-x^2}(-2\pi i\xi)e^{-2\pi i x\xi}dx$$

$$=-2\pi^2\xi\int_{\mathbb{R}} e^{-x^2}e^{-2\pi i x\xi}dx$$

$$=-2\pi^2\xi F(\xi)$$

$$F(0)=\int_{\mathbb{R}} e^{-x^2}=\sqrt{\pi}$$

$$F(\xi)=\sqrt{\pi}\,e^{-\pi^2|\xi|^2}$$

若 $f(x)=e^{-|x|^2}$ $x\in\mathbb{R}^n$

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{-|x|^2}e^{-2\pi i x\cdot\xi}dx$$

$$=\pi^{n/2}e^{-\pi^2|\xi|^2}$$

解方程 $\begin{cases} u_t-\Delta u=0\\ u(x,0)=\varphi(x)\end{cases} \rightsquigarrow \begin{cases}\partial_t\hat{u}(\xi,t)+4\pi^2|\xi|^2\hat{u}(\xi,t)=0\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi)\end{cases}$

$$\rightsquigarrow \hat{u}(\xi,t)=e^{-4\pi^2|\xi|^2t}\hat{\varphi}(\xi)$$

$$\rightsquigarrow u(x,t)=\left(e^{-4\pi^2|\xi|^2t}\hat{\varphi}(\xi)\right)^\vee=\left(e^{-4\pi^2|\xi|^2t}\right)^\vee*\varphi(x)$$

$$\left(e^{-4\pi^2|\xi|^2t}\right)^\vee=\left(e^{-|2\pi\sqrt{t}\xi|^2}\right)^\vee=(2\pi\sqrt{t})^{-n}\pi^{n/2}e^{-\frac{\pi^2|x|^2}{(2\pi\sqrt{t})^2}}$$

$$=\frac{1}{(2\pi\sqrt{t})^n}\pi^{n/2}e^{-\frac{|x|^2}{4t}}$$

Fourier 逆变换和 Fourier 变换是一样的性质

$\widehat{fg}=\hat{f}*\hat{g}$ 的证明:

$$\hat{f}*\hat{g}(\xi)=\int_{\mathbb{R}^n}\hat{f}(\xi-y)\hat{g}(y)dy$$

$$=\iiint f(x_1)e^{-2\pi i x_1\cdot(\xi-y)}\hat{g}(y)e^{2\pi i y\cdot x_1}dx_1\,dy$$

$$=\int f(x_1)g(x_1)e^{-2\pi i x_1\xi}dx_1$$

$$=\widehat{fg}$$

方程的解

$$u(x,t) = \left(\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} * \varphi \right)(x)$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy$$

令 $K(x) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}}$ $\quad K_t(x) = \frac{1}{(\sqrt{t})^n} K\left(\frac{x}{\sqrt{t}}\right)$

$$u(x,t) = (K_t * \varphi)(x)$$

$K_t \rightsquigarrow$ good kernel

(1) $\int_{\mathbb{R}^n} K_t(x) = 1$

(2) $\int_{\mathbb{R}^n} |K_t(x)| < +\infty$

(3) $\int_{|x|>y} K_t \, dx \to 0$ as $t \to 0$

$\forall y > 0$

Claim. 若φ连续有界 则 $\lim_{t\to 0^+} u(x,t) = \varphi(x)$ pointwise.

$$u(x,t) - \varphi(x) = \int_{\mathbb{R}^n} K_t(y)\varphi(x-y)\,dy - \varphi(x)$$

$$= \int_{\mathbb{R}^n} K_t(y)\varphi(x-y)\,dy - \int_{\mathbb{R}^n} \varphi(x) K_t(y)\,dy$$

$$= \int_{\mathbb{R}^n} K_t(y)\left[\varphi(x-y) - \varphi(x)\right] dy$$

$$= \int_{\mathbb{R}^n} K(z)\left[\varphi(x-\sqrt{t}z) - \varphi(x)\right] dz.$$

$K_t(y) = \frac{1}{(\sqrt{t})^n} K\left(\frac{y}{\sqrt{t}}\right)$

$z = \frac{y}{\sqrt{t}}$

$dz = \frac{1}{(\sqrt{t})^n} dy$

$K_t(y)\,dy = K(z)\,dz$

$\forall \varepsilon > 0,\ \exists \delta\ \ |y| < \delta.$

$|\varphi(x-y) - \varphi(x)| < \varepsilon.$

$$|u(x,t) - \varphi(x)| \le \int_{|z|>R} |K(z)||\varphi(x-\sqrt{t}z) - \varphi(x)|\,dz + \int_{|z|\le R} |K(z)|\,|\varphi(x-\sqrt{t}z) - \varphi(x)|\,dA.$$

$\le 2M\varepsilon.$ φ is bdd.

$\le C\varepsilon.$ $t \to 0$ good kernel

$\exists R > 0,\ \int_{|z|>R} K(z)\,dz < \varepsilon$

解的性质（由解的表达式）

(1) $\forall t>0.\ u(x,t)\in C^\infty(\mathbb{R}^n)$

(2) $\sup|u(x,t)|\leq\sup|\varphi|$

(3) $\varphi\geq 0\Rightarrow u\geq 0$

(4) 无限传播速度

(5) 不可反演（不适定）

$$\begin{cases}\partial_t u-\Delta u=f\\ u(x,0)=\varphi(x)\end{cases}\rightsquigarrow\begin{cases}\partial_t\hat{u}(\xi,t)+4\pi^2|\xi|^2\hat{u}(\xi,t)=\hat{f}(\xi,t)\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi)\end{cases}$$

$$\Rightarrow\hat{u}(\xi,t)=e^{-4\pi^2|\xi|^2t}\cdot\hat{\varphi}(\xi)+\int_0^t e^{-4\pi^2|\xi|^2(t-s)}f(\xi,s)\,ds$$

$$u(x,t)=\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy+\int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi(t-s))^{n/2}}e^{-\frac{|x-y|^2}{4t}}f(y,s)\,dy\,ds$$

解的唯一性.

$$\begin{cases}\partial_t u-\Delta u=f & t>0,\ \Omega\\ u(x,0)=\varphi(x)\\ u|_{\partial\Omega}=0\end{cases}$$

$$u\partial_t u-u\Delta u=uf$$

由

$$\frac{1}{2}\partial_t(u^2)-\nabla\cdot(u\nabla u)+|\nabla u|^2=fu$$

~~$u(\Delta u)$~~

在 Ω 上积分

$$\underbrace{\frac{1}{2}\partial_t\int_\Omega u^2}+\int_\Omega|\nabla u|^2=\int_\Omega fu\leq\underbrace{\frac{1}{2}\int_\Omega(u^2+f^2)}$$

Grönwall $\quad\frac{d}{dt}\left(e^{-t}\frac{1}{2}\int_\Omega u^2\right)\leq\frac{1}{2}\int f^2\cdot e^{-t}$

$$e^{-t}\frac{1}{2}\int_\Omega u^2\leq\frac{1}{2}\int_0^t\int_\Omega f^2(x,s)e^{-s}\,dx\,ds+\frac{1}{2}\int_\Omega\varphi^2$$

$$\leq\frac{1}{2}\int_0^t\int_\Omega f^2(x,s)\,dx\,ds+\frac{1}{2}\int_\Omega\varphi^2$$

$$\int_\Omega u^2\leq C_T\left(\int_0^t\int_\Omega f^2+\int\varphi^2\right)$$

不舍去梯度. 那么.

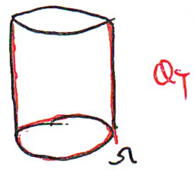
$[0,t]$

$$\cancel{\frac{1}{2}\int_0^t}\ \frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |\nabla u|^2\,dx \le \frac{1}{2}\int_\Omega \varphi^2\,dx + \frac{1}{2}\int_0^{T}\int_\Omega u^2 + f^2$$

$$\le \frac{1}{2}\int_\Omega \varphi^2 + T C_T\left(\int_\Omega |\varphi|^2\,dx + \int_0^T\int_\Omega |f|^2\,dx\,d\tau\right) + \frac{1}{2}\int_0^T\int_\Omega |f|^2$$

$$\le \tilde{C}_T\left(\int_\Omega \varphi^2 + \int_0^T\int_\Omega |f|^2\,dx\,dt\right)$$

$Q_T = \Omega\times[0,t]$



$$\begin{cases} \partial_t u - \Delta u = \cancel{f}\,0 \\ u(x,0) = \varphi(x) & x\in\Omega \\ u(x,t) = h(x,t) & x\in\partial\Omega \end{cases}$$

定义 抛物边界 $\Gamma = \overline{Q_T}\setminus Q_T$ $\qquad Lu = \partial_t u - \Delta u = f \le 0$

$u \in C^{\overset{对t}{1},\overset{对x}{2}}(Q_T)\cap C(\overline{Q_T})$,最大值必在抛物边界达到

1. $f<0$. 设 u 在 Q_T 上达到最大值.

$\Rightarrow \partial_x u(x_*,t_*) = 0.$ $\underline{\Delta u(x_*,t_*) \le 0}$

$\underline{\partial_t u(x_*,t_*) \ge 0}$

不可能 因 $f<0$

2. $f\le 0$. 令 $v = u - \varepsilon t$

$\partial_t v - \Delta v = f - \varepsilon < 0$

$\Rightarrow$ v 在 $\overline{Q_T}$ 上的最大值 在边界上取到

$\max\limits_{\overline{Q_T}} v \le \max\limits_{\Gamma} u$

$\quad\|$

$\max\limits_{\overline{Q_T}} u - \varepsilon T \le \max\limits_{\Gamma} v$

$\Rightarrow \max\limits_{\overline{Q_T}} u - \varepsilon T \le \max\limits_{\Gamma} u$

令 $\varepsilon\to 0$. $\max\limits_{\overline{Q_T}} u = \max\limits_{\Gamma} u$ □

若 $Lu = f \ge 0$. $\Rightarrow \min\limits_{\overline{Q_T}} u = \min\limits_{\Gamma} u$

推论 比较定理

$u, v \in C^{1,2}(Q_T) \cap C(\bar{Q}_T)$

$$\begin{cases} Lu \le Lv \\ u|_\Gamma \le v|_\Gamma \end{cases} \Rightarrow u \le v \quad \text{on } \bar{Q}_T$$

---

let. $w = v - u \ge 0$

$$\Rightarrow \min_{\bar{Q}_T} w = \min_{\Gamma} w \ge 0 \qquad \square$$

$$\Rightarrow \boxed{v \ge u}$$

$$\begin{cases} Lu = \partial_t u - \partial_x^2 u = f & [0,l] \times (0,T) \\ u(x,0) = \varphi(x) & x \in [0,l] \\ u(0,t) = g_1(t) & t \in [0,T] \\ u(l,t) = g_2(t) \end{cases}$$

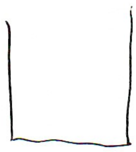
$u \in C^{1,2}(Q_T) \cap C(\bar{Q}_T)$

$$\max_{\bar{Q}_T} |u| \le FT + B$$

$F = \max f$. $\quad B = \max\left\{ \max_{[0,l]} |\varphi|, \max_{[0,T]} |g_1|, \max_{[0,T]} |g_2| \right\}$

令 $v = u - (Ft + B)$

$$\begin{cases} Lv \le 0 \\ v(x,0) \le 0 \\ v(0,t) = g_1(t) - Ft - B \le 0 \\ v(l,t) \le 0 \end{cases}$$



$$\boxed{\Rightarrow v \le 0}$$

导出唯一性

$$\begin{cases} Lu = \partial_t u - \partial_x^2 u = f \\ u(x,0) = \varphi(x) \\ u(0,t) = g_1(t) \\ (u_x + hu)(l,t) = g_2(t) \end{cases} \quad \boxed{h > 0} \longrightarrow \begin{cases} Lu = 0 \\ u(x,0) = 0 \\ u(0,t) = 0 \\ (u_x + hu)(l,t) = 0 \end{cases} \quad (*). \quad 只有零解$$

否则，u 有非零解 $\Rightarrow$ u 有正的最大值或负的最小值

若有正的最大值. $\quad \max_{\bar{Q}_T} u = \max_{\Gamma} u = 0$

$=0$

设

最大值只能在 $x=l$ 上取到

设 u 在 (l,t) 上达到最大值

$$\begin{cases} \cancel{\partial_x(u)}\ \partial_x u > 0 \\ u(l,t) > 0 \end{cases}$$

$$\boxed{\partial_x u + hu > 0}$$ 矛盾

另一边类似。 □

再考虑第二类边值。

$$\begin{cases} Lu = f \\ u(x,0) = \varphi(x) \\ u(0,t) = g_1(t) \quad \partial_x u(l,t) = g_2(t) \end{cases} \rightsquigarrow \begin{cases} Lu = 0 \\ u(x,0) = 0 \\ u(0,t) = 0 \quad \partial_x u(l,t) = 0 \end{cases}$$

将边值转化为第三类

令 $\tilde{u}(x,t) = u(x,t)\,w(x)$ $\qquad u = \dfrac{\tilde{u}}{w}$

$$\partial_t u = \frac{\partial_t \tilde{u}}{w}$$

$$\partial_x u = \frac{\partial_x \tilde{u}}{w} - \frac{\partial_x w \cdot \tilde{u}}{w^2}$$

$$\partial_x^2 u = \frac{\partial_x^2 \tilde{u}}{w} - 2\frac{\partial_x w\,\partial_x \tilde{u}}{w^2} + \frac{(\partial_x w)^2 \tilde{u}}{w^3} - \frac{\partial_x^2 w \cdot \tilde{u}}{w^2}$$

$$\frac{\partial_t \tilde{u}}{w} - \frac{\partial_x^2 \tilde{u}}{w} + 2\frac{\partial_x w}{w^2}\partial_x \tilde{u} - \frac{(\partial_x w)^2 \tilde{u}}{w^3} + \frac{\partial_x^2 w}{w^2}\tilde{u} = 0$$

设 $w>0$. $$\left(\partial_t \tilde{u} - \partial_x^2 \tilde{u}\right) + 2\frac{\partial_x w}{w}\partial_x \tilde{u} - \left(\frac{(\partial_x w)^2}{w^2}\tilde{u} - \frac{\partial_x^2 w}{w}\tilde{u}\right) = 0$$

$$u_x(l,t) = 0 \Rightarrow \left(\tilde{u}_x - \frac{\partial_x w}{w}\tilde{u}\right) = 0$$

$\tilde{u}(0,t) = 0$

$\tilde{u}(x,0) = 0$

$-\dfrac{w_x}{w}$ 在 l 处正.

令. $\boxed{w(x) = l - x + 1} > 1$

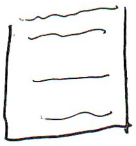
带入计算. $$\left(\partial_t \tilde{u} - \partial_x^2 \tilde{u}\right) - \frac{2}{l-x+1}\partial_x \tilde{u} - \left(\frac{\tilde{u}}{(l-x+1)^2}\right) = 0.$$

令 $v = e^{-\lambda t}\tilde{u}$ $\Rightarrow \partial_t v - \partial_x^2 v - \dfrac{2}{l-x+1}v + \left(\lambda - \dfrac{2}{(l-x+1)^2}v\right) = 0$

令 $\lambda > 2$.

若 $v \not\equiv 0 \Rightarrow v$ 有正的最大值或负的最小值.

设 V 在 $(x_*, t_*) \in Q_T$ 上达到最大



$\partial_t V(x_*, t_*) \geq 0$

$\partial_x V(x_*, t_*) = 0$

$\partial_x^2 V(x_*, t_*) \leq 0$

$V(x_*, t_*) > 0$

$$\underbrace{\partial_t V}_{\geq 0} \underbrace{- \partial_x^2 V}_{\geq 0} \underbrace{\boxed{- \frac{2}{l-x+1}\partial_x V}}_{=0} + \underbrace{\left(\lambda - \frac{2}{(l-x+1)^2}\right) V}_{>0} = 0$$

矛盾!

另一边类似.

$\Rightarrow$ u 的正的最大值只能在边界取到

由边值 且 在 $x=l$ 上取到 $\Rightarrow \partial_x V(x_*, t_*) \geq 0$ 矛盾!

$V(x_*, t_*) > 0$

$\Rightarrow V \equiv 0 \Rightarrow u \equiv 0$

Review

波动方程 $\begin{cases} u_{tt}-\Delta u = f(x,t) & x\in\Omega,\ t\in\mathbb{R} \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \\ \text{边界条件}\ (\Omega\text{有界}) \end{cases}$

1. $\boxed{\Omega=\mathbb{R}^n}$, $n=1$ D'A. 用达式看波的传播性质.

$(\partial_t^2-\partial_x^2)=(\partial_t-\partial_x)(\partial_t+\partial_x)$ ⟶ 特征线法 / 延拓法

2. $n=3$. 球面平均法 ⟶ Kirhoff

3. $n=2$ 降维法 possion 公式

A. 如何看波方程. $\partial_t^2 u-\Delta u=0$

$V=\partial_t u$ 则 $\partial_t\begin{pmatrix}V\\u\end{pmatrix}=\begin{pmatrix}0&1\\\Delta&0\end{pmatrix}\begin{pmatrix}V\\u\end{pmatrix}$

$=\underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{J}\underbrace{\begin{pmatrix}-\Delta&0\\0&1\end{pmatrix}}_{H}\begin{pmatrix}u\\v\end{pmatrix}$

令 $\vec{u}=\begin{pmatrix}u\\v\end{pmatrix}$

$\Rightarrow \partial_t\vec{u}=JHu$ Hamilton系统

$\boxed{J^2=-I}$

B. $-\Delta_x u=f$ $\begin{cases} (t,x)\in\mathbb{R}^{n+1} \\ g=\begin{pmatrix}-1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix}\end{cases}$ M^{1+n} Minkovski 空间

Ω 有界, 特别地 $\Omega=[0,l]$

分离变量法 边值件0. + SL, 后特征函数展开 几种情形

能量估计

① 证明解的唯一性

② 波的传播性质 (在锥台上做能量估计) 考 有限传播速度和能量估计.

二、poission 方程.

1、laplace 方程 $\Delta u=0$. 调和函数的性质，尤其是平均值性质

$\Downarrow$ 估计一点可以转化为积分、

$\Downarrow$ $\begin{cases}\text{Harnack 不等式} \\ \text{梯度估计.}\end{cases}$

2. possion 的解法、基本解与格林函数　　考 求格林函数

①求格林函数

②用格林函数表示 poission 方程的解.

3. 解的唯一性　极值原理与最大模估计.

$\Downarrow$ 引理 强 Hopf　　$\Downarrow$ 唯一性 & 稳定性.

转 $\boxed{\text{能量估计 (乘 } u)}$

三、热传导方程.
$$\begin{cases}\partial_t u - \Delta u = f & x\in\Omega,\ t>0. \\ u(x,0)=\varphi \\ \text{边值 } (\Omega \text{ 有界})\end{cases}$$

1. $\Omega=[0,l]$，分离变量法.

2. $\Omega=\mathbb{R}^n$. Fourier 变换. $u(x,t)=\dfrac{1}{(4\pi t)^{n/2}}\displaystyle\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy+\int_0^t \frac{1}{(4\pi(t-s))^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds$

$$f=0\qquad |u(x,t)|\le \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}|\varphi|$$

$$\begin{cases}\partial_t^2 - \Delta u = 0 \\ u(x,0)=\varphi(x)\quad \partial_t u(x,0)=\psi(x).\end{cases}$$

关于 x 做 Fourier 变换 $\Longrightarrow \begin{cases}\partial_t^2\hat{u}+4\pi^2|\xi|^2\hat{u}=0 \\ \hat{u}(\xi,0)=\hat{\varphi}\quad \partial_t\hat{u}(\xi,0)=\hat{\psi}(\xi).\end{cases}$

$$\hat{u}(\xi)=\hat{\varphi}(\xi)\cos(2\pi|\xi|t)+\frac{\hat{\psi}(\xi)}{2\pi|\xi|}\sin(2\pi|\xi|t)$$

$$\cancel{u(x,t)=\check{\varphi} * (\cos 2\pi|x|t)^{\vee}(x)}$$

$$u(x,t)=\cos t|\nabla|\,\varphi+\frac{\sin t|\nabla|}{|\nabla|}\psi$$

$$\left(\cos 2\pi|\xi|t\,\hat{\varphi}\right)^{\vee} = \left(\frac{e^{i2\pi|\xi|t}+e^{-i2\pi|\xi|t}}{2}\hat{\varphi}\right)^{\vee}$$

$$\int_{\mathbb{R}^n} e^{i2\pi|\xi|t}\cdot e^{2\pi i\xi\cdot x}\hat{\varphi}\,d\xi$$ 振荡积分 哈合…

若 $(t,x)\longrightarrow(\tau,\xi)$ 时空 Fourier 变换

$$\Rightarrow \left(-4\pi^2\tau^2+4\pi^2|\xi|^2\right)\hat{u}(\tau,\xi)=0$$

$$\operatorname{supp}\hat{u}\subseteq\left\{\underbrace{(\tau,\xi)\in\mathbb{R}^{1+n}}\,\middle|\,\tau^2=|\xi|^2\right\}$$ Fourier restriction

解的唯一性.

1. 能量估计. Gronwall 不等式
2. 极值原理. 与最大模估计 辅助函数 (以后)

基本想法: ① 空间分解. $\mathbb{R}^n$ Jordan

② $L^2([0,l])$ 分离变量法

③ $L^2(\mathbb{R}^n)$ Fourier 变换

二、能量估计

$\partial_t u$ $\quad \partial_t^2 u-\Delta u$

u $\quad \partial_t - \Delta u$

u $\quad \Delta u$

Ω 一般区域? 变系数? $\Rightarrow$ 弱解