A SKECHY NOTE OF PDE

Main Ref :

I. ADVANCED PDE JUNYAN ZHANG

II. PDE EVANS

III. FOURIER ANALYSIS AND PDE BAHOURI etc.

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Ch 1. Sobolev Spaces

Sec 1.1 Weak derivatives and Sobolev Spaces.

Weak derivative (uniqueness)

$\downarrow$

Sobolev Spaces $\xrightarrow{\text{Sobolev norm}}$ Banach space

$\boxed{\text{Def 1.1.1}}$ (Weak-derivative) Suppose $u, v \in \boxed{L^1_{loc}(U)}$ and $\alpha = (\alpha_1, \cdots \alpha_d)$.
We say v is the α-th weak derivative of u if

$$\int_U u\, \partial^\alpha \varphi = (-1)^{|\alpha|} \int_U v\, \varphi \quad , \quad \forall \varphi \in C_c^\infty(U).$$

and denote it by $v = \partial^\alpha u$.

If the α-th ~~partial~~ weak partial derivative of u exists, it's "unique".

$\boxed{\text{prop 1.1.2}}$ (Uniqueness of weak derivatives) An α-th weak partial derivative of u, if exists, is uniquely defined up to a set of measure zero.
The result clearly depends on the lemma below.

$\boxed{\text{lem 1.1.3}}$ If $w \in L^1_{loc}(U)$ satisfies $\int_U w \varphi = 0$ for $\forall \varphi \in C_c^\infty(U)$, then $w = 0$ a.e. in U.

$\boxed{\text{Rmk 1.1.4}}$ The lemma should be well-known and it clearly says that C_c^∞ functions are large enough to be ~~test~~ functions. Note that we always consider standard $\boxed{\text{bump functions}}$ when it comes to C_c^∞ functions.

$\boxed{\text{Def 1.1.5}}$ (Sobolev Spaces of integer dimensions) $k \in \mathbb{N}$, $p \in [1, \infty]$

$$W^{k,p}(U) := \{ f \in L^p(U) : \sum_{|\alpha| \le k} \| \partial^\alpha f \|_{L^p(U)} < \infty \}$$

The $\partial^\alpha f$ here should refer to the α-th weak partial derivative of f.
When $p=2$, we denote $H^k(U) := W^{k,2}(U)$.

$\boxed{\text{Def 1.1.6}}$ (norm in $W^{k,p}$) $\quad \|f\|_{W^{k,p}(U)} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^p(U)}$ for $1 \le p \le \infty$

$\boxed{\text{Def 1.1.7}}$ (Convergence) Let $\{f_m\}$, f belong to $W^{k,p}(U)$. We say $f_m \to f$ in $W^{k,p}(U)$ if $\|f_m - f\|_{W^{k,p}(U)} \to 0$ as $m \to \infty$. We say $f_m \to f$ in $W^{k,p}_{loc}(U)$ if $\|f_m - f\|_{W^{k,p}(V)} \to$ as $m \to \infty$ for $\forall\ V \subset\subset U$.

$\boxed{\text{Def 1.1.8}}$ (Closure) We denote by $W^{k,p}_0(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. Thus, $f \in W^{k,p}_0(U)$ iff there exist functions $f_m \in C_c^\infty(U)$ s.t. $f_m \to f$ in $W^{k,p}(U)$.

$\boxed{\text{Rmk 1.1.9}}$ Sobolev spaces equiped with $W^{k,p}$ norm are Banach spaces.

$f \in W^{k,p}_0(U) \iff f \in W^{k,p}(U)$ & $\partial^\alpha f = 0$ on U for $\forall |\alpha| \le k-1$.

The completeness is not hard to verify:

$\|u_m - u_n\|_{W^{k,p}(U)} \to 0$ as $m, n \to \infty$, $\{u_k\} \subseteq W^{k,p}(U)$

$\Rightarrow \partial^\alpha u_m \to$ "u_α" in $\underline{\underline{L^p(U)}}$

Claim: $\partial^\alpha u = u_\alpha$

$$\int_U u\, \partial^\alpha \varphi \overset{DCT}{=} \lim_m \int_U u_m \partial^\alpha \varphi = \lim_m (-1)^{|\alpha|} \int \partial^\alpha u_m\, \varphi$$
$$\overset{DCT}{=} (-1)^\alpha \int u_\alpha \varphi. \qquad \square$$

The second remark is hard to verify right now.

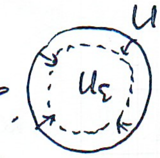
Sec 1.2 Smooth approximation and basic caculus of Sobolev functions.

$$\text{Local} \xrightarrow[\text{bounded}]{\text{PoU}} \text{Global without boundary} \xrightarrow[\text{bounded}]{\text{shift}} \text{Global with boundary}$$

- Chain rule
- composition
- Integral by parts.

Now we fix $k \in \mathbb{N}$, $1 \le p \le \infty$ and an open set $U \subseteq \mathbb{R}^d$.

For each $\varepsilon > 0$, we define $U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) \gneq \varepsilon\} \subseteq U$



$\boxed{\text{Thm } 1.2.1}$ (local) Assume that $f \in W^{k,p}(U)$ for some $1 \le p < \infty$, and set $f_\varepsilon = \eta_\varepsilon * f$ in U_ε. Then $f_\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$ and $f_\varepsilon \to f$ in $W^{k,p}_{loc}(U)$ as $\varepsilon \to 0$.

(1) η_ε is the standard ~~moll~~ mollifier, Bump functions.

$\Rightarrow f_\varepsilon \in C^\infty(\underline{\underline{U_\varepsilon}})$ well-deffineness

(2) $\partial^\alpha f_\varepsilon = \partial^\alpha f * \eta_\varepsilon$:

$$\partial^\alpha f_\varepsilon(x) = \partial_x^\alpha \int_U \eta_\varepsilon(x-y) f(y)\, dy \overset{\text{DCT}}{=\!=} \int_U \partial_x^\alpha \eta_\varepsilon(x-y) f(y)\, dy$$

$$= (-1)^{|\alpha|} \int_U \partial_y^\alpha \eta_\varepsilon(x-y) f(y)\, dy = (-1)^{2|\alpha|} \int_U \eta_\varepsilon(x-y) \partial_y^\alpha f(y)\, dy$$

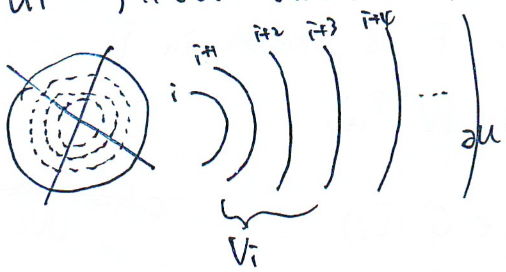
$$= \int_U \eta_\varepsilon(x-y) \partial_y^\alpha f(y)\, dy = \eta_\varepsilon * \partial^\alpha f\,(x).$$

(3) $\forall V \subset\subset U$, $\partial^\alpha f_\varepsilon \xrightarrow{L^p} \partial^\alpha f$ $\square$

$\boxed{\text{Thm } 1.2.2}$ (Global without boundary) Assume U is bounded and $f \in W^{k,p}(U)$ for some $1 \le p < \infty$. Then there exist a sequence of functions $f_m \in C^\infty(U) \cap W^{k,p}(U)$ s.t. $f_m \to f$ in $W^{k,p}(U)$

$U_i := \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{i}\} \xrightarrow{\text{refinement}} V_i = U_{i+3} \setminus \overline{U_{i+1}}$

$V_0 \subset\subset U$



$\Rightarrow U = \bigcup_{i=0}^\infty V_i$ (locally finite)

$\overset{\text{pou}}{\Rightarrow}$ $0 \le \zeta_i \le 1$ $\quad \zeta_i \in C_c^\infty(V_i)$

$\sum_{i=0}^\infty \zeta_i = 1$

$f^i := \eta_{\varepsilon_i} * (\zeta_i f)$ $\quad \operatorname{supp} f^i \subset W_i := W_{i+4} \setminus \overline{W_i}$ $\quad \varepsilon_i$ sufficently small s.t. $\|f^i - \zeta_i f\|_{W^{k,p}(U)} < \frac{\delta}{2^i}$

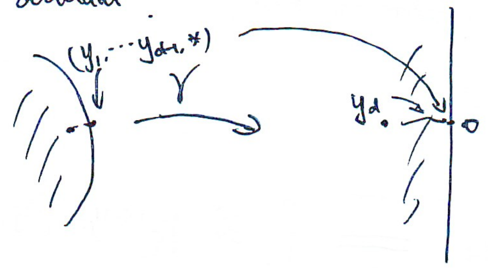
$F := \sum_{i=0}^\infty f^i$ $\quad \Rightarrow \|F - f\|_{W^{k,p}(V)} \le \sum_i \|f^i - \zeta_i f\|_{W^{k,p}(U)} < 2\delta$

locally finite summation $\quad \forall V \subset\subset U$

$\Rightarrow \|F - f\|_{W^{k,p}(U)} < 2\delta \to 0$ $\square$

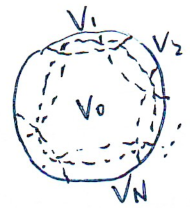
$\boxed{\text{Def 1.2.3}}$ (Boundary Condition) We say the boundary ∂U is Lipschitz continuous if for each $x \in \partial U$, there exists $r>0$ and a Lipschitz continuous mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ s.t. $U \cap B(x,r) = \{ y : y_d > \gamma(y_1, \cdots y_{d-1}) \} \cap B(x,r)$

$\boxed{\text{Rmk 1.2.4}}$ The boundary itself is a ~~d~~ manifold of d-1 dim. We use γ to transform it into a hyperplane in Euclidean space and then the value of this map should ~~denote~~ refer to ~~the~~ height the point in the original domain.



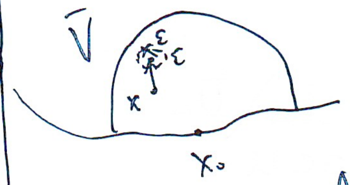
This interpretation may not be correct.

$\boxed{\text{Thm 1.2.5}}$ (Global with boundary) Let $U \subseteq \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary ∂U. Suppose $f \in W^{k,p}(U)$ for some $1 \le p < \infty$. Then there exists functions $f_m \in C^\infty(\bar{U})$ s.t. $f_m \to f$ in $W^{k,p}(U)$



It suffices to construct a smooth approximation in each open cover of ∂U.

$x^\varepsilon = x + \lambda \varepsilon e_d$ $\quad \lambda$ is sufficiently large. $(\lambda \gg \mathrm{lip}(\gamma))$.
ε small



~~E~~ $f^\varepsilon(x) := f(x^\varepsilon)$, $F^\varepsilon = f^\varepsilon * \eta_\varepsilon$

After this shift, F is smooth in $\bar{V}$

If we have shown $F^\varepsilon \to f$ in $W^{k,p}(V)$

$\Rightarrow \exists \|f_i - f\|_{W^{k,p}(V_i)} < \delta \quad f_i \in C^\infty(\bar{V}_i)$ $\quad$ Subcordinate to $\cup V_i$

$U \subseteq \bigcup_{i=0}^N V_i \quad \|f_0 - f\|_{W^{k,p}(V_0)} < \delta \quad . \quad F := \sum_{i=0}^N \zeta_i f_i$

$\Rightarrow \|\partial^\alpha f - \partial^\alpha F\|_{L^p(U)} \le \sum_{i=0}^N \|\partial^\alpha(\zeta_i f) - \partial^\alpha(\zeta_i f_i)\|_{L^p(V_i)}$

$\le \sum_{i=0}^N \|\zeta_i f - \zeta_i f_i\|_{W^{k,p}(V_i)} < (N+1)\delta$

Why ~~the~~ do we have $F^\varepsilon \to f$ in $W^{k,p}(V)$?

$$\|F^\varepsilon - f\|_{L^p(V)} \le \underbrace{\|F^\varepsilon - f^\varepsilon\|_{L^p(V)}}_{\text{mollification}} + \underbrace{\|f^\varepsilon - f\|_{L^p(V)}}_{\text{translation}} \to 0$$

$$\|F^\varepsilon - f^\varepsilon\|_{L^p(V)} = \big\| \, \|\eta(z)[f(x+\lambda\varepsilon e_d - \varepsilon z) - f(x+\lambda\varepsilon e_d)]\|_{L^1_z(B(0,1))} \big\|_{L^p_x(V)}$$

$$\overset{\text{Minkowski}}{\le} \big\| \, |\eta(z)| \, \|f(x+\lambda\varepsilon e_d - \varepsilon z) - f(x+\lambda\varepsilon e_d)\|_{L^p_x(V)} \big\|_{L^1_z(B(0,1))}$$

$$\xrightarrow{\text{DCT}} 0$$

$\square$

$\boxed{\text{prop } 1.2.6}$ (Calculus) Assume $1 \le p < \infty$

(1) If $f, g \in W^{1,p}(U) \cap L^\infty(U)$, then $fg \in W^{1,p}(U) \cap L^\infty(U)$ and $\partial_i(fg) = \partial_i f \cdot g + f \partial_i g$ holds a.e. in U for $i = 1, \cdots d$

(2) If $f \in W^{1,p}(U)$ and $F \in C^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$, $F(0) = 0$, then $F(f) \in W^{1,p}(U)$ and $\partial_i(F(f)) = F'(f)\partial_i f$ a.e. in U for $i = 1, \cdots d$.

Moreover, if U has finite Lebesgue measure in $\mathbb{R}^d$, then $F(0) = 0$ is unnecessary.

(3) If $f \in W^{1,p}(U)$, then $f^+, f^-, |f| \in W^{1,p}(U)$ and

$$\partial f^+ = \begin{cases} \partial f & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f \le 0\} \end{cases}$$

$$\partial f^- = \begin{cases} -\partial f & \text{a.e. on } \{f < 0\} \\ 0 & \text{a.e. on } \{f \ge 0\} \end{cases}$$

$$\partial |f| = \begin{cases} \partial f & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f = 0\} \\ -\partial f & \text{a.e. on } \{f < 0\}. \end{cases}$$

(1) $\varphi \in C_c^\infty(U)$: $\operatorname{supp} \varphi \subseteq V \subset\subset U$ $\quad f_\varepsilon := f * \eta_\varepsilon \quad g_\varepsilon := g * \eta_\varepsilon$

$$\int fg \, \partial_i \varphi \overset{\text{DCT}}{\underset{?}{=}} \lim_{\varepsilon \to 0} \int f_\varepsilon g_\varepsilon \partial_i \varphi \quad \cdots$$

$$\left| \int (fg - f_\varepsilon g_\varepsilon) \partial \varphi \right| \le \int |f(g - g_\varepsilon)\partial \varphi| + \int |g_\varepsilon(f - f_\varepsilon)\partial \varphi|$$

$$\le \|f\|_{L^\infty} \cdot \|g - g_\varepsilon\|_{L^p} \|\partial_i \varphi\|_{L^{p'}} + \cdots \to 0$$

(2) See Sol $\boxed{1.2.3}$

(3) Let $F_\varepsilon(r) = \begin{cases} \sqrt{r^2+\varepsilon^2} - \varepsilon & r \geq 0 \\ 0 & r < 0 \end{cases}$ $\Rightarrow F_\varepsilon(f) \xrightarrow{a.e.} f^+$ as $\varepsilon \to 0$

$$\int_U f^+ \partial_i \varphi \overset{DCT}{=} \lim_\varepsilon \int_U F_\varepsilon(f) \partial_i \varphi = \lim_\varepsilon (-1) \int_U F_\varepsilon'(f) \cdot \partial_i f \varphi$$

$$= \lim_\varepsilon (-1) \int_{U \cap \{f \geq 0\}} F_\varepsilon'(f) \partial f \varphi \overset{W^{1,p}}{\underset{DCT}{=}} (-1) \int_{U \cap \{f > 0\}} \partial f \varphi$$

$$\Rightarrow \partial f^+ = \begin{cases} \partial f & f > 0 \\ 0 & f \leq 0 \end{cases}$$

$\square$

Sec 1.3 Trace and extension

$\boxed{\text{Thm 1.3.1}}$ (Trace theorem) Let $U \subseteq \mathbb{R}^d$ be a bounded open set and ∂U be Lipschitz continuous. Then:

(1) There exists a bounded linear operator $Tr: W^{1,p}(U) \to L^p(\partial U; ds)$ s.t. $Tr f = f$ on ∂U for all $f \in W^{1,p}(U) \cap C(\bar{U})$ and

$$\|Tr f\|_{L^p(\partial U)} \leq C\|f\|_{W^{1,p}(U)}$$

~~(2)~~ for each $f \in W^{1,p}(U)$ with the constant $C>0$ depending only on p, U. Here $ds = \mathcal{H}^{d-1}|_{\partial U}$.

(2) (Integration by parts) For any $\phi \in C^1(\mathbb{R}^d \to \mathbb{R}^d)$ and $f \in W^{1,p}(U)$, there holds

$$\int_U f \,\mathrm{div}\, \phi \, dx = -\int_U \nabla f \cdot \phi \, dx + \int_{\partial U} (\phi \cdot N) \, Tr f \, ds$$

where N denotes the unit outer normal vector to ∂U.

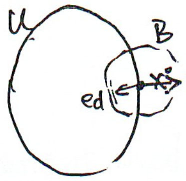
$\boxed{\text{Rmk 1.3.2}}$ $Tr f$ almost provides the boundary value of f on ∂U and it actually satisfies that

$$\lim_{r \to 0} ⨍_{U \cap B(x,r)} |f(y) - Tr f(x)| \, dy = 0 \quad \mathcal{H}^{d-1}\text{-a.e. } x \in \partial U$$

$\boxed{\text{Rmk}}$ Here the conclusion relates to the ~~Rama~~ Rademacher's thm which ~~gar~~ guarantees the outer normal to ∂U exists $\mathcal{H}^{d-1}$-a.e. and the Divergence thm in weak sence.

Since we have Bounded linear transform theorem (BLT thm), so our idea is ~~clearly~~ clear.

U is bounded, so we can assume $f \in C^1(\bar{U})$ and $p=1$ first.



1. $f=0$ on $U \setminus B$ (f is localized near ∂U)

$$-e_d \cdot N = \cos\langle -e_d, N\rangle = \frac{1}{\sqrt{1+\tan^2\langle -e_d, N\rangle}} \gtrsim \frac{1}{\sqrt{1+(\mathrm{Lip}\,\gamma)^2}} \quad (*)$$

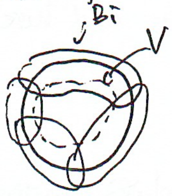
Since $|f|$ is not differentiated everywhere, consider $\beta_\varepsilon(t) = \sqrt{t^2+\varepsilon^2} - \varepsilon$

so that we have $\beta_\varepsilon(f) \longrightarrow |f|$ a.e.

$$\int_{\partial U} \beta_\varepsilon(f)\, dS_x \overset{(*)}{\lesssim} C \int_{\partial U \cap B} \underbrace{\beta_\varepsilon(f)(-e_d \cdot N)}_{(0,\cdots,\beta_\varepsilon(f))} dS_x \overset{\mathrm{div}}{=\!=} -C \int_{U \cap B} \partial_{x_d}(\beta_\varepsilon(f))\, dx$$

$$\lesssim C \int_{U \cap B} |\beta_\varepsilon'(f)| \cdot |\nabla f|\, dx \le C \int_U |\nabla f|\, dx$$

2. $f \not\equiv 0$ on $U \setminus B$



$$\int_{\partial U} \beta_\varepsilon(f) dS_x \overset{\text{finite}}{\lesssim} C \sum_{i=1}^N \int_{\partial U \cap B_i} \zeta_i \beta_\varepsilon(f) (-e_d \cdot N)\, dS_x$$

$$= -C \sum_{i=1}^N \left(\int_{U \cap B_i} \zeta_i \beta_\varepsilon'(f) \partial_{x_d} f\, dx + \int_{U \cap B_i} (\partial_{x_d} \zeta_i) \beta_\varepsilon(f)\, dx \right.$$

$$\lesssim C' \left(\int_U |\beta_\varepsilon'(f)| |\nabla f|\, dx + \int_U |\beta_\varepsilon(f)|\, dx \right)$$

$$\xrightarrow[\text{DCT}]{} C' \|f\|_{W^{1,1}(U)} \quad \text{as } \varepsilon \to 0.$$

3. General index p.

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$\int_{\partial U} |f|^p dS_x \overset{\Downarrow}{\lesssim} C \int_U \Big(\underbrace{|\nabla f| \cdot |f|^{p-1}}_{\text{young}} + |f|^p \Big) dx$$

$$\underbrace{}_{\text{young}} \le \frac{|\nabla f|^p}{p} + \frac{|f|^{(p-1)p'}}{p'}$$

$$\lesssim C \|f\|_{W^{1,p}(U)}$$

$$\Rightarrow \|\mathrm{Tr}\, f\|_{L^p(\partial U)} \lesssim C \|f\|_{W^{1,p}(U)} \quad \text{for } \forall f \in C^1(\bar{U})$$

For function $f \in C^1(\bar{U})$, ~~we~~ we have $\nabla \cdot (f\phi) = \nabla f \cdot \phi + f \nabla \phi$

$$\int_U \nabla \cdot (f\phi)\, dx = \int_{\partial U} f\phi n\, dS_x = \int_{\partial U} \mathrm{tr} f\, \phi N dS_x$$

$$= \int_U \nabla f \cdot \phi + \int_U f \,\mathrm{div}\,\phi$$

$$\Rightarrow \int_U f\, \mathrm{div}\,\phi\, dx = -\int_U \nabla f \cdot \phi\, dx + \int_{\partial U} \mathrm{tr} f\, \phi \cdot N dS_x$$

4. It remains to show the result for $f \in W^{1,p}(U)$.

For a given $f \in W^{1,p}(U)$, there exists a sequence of smooth functions $f_m \in C^\infty(\bar{U})$ s.t. $f_m \to f$ in $W^{1,p}(U)$, and then we have

$$\| tr f_m - tr f_n \|_{L^p(\partial U)} \le C_{p,U} \| f_m - f_n \|_{W^{1,p}(U)}$$

And thus we can define $tr f := \lim\limits_{m} tr f_m$ and it's not hard to prove that the limit doesn't depend the choice of $\{f_m\}$ since we have $f_m \to f$ in $W^{1,p}(U)$ and $\| tr f \|_{L^p(\partial U)} \le C_{p,U} \| f \|_{W^{1,p}(U)}$.

Since $W^{1,p}(U) \cap C^1(\bar{U}) \overset{\text{dense}}{\subseteq} W^{1,p}(U)$, by BLT thm (Simon), we have defined a bounded linear operator from $W^{1,p}(U)$ to $L^p(\partial U)$.

And the integration by parts formula holds still for $W^{1,p}(U)$ by Global smooth approximation. $\square$

The following several theorems are of rather technique difficulties and here I omit the proof but state the conclusions.

$\boxed{\text{Thm } 1.3.3}$ (trace-zero) Assume $U \subseteq \mathbb{R}^d$ is a bounded open set and ∂U is Lipschitz continuous and $f \in W^{1,p}(U)$. Then $f \in W_0^{1,p}(U)$ if and only if $tr f = 0$ on ∂U.

$\boxed{\text{Thm } 1.3.4}$ (Sobolev extension) Let $1 \le p \le \infty$ and $U \subseteq \mathbb{R}^d$ be bounded open set with $\partial U \in C^k$. Assume $G \subseteq \mathbb{R}^d$ is open set with $U \subset\subset G$. Then there exists a constant $C > 0$ depending on d, k, U, G and a bounded linear ~~operation~~ mapping $E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ s.t for any $f \in W^{k,p}(U)$, it holds

(1) $Ef = f$ a.e. in U;

(2) $\operatorname{supp} Ef \subset\subset G$;

(3) $\| Ef \|_{W^{k,p}(\mathbb{R}^d)} \le C \| f \|_{W^{k,p}(U)}$.

Sec 1.4 Sobolev embeddings

Sobolev embeddings $\xrightarrow{\text{dimension}}$

- $p<d$ → GNS — compactness
- regularity — ~~borderline for~~ borderline
- $p>d$ → Morrey — Hölder
- $p=\infty$ Lipschitz

For scaling invariance, we define Sobolev ~~congra~~ conjugate index

$$p^* = \frac{dp}{d-p} \quad \text{for } 1\le p<d \qquad \frac{1}{p^*} = \frac{1}{p}-\frac{1}{d} < \frac{1}{p}.$$
$$p<p^*$$

$\boxed{\text{Thm 1.4.1}}$ (GNS-inequality) Assume $1\le p<d$, there exists a constant $C>0$ only depending on p, d s.t.

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla f\|_{L^p(\mathbb{R}^d)} \quad \forall f\in W^{1,p}(\mathbb{R}^d)$$

$1^* = \frac{d}{d-1}$

1. Assume $f\in C_c^1(\mathbb{R}^d)$ $\Rightarrow |f(x)| \le \int_{\mathbb{R}} |\nabla f|(x_1\cdots t_i \cdots x_d)\,dt_i$

$$|f(x)|^{\frac{d}{d-1}} = \left(|f(x)|^d\right)^{\frac{1}{d-1}} \le \prod_{i=1}^{d} \int_{\mathbb{R}} |\nabla f|(\cdots t_i \cdots)\,dt_i$$

using Hölder d-times $\Rightarrow \left(\int_{\mathbb{R}^d} |f(x)|^{1^*}\right) \le \left(\int_{\mathbb{R}^d} |\nabla f|\,dx\right)^{\frac{d}{d-1}}$

$\Rightarrow \|f\|_{L^{1^*}(\mathbb{R}^d)} \le \|\nabla f\|_{L^1(\mathbb{R}^d)}$

2. $1<p<d \qquad g=|f|^{\gamma} \quad \underline{\underline{\gamma>1}}$

$$\Rightarrow \left(\int_{\mathbb{R}^d} |f|^{\frac{\gamma d}{d-1}}\right)^{\frac{d-1}{d}} \le \gamma\int_{\mathbb{R}^d} |f|^{\gamma-1}|\nabla f|\,dx$$
$$\le \gamma\left(\int_{\mathbb{R}^d} |f|^{(\gamma-1)p'}\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^d}|\nabla f|^p\right)^{\frac{1}{p}}$$

Let $\frac{\gamma d}{d-1} = (\gamma-1)\frac{p}{p-1} \Rightarrow \gamma = \frac{p(d-1)}{d-p} \Rightarrow \frac{\gamma d}{d-1} = p^*$

$\Rightarrow \|f\|_{L^{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla f\|_{L^p(\mathbb{R}^d)}$ for $\underline{\underline{\forall f\in C_c^1(\mathbb{R}^d)}}$

$\Rightarrow$ (*) holds for $\forall f\in W^{1,p}(\mathbb{R}^d)$ $\square$

$\boxed{\text{Thm 1.4.2}}$ (Sobolev embedding) Let $U \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $1 \le p < d$.

(1) Any $f \in W^{1,p}(U)$ belongs to L^{p^*} with the estimation $\|f\|_{L^{p^*}(U)} \le C\|f\|_{W^{1,p}(U)}$ where C only depends on d, p, U.

(2) Any $f \in W_0^{1,p}(U)$ satisfies the estimation $\|f\|_{L^q} \le C\|\nabla f\|_{L^p(U)}$ for all $q \in [1, p^*]$ where the constant C depends on $d\ p\ q\ U$.

Note that we shall avoid zero extension directly since it may break the weak derivative on the boundary.

(1) Since ∂U Lipschitz, we can extend f to Ef s.t.

$\operatorname{supp} Ef$ ~~$\subset\subset V$~~ is compact $\Rightarrow \|Ef\|_{W^{1,p}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(U)}$

Ef has compact support $\Rightarrow \underset{C_c^\infty}{\underset{\Uparrow}{u_m}} \to Ef$ in $W^{1,p}(\mathbb{R}^d)$

$\Rightarrow \|u_k - u_l\|_{L^{p^*}(\mathbb{R}^d)} \le C \cancel{\|u_k - u_l\|}\ \|\nabla u_k - \nabla u_l\|_{L^p(\mathbb{R}^d)} \to 0$

$\Rightarrow u_m \to Ef$ in $L^{p^*}(\mathbb{R}^d)$

$\|u_m\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla u_m\|_{L^p(\mathbb{R}^d)}$ ~~$\to \|\nabla Ef\|_{L^p(\mathbb{R}^d)}$~~

$\Rightarrow$ ~~$Ef \in L^{p^*}(\mathbb{R}^d)$~~ $\|Ef\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla Ef\|_{L^p(\mathbb{R}^d)} \underset{p,d,U}{\le} C\|\cancel{Ef}\, f\|_{W^{1,p}(U)}$

$\|Ef\|_{L^{p^*}(\mathbb{R}^d)} \ge \|f\|_{L^{p^*}(U)}$.

(2) We can use zero extension since $\operatorname{tr} f = 0$. Note $\mathcal{L}(U) < \infty$. $\square$

In the book, there exists a estimation in L^p sence for $W_0^{1,p}(U)$ functions. but here the ~~rate~~ index p is arbitrary so we do not have a direct proof by the embedding thm.

By Banach-~~Ag~~Alaoglu theorem, boundness only implies the weak*-convergence (weak convergence in reflexible space), but we need strong convergence.

$\boxed{\text{Def 1.4.3}}$ (Compact embedding) Let X, Y be two Banach spaces satisfying $X \subseteq Y$. We say X is compactly embedded in Y, denoted by $X \hookrightarrow\hookrightarrow Y$, ~~then~~ if

(1) $\|f\|_Y \leq C\|f\|_X$ for some constant $C>0$;

(2) Each bounded sequence in X is precompact, i.e. any bounded sequence has a convergent subsequence.

$\boxed{\text{Thm 1.4.4}}$ (Rellich–Kondrachov) Let $U \subseteq \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary ∂U. Suppose $1 \leq p < d$. Then $W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U)$ for all $1 \leq q < p^*$.

Embedding $\checkmark$ by GNS + $\mathcal{L}^d(U) < \infty$.

compactness: Assume f_m is uniformly bounded in $W^{1,p}(U)$.

$\sup \|f_m\|_{W^{1,p}(U)} \leq M$. $\boxed{\text{Idea: A-A thm}}$

equicontinuous $\Leftarrow$ gradient estimation

1. mollification: $U \subset\subset V$ extension $\Rightarrow \begin{cases} f_m \in W^{1,p}(\mathbb{R}^d) \\ \operatorname{supp} f_m \subset\subset \underline{V} \text{ bounded.} \\ \sup_m \|f_m\|_{W^{1,p}(\mathbb{R}^d)} < \infty. \end{cases}$

$f_m^\varepsilon := f_m * \eta_\varepsilon \overset{\text{Claim}}{\Rightarrow} \|f_m^\varepsilon - f_m\|_{L^q(V)} \rightrightarrows 0$ in m as $\varepsilon \to 0$.

proof of Claim: $|f_m^\varepsilon - f_m| \leq \int_{B(0,\varepsilon)} \left|\eta_\varepsilon(y)\,(f(x-y)-f(x))\right| dy$

If $\underline{\underline{f_m \in C^1}}$ $\leq \int_{B(0,\varepsilon)} \eta_\varepsilon(y) \left| \int_0^1 \frac{d}{dt} f(x-ty)\, dt \right| dy$

$\|f_m^\varepsilon - f_m\|_{L^1(V)} \leq \int_0^1 \int_{B(0,\varepsilon)} |\eta_\varepsilon(y)|\,|y|\,\|\nabla f(x-ty)\|_{L^1_x(V)}\, dy\, dt$

Maybe refer to evans

$\leq \varepsilon \|\nabla f\|_{L^1(U)} \leq \varepsilon \|1\|_{L^{p'}(V)} \|\nabla f\|_{L^p(V)} \leq CM\varepsilon.$

The estimation holds for $W^{1,p}(V)$

$\|f_m^\varepsilon - f_m\|_{L^q(V)} \overset{\text{Hölder}}{\leq} \|f_m^\varepsilon - f_m\|_{L^1(V)}^\theta \cdot \|f_m^\varepsilon - f_m\|_{L^{p^*}(V)}^{1-\theta}$

$\leq (CM\varepsilon)^\theta \left(\|f_m^\varepsilon\|_{L^{p^*}(V)} + \|f_m\|_{L^{p^*}(V)}\right)^{1-\theta} \leq C'\varepsilon^\theta$

2. Uniform bounded in m

$$|f_m^\varepsilon| \overset{\text{def}}{\le} C\varepsilon^{-d}\|f_m\|_{L^1(V)} \le C'\varepsilon^{-d}\|f_m\|_{L^q(V)} \overset{\text{uniform}}{<\infty}.$$

3. equicontinuity $\Leftarrow$ gradient

$$|\nabla f_m^\varepsilon| \le \|\nabla \eta_\varepsilon\|_{L^\infty(V)}\cdot\|f_m\|_{L^1(V)} \le C\varepsilon^{-1-d} < \infty.$$

$\overset{\text{A-A}}{\Longrightarrow}$ $f^\varepsilon_{m_k} \overset{L^\infty}{\rightrightarrows} *$

$$\limsup_{k,l}\|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^q(V)} \le \limsup_{k,l}\|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^\infty(V)}\|1\|_{L^q(V)} = 0.$$

4. pull back to strong convergence

$$\|f_{m_k} - f_{m_l}\|_{L^q(V)} \le \underbrace{\overset{<\delta_\varepsilon}{\|f_{m_l} - f^\varepsilon_{m_l}\|_{L^q(V)}}}_{\text{uniform in }\varepsilon} + \|f^\varepsilon_{m_l} - f^\varepsilon_{m_k}\|_{L^q(V)} + \underbrace{\overset{<\delta_\varepsilon}{\|f^\varepsilon_{m_k} - f_{m_k}\|_{L^q(V)}}}_{\text{uniform in }\varepsilon}$$

$k,l\to\infty$. $\overline{\lim}_{k,l}\|f_{m_k} - f_{m_l}\|_{L^q} \le C(\varepsilon) \to 0$ as $\varepsilon\to 0$.

Well, ~~there~~ here I made a mistake since the choice of ~~the~~ subsequence depends on ε. Hence we shall use diagonal argument.

~~Let $\delta_n = \frac{1}{n}$, we have $\|f_{m_l} - f^{\varepsilon_n}_{m_l}\| < \delta_n$ when $\varepsilon < \varepsilon_n$~~

~~with $\Rightarrow \{f^{\varepsilon_{i+1}}_{m_k}\} \subseteq \{f^{\varepsilon_i}_{m_k}\}$~~

~~$\forall \varepsilon < \varepsilon_n$ $\overline{\lim}\|f^{\varepsilon_n}_{m_k} - f^{\varepsilon_n}_{m_l}\| = 0 \Rightarrow f^{\varepsilon_n}_{m_k}$ -~~

Let $\delta_n = \frac{1}{n}$. $\exists \varepsilon_n$. $\forall \varepsilon < \varepsilon_n$. $\overline{\lim}_{k,l}\|f^\varepsilon_{m_{k,(n)}} - f^\varepsilon_{m_{l,(n)}}\|_{L^q(V)} = 0$

$$\Rightarrow \overline{\lim}_{k,l}\|f_{m_{k,(n)}} - f_{m_{l,(n)}}\|_{L^q(V)} \le \frac{1}{2^{n-1}}$$

Let $g_k = f_{m_{k,(k)}}$ $\Rightarrow$ $\overline{\lim}_{k,l}\|g_k - g_l\|_{L^q(V)} \to 0$ $\square$

[Rmk 1.4.5] The boundedness assumption is important here.

In Evans's book, he treat an inequality for $W^{1,p}_0(U)$ ~~as Poincáre inequality~~ by Sobolev embedding's corollary. I do not agree with it.

$\boxed{\text{Thm } 1.4.6}$ (Poincáre inequality for zero trace) Assume U is a bounded open set with Lipschitz ∂U. Then we have the estimate

$$\|f\|_{L^p(U)} \le C \|\nabla f\|_{L^p(U)} \quad \text{for } \forall\ 1 \le p \le \infty,\ f \in W^{1,p}_0(U).$$

where C depends on p, d, U.

If $d \le p < \infty$, we can choose an index q s.t. $q^* > p$. Then by GNS ineq, we have

$$\|f\|_{L^p(U)} \lesssim \|f\|_{L^{q^*}(U)} \lesssim_{q,U,d} \|\nabla f\|_{L^q(U)} \lesssim \|\nabla f\|_{L^p(U)}$$

here we can take q relating to p so the constant still only depends on p, U, d.

For the case $p = \infty$, we can extend f to zero outside U. Then we ~~can assume~~ do have a Lipschitz function f by Morrey embedding which is stated in later section. And the proof is direct. $\square$

$\boxed{\text{Thm } 1.4.7}$ (Poincáre inequality) Let $(f)_U$ denote $\frac{1}{\mathcal{L}^d(U)}\int_U f$ if $\mathcal{L}^d(U) < \infty$. Let U be a bounded connected open set with Lipschitz ∂U. Assume $1 \le p \le \infty$. Then there exists a Constant $C > 0$ depending only on d, p, U s.t.

$$\|f - (f)_U\|_{L^p(U)} \le C \|\nabla f\|_{L^p(U)} \quad \forall f \in W^{1,p}(U).$$

prove by contradiction. $\exists \{f_k\}$ s.t. $\|f_k - (f_k)_U\|_{L^p(U)} > k \|\nabla f_k\|_{L^p(U)}$

Renormalization: $g_k = \dfrac{f_k - (f_k)_U}{\|f_k - (f_k)_U\|_{L^p(U)}} \Rightarrow \begin{cases} (g_k)_U = 0 \\ \|g_k\|_{L^p(U)} = 1 \\ \|\nabla g_k\|_{L^p(U)} < \frac{1}{k}. \end{cases}$

By compact embedding (not ~~only~~ only $1 \le p < d$ but for $p \in [1, \infty]$), $\begin{cases} (g)_U = 0 \\ \|g\|_{L^p(U)} = 1 \\ \|\nabla g\|_{L^p(U)} = 0. \end{cases}$ $\square$

Now we're going to deal with the case $p>d$. Let $f^*(x) := \lim_{r\to 0} (f(x))_{x,r}$.

$\boxed{\text{Thm 1.4.8}}$ (Morrey's inequality) Assume $d<p<\infty$. Defined $\alpha := 1-\frac{d}{p}$. There exists a constant $C>0$ depending on ~~only~~ p, d s.t.

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(\mathbb{R}^d)}, \quad \forall f\in \boxed{C_c^1(\mathbb{R}^d)}$$

reminding: $\|u\|_{C^{0,\alpha}(\bar U)} := \sup_{\bar U}|u(x)| + \sup_{\bar U} \frac{|u(x)-u(y)|}{|x-y|^\alpha} = \|u\|_{C(\bar U)} + [u]_{C^{0,\alpha}(\bar U)}$

$$\|u\|_{C^{k,\alpha}(\bar U)} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{C(\bar U)} + \sum_{|\alpha|=k}[\partial^\alpha u]_{C^{0,\alpha}(\bar U)}$$

$C^{k,\alpha}(\bar U)$ equipped with $\|\cdot\|_{C^{k,\alpha}(\bar U)}$ is a Banach space.

For fixed x,y, Let $r:=|x-y|$. $W = B(x,r)\cap B(y,r)$

$$|f(x)-f(y)| = \fint_W |f(x)-f(y)|\,dz \le \fint_W|f(x)-f(z)|\,dz + \fint_W|f(z)-f(y)|\,dz$$

$$\fint_W |f(x)-f(z)|\,dz \overset{z=x+tw}{=} \frac{\mathcal{L}^d(B(x,r))}{\mathcal{L}^d(W)} \fint_W |f(x)-f(z)|\,dz$$

$$\le C_d \frac{1}{\mathcal{L}^d(B(x,r))}\int_{B(x,r)}|f(x)-f(z)|\,dz$$

$$= \frac{C_d}{\mathcal{L}^d(B(x,r))}\int_0^r\int_{\partial B(0,1)}|f(x)-f(x+tw)|\,t^{d-1}\,dS_w\,dt$$

$$= \frac{C_d}{\mathcal{L}^d(B(x,r))}\int_0^r\int_{\partial B(0,1)}\left|\int_0^t \frac{d}{ds} f(x+ws)\,ds\right| dS_w\,dt$$

$$\le \frac{C_d}{\mathcal{L}^d(B(x,r))}\int_0^r\int_{\partial B(0,1)}\int_0^r \frac{|\nabla f(x+sw)|}{s^{d-1}}\, s^{d-1}\,ds\,dS_w\,t^{d-1}\,dt$$

$$= \frac{C_d}{\mathcal{L}^d(B(x,r))}\int_0^r \int_{B(x,r)} \frac{|\nabla f(z)|}{|x-z|^{d-1}}\,dz\; t^{d-1}\,dt$$

$$= \frac{C_d r^d}{d\,\mathcal{L}^d(B(x,r))}\int_{B(x,r)} \frac{|\nabla f(z)|}{|x-z|^{d-1}}\,dz$$

$$\le C_d\|\nabla f\|_{L^p(B(x,r))}\cdot \underbrace{\left\| |x-\cdot|^{1-d}\right\|_{L^{p'}(B(x,r))}}_{\text{finite} \iff (d-1)(p'-1)<1 \iff p>d.\ \checkmark}$$

$$\le C_d\, r^{1-\frac{d}{p}}\|\nabla f\|_{L^p(B(x,r))}. \Rightarrow \alpha = 1-\frac{d}{p} \quad \cdots$$

$\square$

$$|f(x)| \le \underbrace{\fint_{B(x,1)}|f(z)|}_{\|\Omega\|\,\|f\|} + \underbrace{\fint_{B(x,1)}|f(x)-f(z)|\,dz}_{\text{gradient}}$$

$\boxed{\text{Thm 1.4.9}}$ (Morrey's embedding thm) Let $U \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz ∂U. Assume $d < p < \infty$ and $f \in W^{1,p}(U)$. Then f coincide with its precise representative $f^*(x) = \lim_{r \to 0} (f(x))_{x,r}$ a.e. in U. And $f^* \in C^{0,\alpha}(\bar{U})$ with index $\alpha = 1 - \frac{d}{p}$.

By Sobolev extension, $\underset{\substack{\cap \\ C_c^\infty(\mathbb{R}^d)}}{f_m} \longrightarrow f$ in $W^{1,p}(\mathbb{R}^d)$

Using Morrey's Ineq $\|f_m\|_{C^{0,\alpha}(\mathbb{R}^d)} \le C_d \|f\|_{W^{1,p}(\mathbb{R}^d)}$

$\Rightarrow \|\bar{f}\|_{C^{0,\alpha}(\mathbb{R}^d)} \le \|\bar{f}\|_{W^{1,p}(\mathbb{R}^d)}$ (f_m Cauchy in $C^{0,\alpha}(\mathbb{R}^d)$).

Note that $\bar{f} = f$ a.e. (subsequence).

$\Rightarrow f^*(x) \underset{\text{LDT}}{\overset{\text{a.e.}}{=}} \bar{f} \overset{\text{a.e.}}{=} f$. $\square$

Now, we turn to the case $p = \infty$.

$\boxed{\text{Thm 1.4.10}}$ (Lipschitz & $W^{1,\infty}$) Let $U \subseteq \mathbb{R}^d$ be an open set $f : U \to \mathbb{R}$ be given. Then f is locally Lipschitz in U iff $f \in W^{1,\infty}_{loc}(U)$.

Note that ~~Lolo~~ "Locally Lipschitz" denotes f is Lipschitz with restriction to any compact subsets.

$\Rightarrow$ This part is easy.

$\forall x \in V$, $D_i^h(f)(x) := \dfrac{f(x+e_i h) - f(x)}{h}$. Note that $|D_i^h f(x)| \le \mathrm{Lip}(f|W) < \infty$

~~$\Rightarrow$ subsequence $h_j \to 0$ s.t. $D_i^{-h_j}(f)(x) \to v_i$~~

By Banach-Alaoglu thm, $D_i^{-h_j}(f) \longrightarrow v_i$ weakly in $L^p_{loc}(U)$ for $1 < p < \infty$.

It suffices to show $v_i \in L^\infty_{loc}(U)$. Let $L = \mathrm{Lip}(f|W)$. $A = \{v_i \ge L + \varepsilon\}$

$$\underset{\substack{\wedge\!\mid \\ L \cdot \mathcal{L}^d(A)}}{\int_A D_i^{-h_j}(f)\,dx} = \int_W D_i^{-h_j}(f) \cdot \underset{\substack{\cap \\ L^2(W)}}{\chi_A}\,dx \xrightarrow{\text{weak}} \int_A v_i\,dx \ge (L+\varepsilon)\mathcal{L}^d(A)$$

$\Rightarrow \mathcal{L}^d(A) = 0 \Rightarrow v_i \le L$ a.e. similarly, we can prove $v_i \ge -L$

$\Rightarrow v_i \in L^\infty(W)$ for $\forall\, W \subset\subset U$

Now it remains to show v_i is exactly the ~~ith~~ ∂_i-weak partial derivation.

$\forall \varphi \in C_c^\infty(V)$,

$$\int_U f(x) \frac{\varphi(x+he_i)-\varphi(x)}{h} = -\int_U D_i^{-h}(f)(x)\,\varphi(x)$$

$h=h_j \quad j\to\infty$

$$\int_U f(x)\partial_i \varphi(x) = -\int v_i \varphi$$

$\Leftarrow \quad f\in W^{1,\infty}_{Loc}(U) \qquad dist(W,\partial U),\ dist(V,\partial W) < \varepsilon_0$

$$f_\varepsilon := f * \eta_\varepsilon \quad \underline{\underline{\varepsilon<\varepsilon_0}} \qquad (f_\varepsilon \to f$$

Claim: 1. $f_\varepsilon \rightrightarrows F$ in V as $\varepsilon\to 0$

2. F Lipschitz in V

3. $F = f$ a.e. in $\forall\, U \subset\subset V$

$$|f_\varepsilon(x) - f_\delta(x)| = \left|\int_{B(0,1)} \eta(y)\,[\,f(x-\varepsilon y) - f(x-\delta y)\,]\,dy\right|$$

$$\leq \int_{B(0,1)} \eta(y)\,|f(x-\varepsilon y) - f(x-\delta y)|\,dy$$

$$f\in W^{1,\infty}(W) \subseteq W^{1,p}(W) \overset{1<p<\infty}{\Rightarrow} f = f^* \in C^{0,\alpha}(W) \quad \text{a.e. in } W$$

$$\overline{\lim_{\varepsilon,\delta}} |f_\varepsilon - f_\delta| \overset{DCT}{\leq} 0$$

$\Rightarrow f_\varepsilon \rightrightarrows F$ in V, $F\in C(V)$

$$|F(x)-F(y)| \leq \underbrace{|F(x) - f_\varepsilon(x)|}_{\delta} + \underbrace{|f_\varepsilon(x) - f_\varepsilon(y)|} + \underbrace{|f_\varepsilon(y) - F(y)|}_{\delta}$$

$$|f_\varepsilon(x) - f_\varepsilon(y)| = \left|\int_0^1 \nabla f(tx+(1-t)y)\cdot(x-y)\,dt\right| \overset{W^{1,\infty}(V)}{\leq} \|\nabla f\|_{L^\infty(V)} |x-y|.$$

$\delta\to 0 \Rightarrow F$ is Lipschitz in V so f is Lipschitz in V. $\square$

For convenience, we only introduce a special case of general Sobolev ineq which implies $H^k(\Omega) \subseteq L^\infty(\Omega)$ where Ω is bounded and has Lipschitz boundary with the estimate

$$\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{H^k(\Omega)}$$

The proof will be revisited in the later ch.

Ch 2. Linear Elliptic PDEs

1. Divergence form

$$Lu = -\partial_j(a^{ij}(x)\partial_i u) + b^i(x)\partial_i u + c(x)u$$

2. Non-divergence form

$$Lu = -a^{ij}\partial_i\partial_j u + b^i\partial_i u + c(x)u$$

where $a^{ij} = a^{ji}$

L is (uniformly) elliptic if $\exists$ a constant $\theta > 0$ s.t.

$$a^{ij}\xi_i\xi_j \geq \theta|\xi|^2$$

An extra condition in this ch is $u = 0$ on ∂U.

2.1 Weak solution & H^{-1}

$\boxed{\text{Def 2.1.1}}$ (Weak solution) $B[\cdot,\cdot]$ associated with the elliptic operator in the divergence form is

$$B[u,v] := \int_U a^{ij}\partial_i u\,\partial_j v + b^i\partial_i u\, v + cuv \qquad \forall\, u, v \in H_0^1(U)$$

We say that $u \in H_0^1(U)$ is a weak solution if

$$B[u,v] = (f,v)_{L^2(U)} \quad \text{for } \forall\, v \in H_0^1(U)$$

$\boxed{\text{Def 2.1.2}}$ (H^{-1}) We define $H^{-1}(U)$ to be the dual space of $H_0^1(U)$.

$$\|f\|_{H^{-1}(U)} := \sup\{\langle f, u\rangle : u \in H_0^1(U),\ \|u\|_{H_0^1(U)} \leq 1\}$$

$\boxed{\text{Rmk 2.1.3}}$ $H_0^1(U) \subseteq L^2(U) \subseteq H^{-1}(U)$

$\boxed{\text{Thm 2.1.4}}$ (Characterization of H^{-1}) $f \in H^{-1}(U)$ $\exists\, f^0, \cdots, f^d \in L^2(U)$ s.t.

$$\langle f, v\rangle = \int_U f^0 v + f^i\partial_i v\, dx \qquad \forall\, v \in H_0^1(U)$$

If the formula holds, we have $f = f^0 - \sum \partial_i f^i$ ※

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \beta \left(\int_U \sum_{i=0}^{d} (|f^i|^2) \right)^{\frac{1}{2}} \right\}$$

$\Rightarrow (v, u)_{L^2(U)} = \langle v, u \rangle$ for $\forall\, u \in H_0^1(U)$ if we indentify $v \in L^2(U)$ as an element in $H^{-1}(U)$.

Risez : $\Rightarrow$ $\underset{H^{-1}}{\langle f, v\rangle} = \overset{\exists!}{(\underset{H_0^1}{u}, v)}$ $\quad \forall\, v \in H_0^1$

$$= \int_U uv + \nabla u \cdot \nabla v$$

$\Rightarrow f^0 = u \quad f^i = \partial_i u$

$\Rightarrow \langle f, v\rangle = \int_U f^0 v + f^i \partial_i v$

$\Rightarrow$ ~~$\int_U |f^i|^2 =$~~ $\|f\|_{H^{-1}(U)} \le \left(\int |f^i|^2 \right)^{\frac{1}{2}}$

on the other hand, setting $v = \frac{u}{\|u\|_{H_0^1}}$

2.2. Existence theorem I : Lax-Milgram theorem

$\boxed{\text{Thm 2.2.1}}$ (L-M) Bilinear map $B : H \times H \longrightarrow \mathbb{R}$ satisfies

(1) Boundness $|B[u,v]| \le \alpha \|u\| \|v\|$
(2) Coercivity $|B[u,u]| \ge \beta \|u\|^2$

for $\forall\, u, v \in H$ where H is a Hilbert space. Let $f : H \longrightarrow \mathbb{R}$ B a bounded linear functional on H. Then there exists a unique $u \in H$ s.t. $B[u,v] = \langle f, v\rangle$ for $\forall\, v \in H$.

$$\boxed{\begin{aligned} &\langle f, v\rangle = (w, v) \\ &\quad \| \\ &B[u,v] = (Au, v) \end{aligned} \qquad \boxed{u = A^{-1} w}}$$

I. $B[u,v] = (w, v)$ (By risez) $\forall\, v \in H$
$\qquad \| \; (Au, v)$
$A : H \to H$
$\quad u \mapsto w$

II. A is bounded linear trivial
$\|Au\|^2 = (Au, Au) = B[u, Au] \le \alpha \|u\| \|Au\|$

III. Inj & Sur

$$\beta \|u\|^2 \le |B[u,u]| = |(Au, u)| \le \|Au\| \, \|u\| \Rightarrow \text{Inj}$$

$$\forall \underset{\overset{\#}{0}}{w} \in (R(A))^{\perp} \qquad \beta \|w\|^2 \le |(Aw, w)| = 0 \Rightarrow \text{Sur}$$

IV. Existence $\underset{\overset{\#}{\cancel{BC}}}{\langle f, v \rangle} = \underset{!}{\underbrace{(w, v)}} = (Au, v) = B[u,v]$ $\quad u = A^{-1}w$

$\square$

Uniqueness is implied.

$\boxed{\text{Thm 2.2.2}}$ (energy estimate) For the associated case $\boxed{\text{Def 2.1.1}}$, we have

(1) $|B[u,v]| \le \alpha \|u\|_{H_0^1(U)} \cdot \|v\|_{H_0^1(V)}$

(2) $\beta \|u\|_{H_0^1}^2 \le B[u,u] + \gamma \|u\|_{L^2(U)}$

(1) $B[u,v] = \int a^{ij} \partial_i u \partial_j v + b^i \partial_i u v + c u v$

$$\le \|a^{ij}\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|\partial_j v\|_{L^2(U)} + \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|v\|_{L^2(U)}$$

$$+ \|c\|_{L^\infty(U)} \|u\|_{L^2(U)} \|v\|_{L^2(U)}$$

$$\le \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

(2) $B[u,u] = \int \underbrace{a^{ij} \partial_i u \, \partial_j u}_{\text{elliptic}} + b^i \partial_i u \, u + c u^2$

$$\ge \theta \|\nabla u\|_{L^2(U)}^2 - \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|u\|_{L^2(U)} - \|c\|_{L^\infty(U)} \cdot \|u\|_{L^2(U)}^2$$

$$\overset{\text{Young}}{\ge} \theta \|\nabla u\|_{L^2(U)}^2 - \underbrace{\varepsilon \|\nabla u\|_{L^2(U)}^2}_{<\theta} - \left(\frac{C_1}{\varepsilon} + C_2\right) \|u\|_{L^2(U)}^2$$

$$= \underset{>0}{\beta} \|u\|_{H_0^1(U)}^2 - \underset{>0}{\gamma} \|u\|_{L^2(U)}^2 .$$

$\square$

Then, for equation $\begin{cases} Lu + \mu u = f & U \\ u = 0 & \partial U \end{cases}$ where $\underline{\underline{\mu \ge \gamma}}$

There exists a unique weak solution.

2.3 Existence theorem II. Fredholm alternative

A linear bounded operator $K: X \to Y$ is called compact operator if K maps bounded set to precompact set. $K \in \mathcal{C}(X, Y)$

Let X, Y, Z be Banach spaces, then

(1) If $K \in \mathcal{C}(X, Y)$ & $x_n \rightharpoonup x$, then $Kx_n \to Kx$

(2) $K_1 : X \to Y$, $K_2 : Y \to Z \Rightarrow K_2 \circ K_1 \in \mathcal{C}(X, Z)$

(3) $K \in \mathcal{C}(X, Y) \Rightarrow K^* \in \mathcal{C}(Y^*, X^*)$

Fredholm Alternative: X Banach, $K \in \mathcal{C}(X)$

(1) $\dim N(I-K) < \infty$

(2) $R(I-K)$ is closed

(3) $R(I-K) = N(I-K^*)^\perp$ $\quad R(I-K^*) = N(I-K)^\perp$

(4) $N(I-K) = \{0\} \Leftrightarrow R(I-K) = X$

(5) $\dim N(I-K) = \dim N(I-K^*)$

$\sigma(A) =$ A's spectrum

Risez - Schauder $X =$ Banach space $K \in \mathcal{C}(X)$

(1) $0 \in \sigma(K)$ unless $\dim X < \infty$

(2) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$

(3) The accumulation point of $\sigma_p(K)$, if exists, must be $\{0\}$.

$$L = -a^{ij}\partial_j\partial_i u + b^i\partial_i u + cu \quad + b^i \in C^1(\bar{U})$$

$$L^* v := -\partial_i(a^{ij}\partial_j v) - b^i\partial_i v + (c - \partial_i b^i)v$$

$\boxed{\text{Thm 2.3.1}}$ (Second existence theorem)

(1) For any $f \in L^2(U)$, divergence form elliptic equations $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ (*)

admits a unique weak solution $u \in H_0^1(U)$

(2) The homogeneous equation $\begin{cases} Lu = 0 \\ u = 0 & \text{on } \partial U \end{cases}$ admits non-zero solution $u \in H_0^1(U)$

Furthermore, should (2) hold, the solution space N is finite dimensional subspace of $H_0^1(U)$ & $\dim N = \dim N^*$ where N^* denotes the solution space of $\begin{cases} L^* v = 0 & U \\ v = 0 & \partial U \end{cases}$

Finally, (*) has a weak solution $\Leftrightarrow (f, v) = 0$ for $\forall\, v \in N^*$

I. $Lu = f$ $\overset{L_\gamma = L+\gamma I}{\Rightarrow}$ $L_\gamma u = \gamma u + f$ solvable

$\Rightarrow u = L_\gamma^{-1}(\gamma u + f) \Rightarrow (I - \gamma L_\gamma^{-1})u = L_\gamma^{-1} f$

$\Rightarrow K = \gamma L_\gamma^{-1} \quad h = L_\gamma^{-1} f$

$\Rightarrow (I-K)u = h$

II, Verify K is a compact operator on $\underline{L^2(U)}$

We know $B_\gamma[v, \varphi] = (g, \varphi)$ holds for each $g \in L^2(U)$ & $\varphi \in H_0^1(U)$

$\Rightarrow \beta \|v\|^2_{H_0^1(U)} \le |B_\gamma[v, v]| = |(g, v)_{L^2(U)}| \le \|g\|_{L^2(U)} \|v\|_{H_0^1(U)}$

$\Rightarrow \|Kg\|_{H_0^1(U)} \lesssim \|g\|_{L^2(U)}$

$\Rightarrow K : L^2(U) \to H_0^1(U) \ (\hookrightarrow\hookrightarrow L^2(U)) \Rightarrow K$ is compact.

III. Fredholm Alternative $X = L^2(U)$

Case I. $N(I-K) = \{0\}$ $\Rightarrow$ unique solution for $\forall\, h \in L^2(U)$

Case II. $N(I-K) \neq \{0\}$ $\Rightarrow \gamma \neq 0$

homogeneous equation $u - Ku = 0$ has non-zero solutions in $L^2(U)$.

$\dim N(I-K) = \dim N(I-K^*)$

IV. Verify the further argument (existence of solutions $\Leftrightarrow (f, v) = 0$)

The adjoint equation of $Lu = 0$ ($\Leftrightarrow (I-K)u = 0$) is

$L^* v = 0$ ($\Leftrightarrow (I-K^*) v = 0$)

v is a weak solution of $(I-K^*) v = 0$

$(h, v) = \gamma^{-1}(Kf, v) = \gamma^{-1}(f, K^* v) = \gamma^{-1}(f, v)$

but we have known that $(I-K)u = h$ has a solution $\Leftrightarrow (h, v)_{L^2(U)} = 0$

$\Leftrightarrow (f, v) = 0$ where $v \in \ker(I-K^*)$. □

I was confused that we just prove $u \in L^2(U)$, how can we claim $u \in H_0^1$ initially. But I found that $L_\gamma : H_0^1 \to H^{-1}$ is an isomorphism

$\Rightarrow L_\gamma^{-1} : H^{-1} \to H_0^1$

$u = \underset{\sim}{Ku} + \underset{\sim}{h} \in H_0^1$ Done!

$\boxed{\text{Thm 2.3.2}}$ There exists an at most countable set $\Sigma \subseteq \mathbb{R}$ s.t the boundary-value problem $\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has a unique solution (weak) for $\forall f \in L^2(U)$

if & only if $\lambda \notin \Sigma$. Moreover, if Σ is an infinite set, then $\Sigma = \{\lambda_k\}$, then values of a nondecreasing sequence with $\lambda_k \to +\infty$.

We call Σ the spectrum of the operator L

Note in particular that the boundary value problem

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a non-trivial solution $w \not\equiv 0$ if and only if $\lambda \in \Sigma$. in which case λ is called an eigenvalue of L, w a corresponding eigenfunction.

WLOG, let $\lambda > -\gamma$, $\gamma > 0$. By Fredholm theory, $Lu = \lambda u + f$ admits a unique solution if & only if the homogeneous equation

$Lu = \lambda u$ has only zero solution.

$u = \frac{\gamma+\lambda}{\gamma} Ku$ where $k = \gamma L_\gamma^{-1}$ only has the solution $u=0$.

$\Leftrightarrow$ that is to say $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K.

Then we can use Riesz - Shauder thm. $\square$

2.4 The eigenvalue problem of linear elliptic operators

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$ where U is a bounded domain with smooth boundary.

Here we consider $Lu = -\partial_j(a^{ij}\partial_i u)$ $\quad a^{ij} = a^{ji}$ $\quad a^{ij} \in C^\infty(\bar{U})$

A preliminary acknowledgement which was not mentioned for most textbook is Courant-Fisher minimax principle. It states that $A \in \mathcal{L}(H)$ is symmetric and have eigenvalues $\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq 0 \geq \cdots \geq \lambda_2^- \geq \lambda_1^-$. Then

$$\lambda_n^+ = \inf_{E_{n-1}} \sup_{E_{n-1}^\perp} \frac{(Ax,x)}{\|x\|}, \quad \lambda_n^- = \sup_{E_{n-1}} \inf_{E_{n-1}^\perp} \frac{(Ax,x)}{\|x\|}$$

Its idea is kind direct and proof depends on Rayleigh quotient.

$\boxed{\text{Thm } 2.4.1}$ (Eigenvalues of symmetric ~~compact~~ elliptic operators) Each eigenvalue of L is a real num.

Furthermore, if we repeat each eigenvalue according to its multiplicity, we have

$\Sigma = \{\lambda_k\}$ where $0 \le \lambda_1 \le \cdots$ $\lim_{k\to\infty} \lambda_k = +\infty$.

Finally, there exists an ~~ortho~~ orthogonormal basis $\{w_k\}$ of $L^2(U)$ where $w_k \in H_0^1(U)$

s.t $\quad Lw_k = \lambda_k w_k$ in U. $\quad w_k = 0$ on ∂U

> By regularity theory, $a^{ij} \in C^\infty(\bar{U})$ implies that $w_k \in C^\infty(U)$
> and "∂U is smooth" implies $w_k \in C^\infty(\bar{U})$.

I. By Lax-Milgram theorem. L is invertible. Let $S = L^{-1} : L^2(U)$

$\to H_0^1(U) \subset\subset L^2(U)$. That is to say S is a compact operator.

II. $u := Sf$. $v := Sg$ where $f, g \in L^2(U)$.

$$(Sf, g)_{L^2(U)} = (u, g)_{L^2(U)} = B[v, u] \overset{A}{\underset{\text{sym}}{=}} B[u, v] = (f, Sg)_{L^2(U)}$$

$\Rightarrow S$ is symmetric. Since $Lu = -\partial_j(a^{ij}\partial_i u)$, $B[\cdot,\cdot]$ is strictly positive defined by poincáre ineq

$(Sf, f) = B[u, u] \ge 0$ for $\forall f \in L^2(U) \Rightarrow$ All the eigenvalues of

S is positive real numbers.

IV $Sw = \eta w$ for some $w \in H_0^1(U)$ $\Rightarrow Lw = \lambda w$ where $\lambda = \eta^{-1}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \neq 0$

IV . Risez-Schauder.

□

$\boxed{\text{Thm } 2.4.2}$ (~~Prin~~ Variation principle of principal eigenvalue)

Let $\lambda_1 > 0$ be the principal eigenvalue of elliptic operator L with vanishing boundary condition where $Lu = -\partial_j(a^{ij}\partial_i u)$ with $a^{ij} = a^{ji}$.

(1) We have $\quad \lambda_1 = \min\{ B[u,u] : \|u\|_{L^2(U)} = 1 \}$

(2) The minimum is attained for a smooth function w_1 ~~that~~ doesn't change sign within U. Also, $\begin{cases} Lw_1 = \lambda_1 w_1 & \text{in } U \\ w_1 = 0 & \text{on } \partial U. \end{cases}$

(3) Finally, if ~~to solve~~ $u \in H_0^1(U)$ is any solution to $\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

the u is a multiple of w_1. This implies that λ_1 must be simple.
$(0 < \lambda_1 < \lambda_2 \le \cdots)$

(1) the eigenfunctions $\{w_k\}$ of L form an orthogonormal basis of $L^2(U)$

$(\cdot,\cdot)_{H_0^1(U)} := B[\cdot,\cdot]$

Consider $\{w_k/\sqrt{\lambda_k}\}$ $\Rightarrow$ $\{w_k/\sqrt{\lambda_k}\}$ is ~~orth~~ orthonormal

Suppose $\underset{\underset{H_0^1 \subseteq L^2}{\cap}}{u} = \sum_k^\infty d_k w_k$

$B[w_k, u] = \sum_j d_j B[w_k, w_j] = d_k \underbrace{\lambda_k}_{\neq 0}(w_k, w_k) \Rightarrow d_k = 0.$

$\Rightarrow u = \sum_k d_k\sqrt{\lambda_k}\frac{w_k}{\sqrt{\lambda_k}}$ (Here we assume $\sum d_k^2 = 1$)

$B[u,u] = \sum d_j^2 \lambda_j \geq \lambda_1 \sum d_j^2 = \lambda_1$

the equality holds when $u = w_1$.

(2) Claim : $B[u,u] = \lambda_1 \Leftrightarrow Lu = \lambda_1 u$ where $\|u\|_{L^2(U)} = 1$

($\Leftarrow$) trivial

($\Rightarrow$) $u = \sum d_k w_k$ with $\sum d_k^2 = 1$

$\lambda_1 = B[u.u] = \sum \lambda_k d_k^2 \Rightarrow d_k = 0$ if $\lambda_k > \lambda_1$

$\overset{\|}{\lambda_1 \sum d_k^2}$

The multiplicity of λ_1 is finite (Fredholm) ✓

Now $Lu = \lambda_1 u$ $\alpha := \int (u^+)^2$ $\beta = \int (u^-)^2$ $\Rightarrow u^+, u^- \in H_0^1(U)$

$\lambda_1 = B[u,u] = B[u^+,u^+] + B[u^-.u^-] \overset{\|}{\geq} \lambda_1\alpha + \lambda_1\beta = \lambda_1$

$\Rightarrow$ Since the equality holds, u^+, u^- are also eigenfunctions corresponding to λ_1.

Using $\boxed{\text{interior regularity}}$ $u^+. u^- \in C^\infty(U)$

Using Strong maximum principle $\underline{u^+ > 0}$ or $u^+ \equiv 0$

$\Downarrow$ $u > 0$; $\Downarrow$ $u \leq 0$ $\to$ $u < 0$ or $u \equiv 0$

(3) $u, \tilde{u}$ are both nonzero weak solution to $Lu = \lambda_1 u$.

$\Rightarrow \exists\, C$ s.t. $\int \underbrace{u - C\tilde{u}}_{\text{also eigenfunction}} = 0 \Rightarrow u = C\tilde{u}$ □

$\boxed{\text{Rmk 2.4.3}}$ $\lambda_1 = \min_{\substack{u \in H_0^1(U)\\ \neq 0}} \frac{B[u,u]}{\|u\|^2_{L^2(U)}}$

2.5 Elliptic regularity

$$\begin{cases} f \in L^2(U) \Rightarrow u \in H^2(U) \\ f \in C(\bar{U}) \not\Rightarrow u \in C^2(U) \\ \alpha \in (0,1) \quad f \in C^{0,\alpha}(\bar{U}) \Rightarrow u \in C^{2,\alpha}(\bar{U}) \end{cases}$$

$Lu = f$
$\Downarrow$
sufficiently ~~reg~~ regular ?

$$D_i^h f(x) = \frac{f(x+e_i h) - f(x)}{h} \quad \text{where } x \in V \subset\subset U \quad 0 < h < \operatorname{dist}(V, \partial U)$$

and f is locally integrable.

$\boxed{\text{prop 2.5.1}}$ (Difference Quotients & weak derivatives)

1) $1 \le p < \infty$ $\quad f \in W^{1,p}(U)$, $\quad V \subset\subset U$

$$\|D^h f\|_{L^p(V)} \le C \|\nabla f\|_{L^p(U)} \qquad \text{where } 0 < |h| < \tfrac{1}{2}\operatorname{dist}(V, \partial U)$$

2) $1 \boxed{<} p < \infty$ $\quad f \in L^p(V)$
& $\|D^h f\|_{L^p(V)} \le C$ } Condition

Then ~~In particular,~~ if $f \in W^{1,p}(U) \Rightarrow \|\nabla f\|_{L^p(V)} \le C$.

---

1) WLOG, we can assume f is smooth sufficient. $\Rightarrow p \in [1, \infty)$

$$|f(x+he_i) - f(x)| \le |h| \int_0^1 |\nabla f(x+the_i)|\, dt$$

$$\int_V |D^h f|^p dx \lesssim \sum_d \int_V \int_0^1 |\nabla f(x+the_i)|^p dt$$

$$\Rightarrow \lesssim \|\nabla f\|^p_{L^p(U)}.$$

2) $$\int_V D_i^h u \, \varphi = - \int_V u \, D_i^{-h} \varphi \qquad \varphi \in C_c^\infty(V)$$

$$\sup_h \|D_i^h u\|_{L^p(V)} < \infty \xRightarrow{B\text{-}A} D_i^{h_k} u \to v \text{ in } L^p$$

$$\Rightarrow \int u \, \partial \varphi = \lim \int u \, D_i^h \varphi = -\lim \int D_i^{-h} u \, \varphi = - \int v \, \varphi \qquad \square$$

$\boxed{\text{Thm 2.5.2}}$ (Interior regularity) $a^{ij} \in C^1(U)$ $\quad b^i, c \in L^\infty(U)$, $f \in L^2(U)$
u is a weak solution $Lu = f$ in U. Then $u \in H^2_{loc}(U)$ & for $\forall V \subset\subset U$

$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right) \qquad C = C(V, U, L).$$

We shall note that "$a^{ij}, b^i, c \in C^{m+1}(U)$" + "$f \in H^m(U)$" $\Rightarrow$ "$u \in H^{m+2}_{loc}(U)$"

& "$\|u\|_{H^{m+2}(V)} \le c(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$ with $c = c(m, V, U, L)$"

We can prove it by iteration and we shall believe its correctness.

proof of original theorem:

Since we already have the weak solution of the divergence form equation,

WLOG, we can treat $-\partial_j(a^{ij}\partial_i u) = f$ for convenience.

$$\Rightarrow \int_U \cancel{\partial_j(a^{ij}v)}\, a^{ij}\partial_i u\, \partial_j v = \int_U f v$$

weak derivative or difference quotient $\boxed{D(Du)}$

localization

truncate $V \subset\subset W \subset\subset U$ $\begin{cases} \zeta = 1 & \text{in } V \\ \zeta = 0 & \text{in } \mathbb{R}\setminus W \end{cases}$

A choice of v is from the classical book: $v = -D_k^{-h}(\zeta^2 D_k^h u)$

$$LHS = -\int_U a^{ij}\partial_i u\, \partial_j(D_k^{-h}\zeta^2 D_k^h u) \overset{\text{exchange}}{=\!=\!=} -\int_U a^{ij}\partial_i u\, D_k^{-h}\partial_j(\zeta^2 D_k^h u)$$

$$= \int D_k^h(a^{ij}\partial_i u)\, \partial_j(\zeta^2 D_k^h u)$$

four terms

$$= \int_U D_k^h(a^{ij})\, \partial_i u\, \partial_j(\zeta^2)\, (D_k^h u) + a^{ij,h} D_k^h \partial_i u\, \partial_j(\zeta^2)\, D_k^h u$$

$$+ \int_U \underbrace{a^{ij,h} D_k^h(\partial_i u)\, \zeta^2\, \partial_j(D_k^h u)}_{A_1} + \underbrace{D_k^h(a^{ij})\, \partial_i u\, \zeta^2\, \partial_j(D_k^h u)}_{\text{otherwise } A_2}$$

$$= A_1 + A_2$$

$$A_1 = \int_U a^{ij,h} D_k^h(\partial_i u)\, \zeta^2\, \partial_j(D_k^h u) = \int_U a^{ij}\, \cancel{\partial_j(D_k^h u)}\, D_k^h(\partial_i u)\, D_k^h(\partial_j u)\, \zeta^2$$

$$\overset{\text{Elliptic}}{\geq} \theta \int_U |D_k^h(Du)|^2 \zeta^2 \qquad L^2 \text{ form of } D_k^h(Du)\zeta$$

$$|A_2| \le \int_U |D_k^h(a^{ij})\, \partial_i u\, D_k^h u\, \zeta_j \zeta| + |a^{ij,h} D_k^h \partial_i u\, D_k^h u\, \zeta_j \zeta| + |D_k^h a^{ij}\, \partial_i u\, D_k^h \partial_j u\, \zeta^2|$$

Not so important

$a^{ij} \in C^1(U)$ + truncate

$$\le c\|a^{ij}\|_{C^1(U)} \int_U \cancel{|D_k^h(a^{ij})|}\, |\partial_i u|\, |D_k^h u|\, \zeta + \cancel{a^{ij,h}}\, |D_k^h(\partial_i u)|\, |D_k^h(u)|\, \zeta + |\partial_i u|\, |D_k^h \partial_j u|\, \zeta \le \theta\|a^{ij}\|_{C^1(U)}\Big(\varepsilon\|\zeta D_k^h Du\|^2_{L^2(U)} + \frac{c}{\varepsilon}\|D_k^h u\|^2_{L^2(W)} + \frac{\theta c}{\varepsilon}\|Du\|^2_{L^2(W)}\Big)$$

Using $\boxed{\text{prop } 2.5.1}$, we can obtain

$$|A_2| \le \varepsilon \|\zeta D_k^h \nabla u\|_{L^2(U)}^2 + \frac{C}{\varepsilon}\|\nabla u\|_{L^2(U)}^2 \quad \text{Let } \varepsilon = \frac{\theta}{2}$$

$$\Rightarrow A = A_1 + A_2 \ge \frac{\theta}{2}\|D_k^h \nabla u \cdot \zeta\|_{L^2(U)}^2 - \frac{C}{\varepsilon}\|\nabla u\|_{L^2(U)}^2$$

$$B = \int_U f \underbrace{D_k^{-h}(\zeta^2 D_k^h u)}_{v}$$

$$\|D_k^{-h}(\zeta^2 D_k^h u)\|_{L^2(U)}^2 \le \|\nabla(\zeta^2 D_k^h u)\|_{L^2(U)}^2 = \|\nabla(\zeta^2) D_k^h u\|_{L^2(U)}^2 + \|\zeta^2 \nabla D_k^h u\|_{L^2(U)}^2$$

$$= \|\nabla(\zeta^2) D_k^h u\|_{L^2(U)}^2 + \|\zeta^2 D_k^h \nabla u\|_{L^2(U)}^2$$

$$B \le \varepsilon\left(\|\nabla(\zeta^2) \underline{D_k^h} u\|_{L^2(U)}^2 + \|\underline{\zeta^2 D_k^h \nabla u}\|_{L^2(U)}^2\right) + \frac{C}{\varepsilon}\|f\|_{L^2(U)}^2$$

$$\le \frac{\theta}{4}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 + C\left(\|f\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2\right)$$

$$\Rightarrow \frac{\theta}{4}\|\zeta D_k^h \nabla u\|_{L^2(U)} \le C\left(\|f\|_{L^2(U)} + \|\nabla u\|_{L^2(U)}\right)$$

$$\Rightarrow \cancel{\|D_k^h \nabla u\|_{L^2(}} \|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{\underline{H^1(U)}}\right)$$

Since $V \subset\subset W \subset\subset U$

$$\Rightarrow \text{refine: } \|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right)$$

~~Sim~~ likewise: $\|u\|_{H^1(W)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$

$$\Rightarrow \|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right) \Rightarrow u \in H^2_{loc}(U) \qquad \square$$

$\boxed{\text{Thm } 2.5.3}$ (Boundary regularity) U bounded open ∂U is C^2

$a_{ij} \in C^1(\bar{U})$ $b_i, c \in L^\infty(U)$, $f \in L^2(U)$ $u \in H_0^1(U)$ is the weak solution

to $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \Rightarrow u \in H^2(U)$ with $\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$

$C = C(U, L)$.

By interation, we shall see

$$\begin{cases} \partial u \in C^{m+2} \\ a^{ij}, b^i, c \in C^m(\bar{U}) \\ f \in H^m \end{cases} \Rightarrow u \in H^{m+2} + \text{"} \|u\|_{H^{m+2}} \le C\left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\right)\text{"}$$

$\boxed{\text{Rmk 2.5.4}}$ After proving regularity thm, $Lu=f$ a.e holds not only for weak sense.

Before longish proof, we shall see two basic calculus formula.

1. $\Phi : \Omega_x \longrightarrow \Omega_y \qquad \psi(y) = \varphi(x)$

$\qquad x \longmapsto y = \Phi(x)$

$$\Rightarrow \text{"} \nabla_y \psi(y) = D(\Phi^{-1})(y)^T \nabla_x \phi(x) \text{"}$$

Since $\nabla_y \psi(y) = \left(\sum_j \frac{\partial \varphi}{\partial x^j} \frac{\partial x^j}{\partial y^i} \right)_i = \left(\frac{\partial x^j}{\partial y^i} \right) \left(\frac{\partial \varphi}{\partial x^j} \right) = D(\Phi^{-1})^T(y) \nabla_x \varphi(x)$

2. $F = \begin{pmatrix} F_1 \\ \vdots \\ F_d \end{pmatrix}$, $\forall \varphi \in C_c^\infty(\cdot\cdot) \qquad |J(y)| = |\det D(\Phi^{-1})(y)|$

$$\int_{\Omega_x} \nabla_x \cdot F(x) \varphi(x) dx = -\int_{\Omega_x} F(x) \cdot \nabla_x \varphi(x) dx$$

$$\overset{\text{to } y}{\Longrightarrow} \int_{\Omega_y} (\nabla_x \cdot F)(\Phi^{-1}(y))\, \varphi(\Phi^{-1}(y)) \frac{dy}{\cancel{|J(y)|}}|J(y)| = -\int_{\Omega_y} F(\Phi^{-1}(y)) \cdot (\nabla_x \varphi)(\Phi^{-1}(y)) \frac{dy}{\cancel{|J(y)|}}|J(y)|$$

$$\text{RHS} = -\int_{\Omega_y} F(\Phi^{-1}(y)) \cdot (\nabla_x \varphi)(\Phi^{-1}(y)) |J(y)|\, dy$$

$$= -\int_{\Omega_y} F(\Phi^{-1}(y)) \cdot \left[D(\Phi^{-1})(y)\right]^{-T} \nabla_y \varphi(\Phi^{-1}(y))\, |J(y)|\, dy$$

$$= -\int_{\Omega_y} \underbrace{\left(|J(y)| (D(\Phi^{-1})(y))^{-1} F(\Phi^{-1}(y)) \right)}_{G(y)\quad \text{Piola transform}} \nabla_y \varphi(\Phi^{-1}(y))\, dy$$

$$\Rightarrow (\nabla_x \cdot F)(\Phi^{-1}(y)) = \frac{1}{|J(y)|} \nabla_y \cdot G(y)$$

$$\Rightarrow (\nabla_x \cdot F)(\Phi^{-1}(y)) = \frac{1}{|D(\Phi)(y)|} \nabla_y \cdot \left(\cancel{\tfrac{1}{|D}}\, |D(\Phi^{-1})(y)|\, \underbrace{(D(\Phi^{-1})(y))^{-1}}_{D\Phi(x)} F(\Phi^{-1}(y))\right).$$

Proof of $\boxed{\text{thm 2.5.3}}$

The special case of "" is quite easy.

We immediately obtain

$$\|\partial_i\partial_j u\|^2_{L^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{H^1(U)}\right) \quad \text{for } \forall\, i+j<2d.$$

for $\partial_d\partial_d$; Since $Lu=f$ a.e

$$\Rightarrow a^{dd}\partial_d^2 u = \sum_{i+j<2d} -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu - f - \partial_d a^{dd}\partial_d u$$

Elliptic $\Rightarrow$ $a^{dd} \ge \theta$

$$\Rightarrow \|\partial_d^2 u\|^2_{L^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{H^1(U)}\right)$$

Mimicing the previous proof we can get the wanted estimate.

But, How ~~can~~ do we treat the general case?

In Evans's book, he showed longish calculation without motivation, Here I'm going to explain it clearly.

In Ω_x, we have $-\partial_j(a^{ij}\partial_i u) = -\nabla_x\cdot\left(A(x)\nabla_x u(x)\right)$

Using previous two formula, we have $-\nabla_y\cdot\left(\tilde{A}(y)\nabla_y \tilde{u}(y)\right)$

where $\tilde{u}(y) = u(\Phi^{-1}(y))$ and $\tilde{A}(y) = D\Phi(\Phi^{-1}(y))\,A(\Phi^{-1}(y))\,D\Phi(\Phi^{-1}(y))^T$.

We shall always use the notation $\sim$ to represent the pull up map.

The formula of $\tilde{A}(y)$ implies the formula $\tilde{a}^{kl} = \sum a^{rs}\Phi^k_r\Phi^l_s$

which agrees with Evans's book.

likewise. $b^i\partial_i u(x) = b(x)\nabla_x u(x) = b(x)\,D\Phi(\Phi^{-1}(y))^T\nabla_y\tilde{u}(y)$

$$\Rightarrow \tilde{b}^k = \sum b^r\Phi^k_r$$

Since $|D\Phi| = 1$ $\Rightarrow$ $\tilde{L}$ is still uniformly elliptic

Now, it's clear that $\tilde{u}(y)$ is the weak solution of $\tilde{L}\tilde{u} = \tilde{f}$

since the only thing we do is to change variable. $\square$

2.6 Maximum principle (non-divergence form with continuous coefficients)

Energy estimate $\rightsquigarrow$ Pointwise estimate : maximum principle

$\nabla u(x_0)=0$ $\quad \partial^2 u$ is nonpositive definite 

$\boxed{\text{Thm 2.6.1}}$ (Weak maximum principle) $u\in C^2(U)\cap C(\bar U)$, $c=0$ in U

(1) $Lu\le 0$ in U $\Rightarrow$ $\max\limits_{\bar U} u = \max\limits_{\partial U} u$ $\quad$ u is called a subsolution

(2) $Lu\ge 0$ in U $\Rightarrow$ $\min\limits_{\bar U} u = \min\limits_{\partial U} u$ $\quad$ u is called a supersolution.

The most important trick in proving these principles , in my opinion, is the perturbation trick.

" $Lu\le 0 \Rightarrow Lu^{\varepsilon}<0 \Rightarrow u^{\varepsilon}$ satisfies the conclusion "

$\Rightarrow \varepsilon\to 0^+$, u does so.

$$L(\varepsilon e^{\lambda x_1}) = \cancel{\varepsilon a^{11}}\, a^{11}\varepsilon e^{\lambda x_1}(-a^{11}\lambda_1^2+b^1\lambda) <0 \quad \text{for } \lambda \text{ sufficient large}$$
$$\le \varepsilon e^{\lambda x_1}\left(-\theta\lambda^2+\|b^i\|_{L^\infty(U)}\lambda\right)$$

Now, it remains to show the case $Lu<0$. We shall see that a non-divergence form elliptic operator is very similar to Laplacian. "$-\Delta$".

Suppose $\quad OAO^T = \Lambda = \mathrm{diag}(\lambda_1,\cdots\lambda_d) \qquad \lambda_i\ge 0$

$$y = x_0 + O(x-x_0)$$

$$\Rightarrow\begin{cases}\partial_{x_i}u = \sum\limits_k \partial_{y^k}u\,\dfrac{\partial y^k}{\partial x^i} = \sum\limits_k \partial_{y_k}u\, O_{ki}\\[2ex] \partial_{x_i}\partial_{x_j}u = \sum\limits_{k,l} O_{ki}\,\partial_{y_k y_l}u\, O_{lj}\end{cases}$$

Then
$$-a^{ij}\partial_i u\,\partial_j u = \sum_{i,j} -a^{ij}\sum_{k,l} O_{ki}\,\partial_{y_k y_l}u\, O_{lj}$$
$$= -\sum_{k,l}\Big(\sum_{i,j} O_{ki}a^{ij}O_{lj}\Big)\partial_{y_k}\partial_{y_l}u = -\sum_{k,l}\delta_{kl}\lambda_k\,\partial_{y_k}\partial_{y_l}u$$

if it's an interior point,
$$= \sum_k -\lambda_k\,\partial^2_{y_k}u$$

$$\downarrow$$

$$Lu(x_0) = -a^{ij}\partial_{x_i}\partial_{x_j}u = \sum_k \underbrace{-\lambda^2}_{\ge 0}\underbrace{\partial^2_{y_k}u}_{<0} <0 \qquad \text{contradiction!}$$

$\square$

From the previous proof, it's not hard to see the thm below.

$\boxed{\text{Thm } 2.6.2}$ (Weak maximum principle for $c \geq 0$) The same conditions as the last thm except $c \geq 0$ in U. We have

$$Lu \leq 0 \Rightarrow \max_{\bar{U}} u \leq \max_{\partial U} u^+$$

Now, we want to have a certain expression ~~about~~ when we can obtain the maximal value. We need a "geometric lemma" to state it precisely.

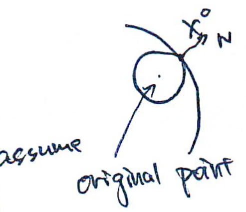
$\boxed{\text{Lem } 2.6.3}$ (Hopf's lemma) $u \in C^2(U) \cap C(\bar{U})$. $c \equiv 0$ in U.

If

$$\begin{cases} (1) \ Lu \leq 0 \\ (2) \ \exists\, x_0 \in \partial U \text{ s.t. } u(x_0) > u(x) \text{ for } \forall x \in U \\ (3) \text{ "interior ball condition" at } x_0.\ \exists\, B \subseteq U \text{ with } x_0 \in \partial B \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial N}(x_0) > 0 \quad \text{where } N \text{ is the outer normal unit}$$

vector to B at x^0. If $c \geq 0$, the same conclusion holds if $u(x_0) \geq 0$



assume

Our idea it's that $u(x^0) > u(x)$ implies we can make a perturbation and after the perturbation we still have $\frac{\partial u^\varepsilon}{\partial N} \geq 0$ which it's trivial, but $\frac{\partial v}{\partial N} < 0$.

And here our choice is $u + \varepsilon v = u(x) + \varepsilon\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right)$ $\square$

(To obtain $Lu^\varepsilon \leq 0$, we may consider the annulus $B(0,r) \setminus B(0,r/2)$)

$\boxed{\text{Thm } 2.6.4}$ (Strong maximum principle) $U \subseteq \mathbb{R}^d$ is a bounded domain, $u \in C^2(U) \cap C(\bar{U})$ $c \equiv 0$ in U. If $Lu \leq 0$ + u attains its maximum over $\bar{U}$ at $x^0 \in U$

$$\Rightarrow u \text{ is a constant in } U.$$

If $c \geq 0$, $Lu \leq 0$ & u attains its nonnegative maximum over $\bar{U}$ at $x^0 \in U$

$$\Rightarrow u \text{ is a constant in } U.$$

$Lu \leq 0$. $M = \max_{\bar{U}} u$. $C := \{x \in U \mid u(x) = M\}$. $V = \{x \in U \mid u(x) < M\} \neq \emptyset$

open (not a constant)

set $y \in V$. $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial U)$

B is the largest ball centered at y s.t.

$B \subseteq V$ & $\exists\, x_0 \in C \cap B$ $\Rightarrow \frac{\partial u}{\partial N}(x_0) > 0$ ↯

$(u(x_0) = M)$

$\square$