We already know the upper bound information of elliptic PDEs, but what about the oscallotory information? Well, ~~from~~ Harnack ineq. we can understand a part of information at least.

$\boxed{\text{Thm 2.6.5}}$ (Harnack's ineq.) Assume $u \geq 0$ is a C^2 solution ~~of~~ to $Lu = 0$ in U with L defined as ~~a~~ the non-divergence form. Suppose $V \subset\subset U$ is connected. Then there exists $C = C(V, L) > 0$ s.t.

$$\sup_V u \leq C \inf_V u$$

We only prove for the case $b^i \equiv c \equiv 0$ & $\boxed{u > 0}$

" For the case $u \geq 0$. Evans said that we can consider $u + \varepsilon$ and let $\varepsilon \to 0^+$, but the later estimate actually relies on the original $Lu = 0$. So, let's just omit this argument."

The proof of Harnack ineq.:

The idea is that, letting $v = \ln u$, then it suffices to show $|v(x) - v(y)|$ is bounded. By quasi-mean value thm, it suffices to show the gradient of v is bounded.

Note that $Lu = 0$ gives that $a^{ij}(\partial_i v \partial_j v + \partial_{ij} v) = 0$

Let $w = \overbrace{\underbrace{a^{ij}\partial_i v \partial_j v}_{\overset{?}{\to} |\nabla v|^2}}^{\geq 0} = -\underbrace{a^{ij}\partial_{ij} v}_{\overset{?}{\to} \partial^2 v}$

$z = \zeta^4 w$ where $\zeta = \begin{cases} 1 & \text{in } V \\ \dots & \end{cases} \in C_c^\infty(U)$ is a cutoff function.

Assume z attains its maximal value at $x^0 \in U$

$$0 = \partial_k z \Rightarrow \zeta \partial_k w + 4(\partial_k \zeta) z = 0$$

$$\partial_k \partial_l z = \zeta^4 \partial_k \partial_l w + (4\zeta^3 \partial_k \zeta \partial_l w + 4\zeta^3 \partial_l \zeta \partial_k w + 12\zeta^2 \partial_k \partial_l \zeta\, w)$$

A useful claim in this proof is that $b^k := -2a^{kl}\partial_l v$, then $-a^{kl}\partial_k \partial_l w + b^k \partial_k w \leq -\frac{\theta^2}{2}|\partial^2 v|^2 + C|\nabla v|^2$

Using this claim, we can compute

$$\Rightarrow -a^{kl}\partial_k\partial_l z + b^k\partial_k z$$

$$= \xi^4\left(-a^{kl}\partial_k\partial_l w + b^k\partial_k w\right) + \begin{pmatrix} -12a^{kl}\xi^2\partial_k\xi\partial_l\xi w - 4a^{kl}\xi^3\partial_k\xi\partial_l w \\ -4a^{kl}\xi^3\partial_l\xi\partial_k w + 4\xi^3\underbrace{b^k}\partial_k\xi w \end{pmatrix}$$

$$= \xi^4\left(-a^{kl}\partial_k\partial_l w + b^k\partial_k w\right) + O\left(\xi^2|w| + \xi^3|\nabla w| + |\nabla v| w\right)$$

$$\le \xi^4\left(-\frac{\theta}{2}|\partial^2 v|^2 + C|\nabla v|^2\right) + O\left(\xi^2|w| + \xi^3|\nabla w| + \xi^3|\nabla v| w\right)$$

$$\Rightarrow \quad \xi^4|\partial^2 v|^2 \le C\left(\xi^4\underset{\sim}{|\nabla v|^2} + \xi^3|\nabla w| + \xi^3|\nabla v| w + \xi^2 w\right) \quad (*)$$

Note that we have $\theta|\nabla v|^2 \le w$ since uniformly elliptic.

Also $w = -a^{ij}\partial_i\partial_j v \Rightarrow w^2 \le C|\partial^2 v|^2$

$$\overset{(*)}{\Rightarrow} \quad \xi^4 w^2 \le C\left(\xi^4|\nabla v|^2 + \xi^3|\nabla w|^2 + \xi^3|\nabla v| w + \xi^2 w\right)$$

$$\le C\left(\xi^{4}\frac{w^1}{\theta} + \xi^2|w| + \underbrace{C(\varepsilon)\xi^2|\nabla v|^2 + \varepsilon\xi^4 w^2}_{\text{young}} + \xi^2 w\right)$$

$$\xi\partial_k w + 4\partial_k\xi w = 0$$

$$\le C\left(\xi^2 w + \varepsilon\xi^4 w^2\right)$$

$$\Rightarrow \quad \xi^4 w^2 \le C_\varepsilon \xi^2 w \quad \Rightarrow z = \xi^4 w \le C_\varepsilon\xi^2 \text{ where } \varepsilon \text{ is fixed.}$$

$\Rightarrow z(x^0)$ is bounded $\Rightarrow$ w is bounded in $V \Rightarrow |\nabla v| < \infty$

It remains to show the claim. $\quad b^k = -2a^{kl}\partial_l v$

$$\text{"} -a^{kl}\partial_k\partial_l w + b^k\partial_k w \le -\frac{\theta}{2}|\partial^2 v|^2 + C|\nabla v|^2 \text{"}$$

$$\partial_l w = \partial_l a^{ij}\partial_i v\partial_j v + 2a^{ij}\partial_l\partial_i v\partial_j v$$

$$\cancel{\partial_k\partial_l w = \underbrace{\partial_k\partial_l a^{ij}\partial_i v\partial_j v} + 4\partial_k a^{ij}\partial_l\partial_i v\partial_j v + 2a^{ij}\partial_l\partial_i v\partial_k\partial_j v + \underbrace{2a^{ij}\partial_k\partial_l\partial_i v\partial_j v}}$$

$$\partial_k\partial_l w = 2a^{ij}\partial_l\partial_i v\partial_k\partial_j v + 2a^{ij}\partial_k\partial_l\partial_i v\partial_j v + R$$

$$R = \partial_k\partial_l a^{ij}\partial_i v\partial_j v + 2\partial_l a^{ij}\partial_k\partial_i v\partial_j v + 2\partial_k a^{ij}\partial_l\partial_i v\partial_j v$$

$$|R| \le C\left(|\nabla v|^2 + |\nabla v||\partial^2 v|\right) \le \left(\varepsilon|\partial^2 v|^2 + C(\varepsilon)|\nabla v|^2\right)$$

$$\boxed{a^{kl}a^{ij}\partial_l\partial_i v\partial_k\partial_j v} \;?$$

$A = (a^{ij}) = P^T P \Rightarrow (P\xi)^T(P\xi) \geq \theta|\xi|^2 \Rightarrow \sigma_{min} P \geq \sqrt{\theta}$

Let $\|\cdot\|_F$ denote the Frobenius norm of ~~a~~ ~~a~~ the matrix.

Let ~~$B = AA$~~ $M = P \cdot D^2 V \cdot P^T$

$\Rightarrow \|M\|_F^2 \geq \theta^2 |D^2 V|^2$

But $\|M\|_F^2 = \sum_{\alpha,\beta} (M_{\alpha\beta})^2 \overset{Ein}{=} \sum_{\alpha,\beta} (P_{\alpha i} V_{il} P_{\beta l})(P_{\alpha j} V_{jk} P_{\beta k})$ ~~$P_{\alpha i} \partial_{ij} V \partial_j$~~

$= \sum_{\alpha,\beta} P_{\alpha i} P_{\alpha j} P_{\beta l} P_{\beta k} V_{il} V_{jk}$

$= a^{ij} a^{lk} V_{il} V_{jk}$

$\Rightarrow -a^{kl} a^{ij} \partial_l \partial_i V \partial_k \partial_j V \leq -\theta^2 |D^2 V|^2$ ✓

$-a^{kl} a^{ij} \partial_k \partial_l \partial_i V \partial_j V \overset{\text{exchange}}{=} -a^{ij} \partial_j V (a^{kl} \partial_k \partial_l \partial_i V)$

$= -a^{ij} \partial_j V \left(\partial_i (\underbrace{a^{kl} \partial_k \partial_l V}_{-W}) - \partial_i a^{kl} \partial_k \partial_l V \right)$

$= \underbrace{a^{ij} \partial_j V}_{-\frac{1}{2} b^i} \left(\partial_i W + \partial_i a^{kl} \partial_k \partial_l V \right)$

$= -\frac{1}{2} b^i \partial_i W + a^{ij} \partial_j V \, \partial_i a^{kl} \, \partial_k \partial_l V$

$\Rightarrow LHS = -2 a^{kl} a^{ij} \partial_l \partial_i V \partial_k \partial_j V + 2 a^{ij} \partial_i a^{kl} \partial_k \partial_l \partial_j V + a^{ij} R$

$\leq -2\theta^2 |D^2 V|^2 + |R| + C |\nabla V| |\partial^2 V|$

$\leq -2\theta^2 |\partial^2 V|^2 + \varepsilon |\partial^2 V|^2 + C(\varepsilon) |\nabla V|^2 + \varepsilon |\partial^2 V|^2 + C'(\varepsilon) |\nabla V|^2$

$\underset{\text{choose } \varepsilon}{=} -\frac{\theta}{2} |\partial^2 V|^2 + C |\nabla V|^2.$ $\square$

After the tedious computation, we can see that the choice of ζ^4 can be explained precisely, but, it's quite ~~techiq~~ technical so I choose to omit it now.

Now, the study of elliptic equations ends temporarily. Some interesting content would be added later, such as:

elliptic equations on manifolds

calculus of variation $\sim$ eigenvalue problem

~~Sw~~ Schwarz & Steiner Symmetrization method $\sim$ Faber-Krahn ineq

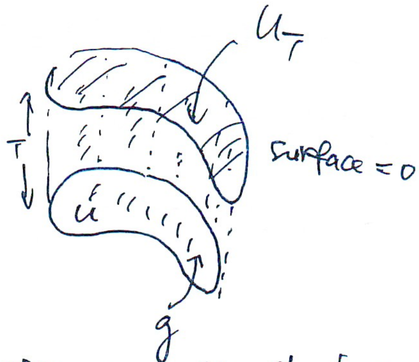
Moser interation

Ch 3. Linear Parabolic Equation

U open bounded set in $\mathbb{R}^d$, $T>0$

$U_T := [0,T] \times U$

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T \\ u = 0 & \text{on } [0,T]\times\partial U \\ u = g & \text{on } \{t=0\}\times U \end{cases}$$



$f : U_T \to \mathbb{R}$

$g : U \to \mathbb{R}$

$u : \bar{U}_T \to \mathbb{R} = u(t,x)$

L is the elliptic operator appeared in the last ch.

3.1 Space-time Sobolev spaces

$u(t,x) \longrightarrow [u(t)](x) \longrightarrow \mathbb{R}$ ~~$\rightsquigarrow [u(t)](x)$~~

$[u(t)] \in X$ (A Banach space)

$u : [0,T] \longrightarrow X$

$t \ [u(\cdot)] \longmapsto [u(t)](x)$

$\rightsquigarrow$ Banach space-valued functions & space-time Sobolev spaces.

We shall introduce some natural definitions to deal with our problems.

$\boxed{\text{Def 3.1.1}}$ (extended measure theory)

(1) Simple functions : $S : [0,T] \longrightarrow X$, $S(t) = \sum_{i=1}^{m} \chi_{E_i} u_i$

where $u_i \in X$, $E_i \in \mathcal{L}([0,T])$

(2) Strongly measurable : $f : [0,T] \longrightarrow X$, there exists a seq of simple functions s_k s.t. $s_k \to f$ for $\mathcal{L}$-a.e. $t \in [0,T]$.

(3) Weakly measurable : $f : [0,T] \longrightarrow X$, $\forall u^* \in X^*$

$t \longmapsto \langle u^*, f(t)\rangle$ is Lebesgue measurable.

(4) Almost separable valued : $\exists$ A null set $N \subseteq [0,T]$ s.t.

$\{ f(t) : [0,T] \setminus N \}$ is separable.

It's clear the strong measurability implies weak measurability and we naturally ask what relation lies between the two definitions.

A useful answer is "Pettis's lemma".

$\boxed{\text{Thm } 3.1.2}$ Strongly measurable $\Leftrightarrow$ Weakly measurable + almost separable valued.
$f : [0,T] \to X$

Then we shall define integral for the functions we defined.

$\boxed{\text{Def } 3.1.3}$ (1) $s(t) = \sum_{i=1}^{m} \chi_{E_i} u_i \qquad \int_0^T s(t)\,dt := \sum_{i=1}^{m} \mathcal{L}(E_i) u_i$

(2) A strongly measurable functions $f : [0,T] \to X$ is Bochner integrable if $\exists$ a seq. of $\{s_k\}$ s.t. $\int_0^T \| s_k - f \| \, dt \to 0$ as $k \to \infty$

Then $\int_0^T f(t)\,dt := \lim_{k\to\infty} \int_0^T s_k(t)\,dt$

$\boxed{\text{Thm } 3.1.4}$ (Bochner Lemma) A strongly measurable function f is Bochner integrable if and only if $t \mapsto \|f\|$ is integrable. In that case, we have

$$\begin{cases} \left\| \int_0^T f(t)\,dt \right\| \le \int_0^T \|f(t)\|\,dt \\ \langle u^*, \int_0^T f(t)\,dt \rangle = \int_0^T \langle u^*, f(t) \rangle\,dt. \end{cases}$$

Now we can define space-time ~~space~~ sobolev spaces.

$\boxed{\text{Def } 3.1.5}$ (L^p) (1) $L^p(0,T;X) = \left\{ \text{strongly measurable function } u : [0,T] \to X \;\middle|\; \|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|^p\,dt \right)^{\frac{1}{p}} < \infty \right\}$

$T > 0$

$1 \le p < \infty$

(2) $\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0 \le t \le T} \|u(t)\| < \infty$

(3) $\|u\|_{C([0,T];X)} := \max_{0 \le t \le T} \|u(t)\| < \infty$

$\boxed{\text{Def } 3.1.6}$ (Weak derivatives of time) $u \in L^1(0,T;X)$, $v \in L^1(0,T;X)$ is the weak (time) derivative of u, written $v = u'$, if

$$\int_0^T u(t)\varphi'(t)\,dt = -\int_0^T v(t)\varphi(t)\,dt \quad \text{for } \forall \varphi \in C_c^\infty(0,T)$$

$\boxed{\text{Def 3.1.7}}$ (Sobolev Spaces) $W^{1,p}(0,T;X) = \Big\{ u \in L^p(0,T;X) \;\Big|\; \|u\|_{W^{1,p}(0,T;X)} :=$

$$\left\{ \begin{array}{ll} \left(\int_0^T \|u(t)\|^p \, dt + \int_0^T \|u'(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty & \text{for } 1 \le p < \infty \\ \operatorname*{ess\,sup}_{0 \le t \le T} \left(\|u(t)\| + \|u'(t)\| \right) < \infty & \text{for } p = \infty \end{array} \right\} \Big\}$$

Now, we need some basic properties of caculus to end this section.

$\boxed{\text{prop 3.1.8}}$ $u \in W^{1,p}(0,T;X)$ for some $1 \le p \le \infty$

(1) $u \in C([0,T];X)$ after some refinements on null sets.

(2) $u(t) = u(s) + \int_s^t u'(\tau)\, d\tau$

(3) $\max_{0 \le t \le T} \|u(t)\| \lesssim_T \|u\|_{W^{1,p}(0,T;X)}$

The proof is not essential so we just omit here. The key idea is that, in dimension 1, $W^{1,p} \equiv$ AC functions.

$\boxed{\text{prop 3.1.9}}$ $u \in L^2(0,T;H_0^1(U))$ with $u' \in L^2(0,T;H^{-1}(U))$

(1) $u \in C(0,T;L^2(U))$ after refinements.

(2) $t \mapsto \|u(t)\|^2_{L^2(U)}$ is AC with $\frac{d}{dt}\|u(t)\|^2_{L^2(U)} = 2\langle u'(t), u(t)\rangle$

for a.e. $t \in [0,T]$.

(3) $\max_{0 \le t \le T} \|u(t)\|_{L^2(U)} \le C_T \left(\|u\|_{L^2(0,T;H_0^1(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))} \right)$

The idea is still by mollification and we construct a uniformly convergent seq to attain our purpose.

$\boxed{\text{prop 3.1.10}}$ U is bounded, ∂U is $\boxed{\text{smooth}}$, $m \in \mathbb{N}$, $u \in L^2(0,T;H^{m+2}(U))$ with $u' \in L^2(0,T;H^m(U))$

$\Rightarrow$ (1) $u \in C([0,T];H^{m+1}(U))$ after refining.

(2) $\max_{0 \le t \le T} \|u(t)\|_{H^{m+1}(U)} \lesssim_T \|u\|_{L^2(0,T;H^{m+2}(U))} + \|u'\|_{L^2(0,T;H^m(U))}$.

3.2 Existence of weak solutions : Galerkin's method

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T \\ u = 0 & [0,T]\times\partial U \\ u = g & \{t=0\}\times U \end{cases}$$

$$\partial_t u\, v + \left(-\partial_j(a^{ij}\partial_i u)\, v + b^i \partial_i u\, v + c u v\right) = f v$$

$$\Rightarrow \int_U u' v + \underbrace{\int_U a^{ij}\partial_i u\, \partial_j v + b^i \partial_i u\, v + c u v} = \int_U f v$$

$$\Rightarrow (u', v)_{L^2(U)} + B[u, v, t] = (f, v)_{L^2(U)} \quad \forall\, 0 \le t \le T \qquad u' := \frac{d}{dt} u$$

$\boxed{\text{Def } 3.2.1}$ (Weak solution) $u \in L^2(0,T; H_0^1(U))$ with $u' \in L^2(0,T; H^{-1}(U))$ is a weak solution ~~the equations above~~ of

(1) $\langle u', v\rangle + B[u, v; t] = (f, v)_{L^2(U)}$ for $\forall\, v \in H_0^1(U)$ & a.e. $t \in [0,T]$.

(2) $u(0) = g$

By $\boxed{\text{Prop } 3.1.9}$, u can be refined to be belonged to $C([0,T]; L^2(U))$

Our purpose is to state the existence and uniqueness of the weak solution to Parabolic equation written above. And the left ~~of~~ content of this section is all about Galerkin's approximation.

① Constructing a finite-dimensional truncation

② Uniformly energy estimates

③ Verifying the limit ④ Uniqueness

---

①: Claim : $\forall m \in \mathbb{N}$, $\exists!$ $u_m := \sum_{k=1}^{m} d_m^k(t) w_k$ satisfying

$$\begin{cases} d_m^k(0) = (g, w_k)_{L^2(U)} \\ (u_m', w_k)_{L^2(U)} + B[u_m, w_k; t] = (f, w_k) \end{cases} \quad \begin{cases} 0 \le t \le T \\ 1 \le k \le m. \end{cases}$$

where $\{w_k\}$ is a orthonormal basis of $H_0^1(U)$

This part is not so hard.

Assuming u_m has the structure $u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k$

$$\begin{cases} (u_m', w_k) = d_m^{k\,\prime}(t) \\ B[u_m, w_k; t] = \sum_{l=1}^{m} d_m^l \underbrace{B[w_l, w_k : t]}_{e^{lk}} = \sum_{l=1}^{m} e^{lk}(t)\, d_m^l(t) \\ f^k(t) = (f(t), w_k) \end{cases}$$

$$\Rightarrow \quad d_m^{k\,\prime}(t) + \sum_{l=1}^{m} \underbrace{e^{lk}}_{\text{known}}(t)\, d_m^l(t) = \underbrace{f^k}_{\text{known}} \qquad k = 1, \cdots m \qquad (*)$$

$(*)$ forms a system of ODE. $\longrightarrow$ Absolutely cts solution $\square$

② Claim: $\boxed{\sup_{0\le t\le T} \|u_m(t)\|^2_{L^2(U)}} + \|u_m\|^2_{L^2(0,T;H_0^1(U))} + \|u_m'\|^2_{L^2(0,T;H^{-1}(U))} \le C(U,T,L)\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right)$

The key technique here is Grönwall's estimate

We have $(u_m', w_k)_{L^2(U)} + B[u_m, w_k; t] = (f, w_k)_{L^2(U)}$

$$\Rightarrow (u_m', u_m)_{L^2(U)} + B[u_m, u_m; t] = (f, u_m)_{L^2(U)}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(U)} + \int_U a^{ij}\partial_i u\, \partial_j u\, dx = \cancel{\int_U f, u_m}$$

$$(f, u_m)_{L^2(U)} - \int_U b^i \partial_i u + cu\, dx$$

$$\text{LHS} \ge \frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(U)} + \theta\|\nabla u\|_{L^2(U)}$$

$$\text{RHS} \le \|f\|_{L^2(U)}\|u_m\|_{L^2(U)} + \|b^i\|_{L^\infty(U)}\|\nabla u\|_{L^2(U)} + \|c\|_{L^\infty(U)}\|u\|_{L^2(U)}$$

$$\le \delta\|\nabla u\|^2_{L^2(U)} + C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right)$$

Let $\delta = \frac{\theta}{2}$ $\Rightarrow \frac{d}{dt}\left(\|u_m\|^2_{L^2(U)} + \theta\int_0^t \|\nabla u_m(\tau)\|^2_{L^2(U)} d\tau\right)$

$$\le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right)$$

Grönwall $\Rightarrow \|u_m\|^2_{L^2(U)} + \theta\int_0^t \|\nabla u_m(\tau)\|^2_{L^2(U)} d\tau \le C_T\left(\underbrace{\|u(0)\|^2_{L^2(U)}}_{\|g\|^2_{L^2(U)}} + \int_0^{\underset{\sim}{t\le T}} \|f(t,\cdot)\|^2_{L^2(U)} d\tau\right) \quad (**)$

$(**)$ provides the estimate for the term $\sup_{0\le t\le T} \|u_m(t)\|^2_{L^2(U)}$, $\|u_m\|^2_{L^2(0,T;H_0^1(U))}$.

It remains to show the part of $\|u_m'\|^2_{L^2(0,T;H^{-1}(U))}$.

Since $u_m(t) = \sum_{k=1}^m d_m^k(t) w_k \Rightarrow u_m'(t) = \sum_{k=1}^m d_m^{k\prime}(t) w_k$.

Let φ be the test function. $\|\varphi\|_{H_0^1(U)} \le 1$

$$\langle u_m', \varphi\rangle \xrightarrow[\operatorname{span}\{w_1,\dots w_k\}]{\varphi=\varphi_1+\varphi_2} \langle u_m', \varphi_1\rangle = (f,\varphi_1) - B[u_m, \varphi_1; t]$$

$$\Rightarrow \|u_m'\|_{H^{-1}(U)} \le C(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)})$$

$$\Rightarrow \int_0^T \|u_m'\|^2_{H^{-1}(U)}\, dt \le C\int_0^T \|f\|^2_{L^2(U)} + \underbrace{\|u_m\|^2_{H_0^1(U)}}_{\text{has been treated. (✓)}}$$

$$\le C(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))}). \quad \square$$

③ From uniform boundness $\Rightarrow \begin{cases} u_{m_l} \to u & \text{in } L^2(0,T;H_0^1(U)) \\ u_{m_l}' \to v & \text{in } L^2(0,T;H^{-1}(U)) \end{cases}$

$\boxed{\text{Lem 3.2.2}}$ $u_{m_l}' \to v = u'$

Consider $\phi \in C_c^\infty(0,T)$, $w \in H_0^1(U)$

$$\Rightarrow \int_0^T \langle u_{m_l}, \phi' w\rangle\, dt = \int_0^T \langle u, \phi' w\rangle\, dt$$

$$\| $$

$$-\int_0^T \langle u_{m_l}', \phi w\rangle = -\int_0^T \langle v, \phi w\rangle\, dt$$

$\Rightarrow v = u'$. $\square$

Also by $\boxed{\text{prop 3.1.9}}$ $u \in C([0,T]; H_0^1(U))$

since ~~$C([0,T]; H_0^1(U))$~~ functions of the form $\sum_{\text{finite}} d^k(t) w_k$ (smooth, $\in C^1([0,T]; H_0^1(U))$) are dense in $L^2([0,T]; H_0^1(U)) \Rightarrow u$ satisfies bilinear form ~~function~~ equation, It remains to verify that u satisfies the initial data condition.

$$\langle u_m', v\rangle + B[u_m, v; t] = (f_m, v) \quad \text{for } \forall\, v \in C^1([0,T]; H_0^1(U)),\ v(T)=0$$

$$\Rightarrow \int_0^T \left(\langle u_m, v'\rangle + B[u_m, v; t]\right) = \int (f, v) + \langle u_m(0), v(0)\rangle.$$

$u_m(0) \to g$ in L^2

$$\Rightarrow \int_0^T \langle u^a, v'\rangle + B[u, v; t] = \int (f, v) + \langle g, v(0)\rangle$$

$$\int_0^T \langle u, v'\rangle + B[u, v; t] = \int (f, v) + \langle u(0), v(0)\rangle$$

$v(0)$ is ~~ar~~ arbitrary. $\Rightarrow u(0) = g$ $\square$

④ Uniqueness $\quad f = g = 0$

$$v = u \Rightarrow \int \langle u', u\rangle + B[u, u; t] = \int (0, v) + \cancel{\langle 0, v(0)\rangle} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{1}{2}\|u(t)\|^2_{L^2(U)} + B[u, u; t] = 0$$

Since $B[u, u; t] \geq \beta \|u\|^2_{H_0^1(U)} - \gamma \|u\|^2_{L^2(U)} \geq -\gamma \|u\|^2_{L^2(U)}$

$$\Rightarrow \frac{d}{dt}\frac{1}{2}\|u(t)\|^2_{L^2(U)} \leq \gamma \|u\|^2_{L^2(U)} \Rightarrow u \equiv 0 \quad \square$$

3.3 Regularity of parabolic equations.

I think it's necessary to show some ~~technique~~ technical details in the simplest equation, i.e., the heat equation.

$$\begin{cases} \partial_t u - \Delta u = f & [0,T] \times \mathbb{R}^d \\ u(0, x) = g(x) & \{t = 0\} \times \mathbb{R}^d \end{cases}$$

① priori estimate (Existence) :
$$\sup_{0 \leq t \leq T} \|u(t)\|^2_{L^2(\mathbb{R}^d)} + \|Du\|^2_{L^2(0,T; L^2(\mathbb{R}^d))} \leq C\left(\|f\|_{L^2(0,T; L^2(\mathbb{R}^d))} + \|g\|^2_{L^2(\mathbb{R}^d)}\right)$$

② Assuming u is smooth & decrease rapidly.

$$\int_{\mathbb{R}^d} f^2 = \int_{\mathbb{R}^d} (\partial_t u - \Delta u)^2 = \int (\partial_t u)^2 + \underbrace{\int (\Delta u)^2}_{\text{By IBP } = \int |\partial^2 u|^2} - 2\int \partial_t u \Delta u$$

$$= \int_{\mathbb{R}^d} (\partial_t u)^2 + \int_{\mathbb{R}^d} |\partial^2 u|^2 + \underbrace{2\int_{\mathbb{R}^d} \nabla u \cdot \nabla(\partial_t u)}_{= \frac{d}{dt}\int |\nabla u|^2}$$

$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} (\partial_t u)^2 + \int_{\mathbb{R}^d} |\partial^2 u|^2 dx \leq \int_{\mathbb{R}^d} f^2 dx$$

$$\Rightarrow \underbrace{\sup_{0\leq t\leq T} \int_{\mathbb{R}^d} |\nabla u|^2 dx}_{L^\infty_t L^2_x} + \underbrace{\int_0^T \left(\int_{\mathbb{R}^d} (\partial_t u)^2 + |\partial^2 u|^2 dx\right) dt}_{\boxed{L^2_t} H^2_x} \leq C\left(\int_0^T \int_{\mathbb{R}^d} f^2 dx + \int_{\mathbb{R}^d} |\nabla g|^2 dx\right)$$

$$\longrightarrow L^\infty_t L^2_x \cap L^2_t H^2_x$$

Energy type $\xrightarrow{?}$ pointwise

③
$$\xrightarrow[\text{Heat}]{\partial_t} \begin{cases} \partial_t^2 u - \Delta(\partial_t u) = \partial_t f & (0,T]\times\mathbb{R}^d \\ \partial_t u(0,x) = f(0,x) + \Delta g(x) & \{t=0\}\times\mathbb{R}^d \end{cases}$$

$$\xrightarrow{\int \partial_t u} \frac{d}{dt}\underbrace{\left(\frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u|^2 dx\right)}_{\int_{\mathbb{R}^d} |\partial_t u(t,\cdot)|^2 dx} + \int_{\mathbb{R}^d} |\nabla(\partial_t u)|^2 dx \leq \|\partial_t f\|_{L^2(\mathbb{R}^d)} \underbrace{\|\partial_t u\|_{L^2(\mathbb{R}^d)}}$$

$$\xrightarrow[\text{variable}]{\text{change}} \int_{\mathbb{R}^d} |\partial_t u(t,\cdot)|^2 dx \leq C\left(\int_0^T \int_{\mathbb{R}^d} (\partial_t f)^2 + \int_{\mathbb{R}} |\partial^2 g|^2 + f^2(0,x)\, dx\right)$$

$$\Rightarrow \sup_t \int |\partial_t u|^2 dx + \int_0^T \int_{\mathbb{R}^d} |\nabla(\partial_t u)|^2 \leq C\left(\int_0^T \int_{\mathbb{R}^d} (\partial_t f)^2 + \int_{\mathbb{R}} |\partial^2 g|^2 + f^2\right)$$

$u' \in L^\infty_t L^2_x \cap L^2_t H^1_x \leftarrow$ Want H^2_x $\quad g \in H^2$

④ By ~~priori~~ Sobolev estimate $\quad \sup_t \|f\|^2_{L^2(U)} \leq C\left(\|f\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \|\partial_t f\|^2_{L^2(0,T;L^2(\mathbb{R}^d))}\right)$

$\Rightarrow$ ~~$\sup_t \|\partial_t u\|_{L^2(U)} \leq$~~

$L^\infty L^2$

time stepping $\quad$ ~~$\partial_t Lu$~~ $-\Delta u = \underset{L^\infty_t L^2_x}{f} - \underset{L^\infty_t L^2_x}{u_t}$ $\quad \xrightarrow{\text{regularity}} \begin{cases} u \in L^\infty_t H^2_x \\ \|u\|^2_{L^\infty_t H^2_x} \lesssim \|\partial_t f\|^2_{L^2_t L^2_x} + \|g\|^2_{H^2} + \|f\|^2_{L^2_t L^2_x} \end{cases}$

$L^\infty_t H^2_x$ $\quad H^1_t L^2_x$

Maybe I had mistaken some details in the tedious computation above. But I thought I DO NOT miss the point, so just keep on.

$\boxed{\text{Thm 3.3.1}}$ (Parabolic regularity)

(1) Condition: $\begin{cases} u \in L^2(0,T; H_0^1(U)), & u' \in L^2(0,T; H^{-1}(U)) \\ \partial_t u + Lu = f, & f \in L^2(0,T;L^2) \\ u = g \quad \{t=0\} \times U & g \in H_0^1(U) \\ u = 0 \quad [0,T] \times \partial U \end{cases}$

$$\Rightarrow \begin{cases} u \in L^2(0,T; H^2(U)) \cap L^\infty(0,T; H_0^1(U)) \\ u' \in L^2(0,T; L^2(U)) \\ \operatorname*{ess\,sup}_{0 \le t \le T} \|u(t)\|^2_{H_0^1(U)} + \|u\|^2_{L^2(0,T;H^2(U))} + \|u'\|^2_{L^2(0,T;L^2(U))} \le \\ C\left(\|f\|^2_{L^2(0,T;H^2(U))} + \|g\|^2_{H_0^1(U)}\right) \qquad C \text{ depends on } U, T, L \end{cases}$$

(2) Conditions improved by $g \in H^2(U)$ $f' \in L^2(0,T;L^2(U))$

$$\Rightarrow \begin{cases} u \in L^\infty(0,T; H^2(U)) \\ u' \in L^\infty(0,T; L^2(U)) \cap L^2(0,T; H_0^1(U)) \\ u'' \in L^2(0,T; H^{-1}(U)) \end{cases}$$

$$\operatorname*{ess\,sup}_{0 \le t \le T} \left(\|u(t)\|^2_{H^2(U)} + \|u'\|^2_{L^2(U)}\right) + \|u'\|_{L^2(0,T;H_0^1(U))} + \|u''\|_{L^2(0,T;H^{-1}(U))}$$

$$\le C\left(\|f\|^2_{H^1(0,T;L^2(U))} + \|g\|^2_{H^2(U)}\right)$$

---

(1) We DO NOT have "$u' \in L^2(0,T;L^2(U))$", so we should be careful.

$\{u_m\}$ is the approximation seq in the proof ~~of~~ of existence.

$$\Rightarrow (u_m', w_k) + B[u_m, w_k; t] = (f, w_k) \qquad 1 \le k \le m$$

$$\xRightarrow{d_m^{k\prime}(t)} (u_m', u_m') + B[u_m, u_m'; t] = (f, u_m')$$

where $B[u_m, u_m'; t] = \int a^{ij} \partial_i u_m \partial_j u_m' + b^i \partial_i u_m u_m' + c u_m u_m'$

$$= \left(\frac{1}{2} \frac{d}{dt} \int a^{ij} \partial_i u_m \partial_j u_m\right) + \left(\qquad \right)$$

$\left|\int b^i \partial_i u_m u_m' + c u_m u_m'\right| \le \frac{C}{\delta} \|u_m\|^2_{H_0^1(U)} + \underline{\delta} \|u_m'\|^2_{L^2(U)}$; $|(f, u_m')| \le \delta \|u_m'\|^2_{L^2(U)} + \frac{C}{\delta}\|f\|^2_{L^2(U)}$

$\Rightarrow \underset{\sim}{?} \|u_m'\|^2_{L^2(U)} + \cdots$ Let $\delta < \frac{1}{4}$

$$\Rightarrow \|u_m'\|^2_{L^2(U)} + \frac{1}{2} \frac{d}{dt} \Big(\underbrace{\int a^{ij} \partial_i u_m \partial_j u_m}_{\ge 0} \Big) \le C\left[\|f\|^2_{L^2(U)} + \|u_m\|^2_{H_0^1(U)}\right]$$

$\overset{\text{Int}}{\Longrightarrow}$ $\int_0^T \|u_m'\|^2_{L^2(U)} + \sup_{0\le t\le T} \int a^{ij}\partial_i u_m \partial_j u_m \le C\left(\left(\int_0 a^{ij}\partial_i u_m \partial_j u_m\right)_{t=0} + \int_0^T\int_U f^2\,dxdt\right.$

$\left. + \int_0^T \|u_m\|^2_{H_0^1(U)}\,dt\right)$

$\underbrace{}_{\cancel{\int a^{ij}\partial_i g\,\partial_j g}}$ Absorb by g

$\boxed{L^2(0,T;H_0^1(U))}$ priori

$\Rightarrow$ $\int_0^T \|u_m'\|^2_{L^2(U)} + \sup_{0\le t\le T}\int a^{ij}\partial_i u_m\partial_j u_m \le C\ \cancel{f\,g^2}\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right).$

$\overset{\text{Uniformly}}{\underset{\text{elliptic}}{\Rightarrow}}$ $\|u_m'\|^2_{L^2(0,T;L^2(U))} + \sup\|\nabla u_m\|_{L^\infty(0,T;L^2(U))} \le C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right).$

$\boxed{\text{Lem 3.3.2}}$ H = a real Hilbert space, $u_k \to u$ in $L^2(0,T;H)$. If we have $\operatorname{ess\,sup}\|u_k(t)\|\le c$

$\Rightarrow$ $\operatorname{ess\,sup}\|u(t)\|\le c$

proof. for $\forall\, v\in H$. We have $\int_0^T (v,u_k)_H \le \int_0^T c\|v\|_H = cT\|v\|_H$

More precisely, we have $\frac{1}{(b-a)}\int_a^b (v,u_k) \le c\|v\|_H$.

$$\begin{array}{ccccc} \Rightarrow \ \frac{1}{|I_j|}\int_{I_j}(u,u_k) & \xrightarrow{j} & (u,u_k) & \le c\|u\|_H & \\ \big\downarrow k & \searrow^{\text{diag}} & & & \Rightarrow \operatorname{ess\,sup}\|u\| \le c \\ \frac{1}{|I_j|}\int_{I_j}(u,u) & & \|u\|_H^2 & & \end{array}$$

□

$\Rightarrow$ Here the limit does NOT pass to directly as Evans's book.

① term $\sup\|\nabla u_m\|_{L^\infty(0,T;L^2(U))}$ easy

② $\|u_m'\|_{L^2(0,T;L^2(U))}$ subtle ~~In Galerkin $u_m \to u$ in $L^2(0,T;H^1(U))$~~

$u_m' \to v$ in $L^2(0,T;L^2(U)) \subseteq L^2(0,T;H^{-1}(U))$

$v = u'$

Weak lower semi-cts $\Rightarrow$ $\|u\|_{L^2(0,T;L^2(U))} \le \cdots$

$\Rightarrow$ $\underbrace{\|u'\|^2_{L^2(0,T;L^2(U))}}_{L^2_tL^2_x} + \underbrace{\sup\|\nabla u\|_{L^\infty(0,T;L^2(U))}}_{L^\infty_tH^1_{0x}} \le C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right)$

Now, $(u',\varphi) + B[u,\varphi] = (f,\varphi)$ for $\forall\varphi\in H_0^1$. a.e. $t\in[0,T]$

$\Rightarrow$ $B[u,\varphi] = (\underbrace{f-u'}_{\in L^2 \text{ for a.e. } t},\varphi)$

By elliptic regularity $\Rightarrow u(t) \in H^2(U)$ for a.e. $t \in [0,T]$.

And $\|u(t)\|^2_{H^2(U)} \le C\left(\|u'\|^2_{L^2(U)} + \|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right)$

$$\Rightarrow \|u\|_{L^2(0,T;H^2(U))} \le C\left(\underbrace{\|u'\|_{L^2(0,T;L^2(U))} + \|u\|_{L^2(0,T;L^2(U))}}_{} + \|f\|^2_{L^2(0,T;L^2(U))}\right)$$

Absorb $L^2_t L^2_x$ $L^\infty_t H^1_{0x}$

(2) $(u_m'', w_k) + B[u_m', w_k] = (f', w_k) \Rightarrow (u_m'', u_m') + B[u_m', u_m'] = (f', u_m')$

diff

$$\Rightarrow \sup_{0\le t\le T} \|u_m'\|^2_{L^2(U)} + \int_0^T \|u_m'\|^2_{H_0^1(U)}\,dt \le C\left(\|u_m'(0)\|^2_{L^2(U)} + \|f'\|^2_{L^2(U)}\right)$$

$$\le C\left(\|u_m(0)\|^2_{H^2(U)} + \|f\|^2_{H^1(0,T;L^2(U))}\right)$$

Why can we take second derivative to t on u_m.

Note that $\begin{cases} d_m^k(0) = (g, w_k) \quad (w_k \in H_0^1) \\ d_m^{k\,\prime}(t) + \sum e^{kl}(t)\, d_m^l(t) = f^k(t) = (f, w_k) \end{cases}$

e^{kl}: $B[w_l, w_k, t]$ $\qquad$ $f \in H^1(0,T,L^2(U))$

It remains to show the estimate of $\|u_m(0)\|_{H^2(U)}$.

~~$\partial_t u_m(0)$~~ $\{w_k\}$ is the collection of eigenfunctions of $-\Delta$. But L is not necessarily symmetric.

$w_k \in H_0^1(U)$

$\Rightarrow$ ~~$Lu_m = f$~~ Using elliptic regularity (u_m can be spanned into $\{w_k\}$)

$$\|u_m(0)\|_{H^2(U)} \le C\left(\|\Delta u_m(0)\|_{L^2(U)} + \|u_m(0)\|_{L^2(U)}\right)$$

$$-\Delta u_m(0) = \sum d_k^m(0)\, \lambda_k w_k$$

$$\Rightarrow \|u_m(0)\|^2_{L^2(U)} = \sum_{k=1}^m (d_k^m(0))^2 \le \frac{1}{\lambda_1^2} \sum_{k=1}^m (\lambda_k d_k^m(0))^2 = \frac{1}{\lambda_1^2}\|\Delta u_m(0)\|^2_{L^2(U)}$$

$$\Rightarrow \|u_m(0)\|^2_{H^2(U)} \le C\left(1 + \frac{1}{\lambda_1}\right)\|\Delta u_m(0)\|^2_{L^2(U)}$$

Using $u_m|_{\partial U} = \Delta u_m|_{\partial U} = 0 \Rightarrow \|u_m\|^2_{H^2(U)} \le C(u_m(0), \Delta^2 u_m(0))$

$\Delta^2 u_m(0) \in \operatorname{span}\{w_1, \cdots w_m\}$

$$\Rightarrow \|u_m(0)\|^2_{H^2(U)} \le C(g, \Delta^2 u_m(0)) = C(\Delta g, \Delta u_m(0))$$

$$\overset{\text{Young}}{\le} \frac{1}{2}\|u_m(0)\|^2_{H^2(U)} + C\|g\|^2_{H^2(U)}$$

$$\Rightarrow \|u_m(0)\|^2_{H^2(U)} \le C\|g\|^2_{H^2(U)}.$$

Now. $\sup \|u_m'\|_{L^2(U)}^2 + \int_0^T \|u_m'\|_{H_0^1(U)}^2\,dt \le C\left(\|f\|_{H^1(0,T;L^2(U))}^2 + \|g\|_{H^2(U)}^2\right)$.

$L_t^\infty H_x^2$: $B[u_m, w_k] = (f - u_m', w_k)$

$$\Rightarrow B[u_m, -\Delta u_m] = (f - u_m', -\Delta u_m)$$

since $-\Delta u_m|_{\partial U} = 0 \Rightarrow B[u_m, -\Delta u_m] = (Lu_m, -\Delta u_m)$

$\boxed{\text{Lem 3.3.3}}$ $u \in C^\infty(U)$. $u = 0$ $\Delta u = 0$ on ∂U

$$\Rightarrow \beta \|u\|_{H^2(U)}^2 \le (Lu, -\Delta u)_{L^2(U)} + \gamma \|u\|_{L^2(U)}$$

proof: First, we consider $Lu = -\partial_j(a^{ij}\partial_i u)$

$$(Lu, -\Delta u) = \int_U \partial_j(a^{ij}\partial_i u)\,\partial_k^2 u = -\underbrace{\int_U a^{ij}\partial_i u\,\partial_j\partial_k^2 u} - \int_{\partial U}(a^{ij}\partial_i u\cdot n)\,\partial_k^2 u\,ds$$

$$= \underbrace{\int_U a^{ij}\partial_i\partial_k u\,\partial_j\partial_k u}_{\ge \|\partial^2 u\|_{L^2(U)}^2} - \underbrace{\int_{\partial U}(a^{ij}\partial_i u\cdot n)\underline{\underline{\partial_k^2 u}}}_{\text{Boundary, Vanishing}} - \underbrace{\int_{\partial U} a^{ij}\partial_i u(\partial_j\partial_k u\cdot n)}_{\text{trace ?}}$$

$$+ \underbrace{\int_U \partial_k a^{ij}\,\partial_i\partial_k u\,\partial_j u}_{\text{low order}} \ge -\varepsilon\|\partial^2 u\|_{L^2(U)}^2 - \underset{?}{\underline{C_\varepsilon\|\nabla u\|_{L^2(U)}^2}}.$$

Boundary term :

$$\left|\int_{\partial U} a^{ij}\partial_i u(\partial_j\partial_k u, n)\right| \approx \int |\nabla u||\partial^2 u|$$

这里的本质困难项在于边界的高阶导数项，根据 Evans，需要将边界拉平处理，经过冗长的计算，能够使用 trace thm（参考陈亚浙《二阶抛物型偏微分方程》）核心在于凑出 边界项 $I \lesssim \int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2 dS_x$

一个更好的办法是，只考虑其本质困难项 a^{ij}，然后特征函数展开.

$\boxed{\text{Thm 3.3.4}}$ (Higher)

$$g \in H^{2m+1}(U)$$
$$f^{(k)} \in L^2(0,T;H^{2m-2k}(U))$$
$$\Rightarrow u^{(k)} \in L^2(0,T;H^{2m+2-2k}(U))$$
$$u^{(j)} \in L^2(0,T;H^{2m+2-2j}(U)) \quad 0 \le j \le k.$$

By $\boxed{\text{lem 3.3.3}}$, we can see $\|u_m\|_{H^2(U)}^2 \le C\left(\|f\|_{L^2(U)}^2 + \|u_m'\|_{L^2(U)}^2 + \|u_m\|_{L^2(U)}^2\right)$ ✓.

It remains to show the estimate for u''.

By def, $$\|u_m''\|_{H^{-1}(U)} = \sup_{\substack{\varphi\in H_0^1\\ \|\varphi\|\le 1}} \langle u_m'', \varphi\rangle = \sup \left| \langle f', \varphi_m\rangle - B[u_m', \varphi_m; t]\right|$$

$$\le \|f'\|_{L^2(U)} + \|u_m'\|_{H_0^1(U)}$$ □

Higher regularity can be achieved by induction.

3.4 Maximum principles of Parabolic equations.

$\Gamma_T = \overline{U_T}\setminus U_T$

Γ_T

$Lu = \cancel{a^{ij}\partial_i u \partial_j u}\; -a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu$ $\qquad a^{ij}=a^{ji},\ b^i\ c \in C(\overline{U_T})$

∂U is smooth ~~suffiencie~~ sufficiently.

a. Weak

$\boxed{\text{Thm 3.4.1}}$ $u\in C_1^2(U_T)\cap C(\overline{U_T})$ $\quad c=0$ in U_T

$\partial_t u + Lu \le 0 \Rightarrow \max\limits_{\overline{U_T}} u = \max\limits_{\Gamma_T} u$ $\qquad$ (subsolution)

The argument is similar.

① $\partial_t u + Lu < 0$

$\partial_t u = 0 \quad Lu \ge 0$ ↯

② $\partial_t u + Lu \le 0$

$u^\varepsilon = u - \varepsilon t$

$\Rightarrow \max\limits_{\overline{U_T}} u^\varepsilon = \max\limits_{\Gamma_T} u^\varepsilon$

$$\max_{\overline{U_T}} u = \lim \max_{\overline{U_T}}(u-\varepsilon t) = \lim \max_{\Gamma_T} u - \varepsilon t \le \max_{\Gamma_T} u \le \max_{\overline{U_T}} u.$$ □

$\boxed{\text{Rmk 3.4.2}}$ $C_1^2(U_T)$ is not a standard notation.

$C_1^2(U_T) = \{ u \mid u,\ \partial_x u,\ \partial_x^2 u,\ u_t \in C(U_T)$

$\boxed{\text{Thm 3.4.3}}$ $(c\ge 0)$ Then we have $\max\limits_{\overline{U_T}} \le \max\limits_{\Gamma_T} u^+$.

In particular, $(\partial_t + L)u = 0 \Rightarrow \max\limits_{\overline{U_T}} |u| = \max\limits_{\Gamma_T} |u|$.

b. Harnack ineq.

The conclusion is easy to be accepted but the proof is hard to read. So I omit here.

$\boxed{\text{Thm 3.4.4}}$ $u \in C_1^2(U_T)$ $\partial_t u + Lu = 0$ in U_T. $u \geq 0$ in U_T. $V \subset\subset U$ and V is connected. $\Rightarrow$ $0 < t_1 < t_2 \leq T$

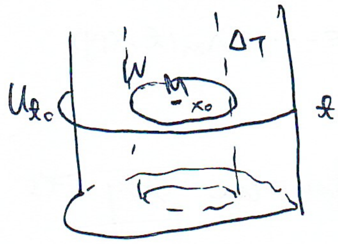
$$\sup_V u(t_1, \cdot) \leq C(L, t_1, t_2, V) \inf_V u(t_2, \cdot)$$

c. Strong maximum principle

$\boxed{\text{Thm 3.4.5}}$ $u \in C_1^2(U_T) \cap C(\overline{U_T})$ $c = 0$. U is connected

$(\partial_t + L)u \leq 0$ $\Rightarrow$ If u attains its maximum in $(t_0, x_0) \in U_T$

$\Rightarrow u$ is a constant on U_{t_0}



$$\begin{cases} \partial_t v + Lv = 0 & \text{in } W_T \\ v = u & \text{in } \Delta_T \end{cases}$$

By weak maximum principle,

$$\partial_t(u - v) + L(u - v) \leq 0 \Rightarrow \max_{\overline{W_T}} (u - v) = \max_{\Delta_T} (u - v) = 0$$

$$\Rightarrow u \leq v \leq M := \max_{\overline{U_T}} u$$

$\hat{v} = M - v \geq 0$ $\quad$ $\hat{v}_t + L\hat{v} = 0$ in W_T

By Harnack ineq: $\sup_{V} \hat{v}(t, \cdot) \leq C \inf_V \hat{v}(t_0, \cdot) \leq 0$

$V \subset\subset W$

$\Rightarrow \hat{v} \equiv 0$ on $V \times \{t\}$ $\Rightarrow$ $\hat{v} \equiv 0$ on W_{t_0}

$\Rightarrow u \equiv M$ on $\partial W \times [0, t_0]$ $\Rightarrow$ u is a constant on U_{t_0}. $\square$

$\boxed{\text{Thm 3.4.6}}$ $\cancel{u \in C_1^2}$ $(c \geq 0)$ u attains its non-negative maximum at (t_0, x_0), and then we can get the same conclusion.

Something is worth of adding:

Semigroup estimate

ch4. Linear Wave equations

4.1. Hyperbolic equations

$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$ $B_j : \mathbb{R}^n \times [0,\infty) \longrightarrow \mathbb{M}^{m\times m}$, $\vec{f} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$, $\vec{g} : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{u} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$

system: $$\begin{cases} \vec{u}_t + \sum_{j=1}^{n} B_j \partial_j \vec{u} = \vec{f} & \mathbb{R}^n \times (0,\infty) \\ \vec{u} = \vec{g} & \mathbb{R}^n \times \{t=0\} \end{cases} \quad (*)$$

$\boxed{\text{Def 4.1.1}}$ (Hyperbolicity) $\forall y \in \mathbb{R}^n$ $B(t,x;y) := \sum_{j=1}^{n} y_j B_j(t,x)$, $t \geq 0$, $x \in \mathbb{R}^n$

the system $(*)$ is called hyperbolic if the matrix $B(t,x;y)$ is always diagonalizable for $\forall x, y \in \mathbb{R}^n$, $t \geq 0$.

If $\forall B_j$ is symmetric, then we say that $(*)$ is symmetric hyperbolic.

If $B(t,x;y)$ has m real eigenvalues $\lambda_1(t,x;y) \leq \cdots \leq \lambda_m(t,x;y)$, then we say $B(t,x;y)$ is hyperbolic equivalently.

If all "$\leq$" can be replaced by "$<$", then we say it's strictly hyperbolic.

Aim: Local existence.

Condition $$\begin{cases} B_j \in C^2([0,T] \times \mathbb{R}^d ; \mathbb{M}^{d\times d}) \\ \sup_{[0,t]\times\mathbb{R}^d} |B_j| + |\nabla_{t,x} B_j| + |\nabla^2_{t,x} B_j| < \infty \\ g \in H^1(\mathbb{R}^d \to \mathbb{R}^m) \quad f \in H^1((0,T)\times\mathbb{R}^d \to \mathbb{R}^m) \end{cases}$$

$\boxed{\text{Def 4.1.2}}$ (Weak solution) $u, v \in H^1(\mathbb{R}^d \to \mathbb{R}^m)$, $0 \leq t \leq T$

$$B[u,v;t] := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (B_j(t,\cdot)\partial_j u) \cdot v \, dx$$

$u \in L^2(0,T; H^1(\mathbb{R}^d \to \mathbb{R}^m))$ with $u' \in L^2(0,T; L^2(\mathbb{R}^d \to \mathbb{R}^m))$ is the weak solution to $(*)$ if it satisfies $\begin{cases} (u',v) + B[u,v;t] = (f,v) \quad \forall v, t \\ u(0) = g \end{cases}$

From $\boxed{\text{Prop 3.1.8}}$, We can see $u \in C([0,T]; L^2(\mathbb{R}^d \to \mathbb{R}^m))$.

Vanishing Voscosity method

regularized : $\begin{cases} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + \sum_{j=1}^{m} B_j \partial_j u^\varepsilon = f \\ u^\varepsilon = g^\varepsilon \end{cases}$ $(**)$ $\quad g^\varepsilon = \eta_\varepsilon * g$

$\boxed{\text{Thm 4.1.3}}$ There exists a unique weak solution to $(**)$ satisfying

$$u^\varepsilon \in L^2(0,T; H^3(\mathbb{R}^d \to \mathbb{R}^m)), \quad u^{\varepsilon\prime} \in L^2(0,T; H^1(\mathbb{R}^d \to \mathbb{R}^m))$$

By some arguments in Harmonic Analysis, We can predict the space where the solution $u(t,x)$ lies.

Duhamel : $u(t) = \underbrace{e^{\varepsilon t \Delta} g}_{H^1} + \int_0^t e^{\varepsilon(t-\tau)\Delta} f(\tau)\, d\tau \quad (\partial_t u - \varepsilon \Delta u = f)$

$$\left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} f(\tau)\, d\tau \right\|_{L^2} \le \begin{cases} T \|f\|_{L^\infty_t L^2_x} \\ T^{\frac{1}{2}} \|f\|_{L^2_t L^2_x} \end{cases}$$

$$\|\nabla_x u\|_{L^2} \le \underset{\downarrow \frac{1}{\sqrt{\varepsilon}}}{C_\varepsilon} T^{\frac{1}{2}} \|f\|_{L^\infty_t L^2_x}$$

$f \in H^1$ is given $\Rightarrow f \in L^\infty(0,T; L^2(\mathbb{R}^d)) \cap L^2((0,T) \times \mathbb{R}^d)$ For $T < \infty$

$\Rightarrow u \in L^\infty(0,T; H^1(\mathbb{R}^d))$.

solution operator $\mathcal{T} : w \mapsto e^{\varepsilon t \Delta} g + \int_0^t e^{\varepsilon(t-\tau)\Delta} \left(f(\tau,\cdot) - \sum_{j=1}^{d} B_j(\tau,\cdot) \partial_j w(\tau,\cdot) \right) d\tau$

$X = L^\infty(0,T; H^1(\mathbb{R}^d))$.

$$\|\mathcal{T} w\|_X \le e \|g\|_{H^1} + C_\varepsilon T^{\frac{1}{2}} \left(\|f\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L^\infty_t L^2_x} \right) < \infty$$

Choose T to be small $\Rightarrow$ Contraction Map thm leads the ~~constr~~ conclusion of existence.

Now, $u^\varepsilon \in L^\infty(0,T; H^1)$, $f \in L^2_t L^2_x$, $g \in H^1 \Rightarrow u^\varepsilon \in L^\infty_t H^2_x$, $u^{\varepsilon\prime} \in L^2_t H^1$

$\Rightarrow f - \sum B_j \partial_j u^\varepsilon \in L^\infty_t H^1 \Rightarrow$ Desired. $\square$

The priori estimate is not uniform with respect to ε.

$\boxed{\text{Thm 4.1.4}}$ ~~$\|u^\varepsilon\|^2_{L^\infty H^1}$~~ $\operatorname{ess\,sup}\left(\underbrace{\|u^\varepsilon\|_{H^1}}_{①} + \underbrace{\|u^{\varepsilon\prime}\|^2_{L^2}}_{②}\right) \le \underset{\text{independant of } \varepsilon}{\underline{C}} \left(\|g\|^2_{H^1} + \|f\|^2_{L^2(0,T;H^1)} + \|f'\|^2_{L^2(0,T;L^2)}\right)$

$$① : \frac{d}{dt}\frac{1}{2}\int |u^\varepsilon|^2 = \underbrace{\int u^\varepsilon(\varepsilon \Delta u^\varepsilon)}_{\mathrm{I}} + \underbrace{\int u^\varepsilon f}_{\mathrm{II}} - \underbrace{\sum \int u^\varepsilon (B_j \partial_j u^\varepsilon)}_{\mathrm{III}}$$

$$\mathrm{I} = -\varepsilon \int |\nabla u^\varepsilon|^2$$

$$|\mathrm{II}| \le \|u^\varepsilon\|_{L^2}\|f\|_{L^2}$$

$$\mathrm{III}: \underline{\int u^\varepsilon \cdot (B_j \partial_j u^\varepsilon)}\; dx = \int u^\varepsilon \left(\partial_j (B_j u^\varepsilon) - (\partial_j B_j) u^\varepsilon\right)$$

$$= -\underbrace{\int \partial_j u^\varepsilon \cdot B_j u^\varepsilon}_{\text{sysmetric } = \int u^\varepsilon B_j \partial_j u^\varepsilon} - \int |u^\varepsilon|^2 \partial_j B_j$$

$$\Rightarrow |\mathrm{III}| \le \frac{1}{2}\left|\int |u^\varepsilon|^2 (\partial_j B_j)\right| \le C\|u^\varepsilon\|^2_{L^2}$$

$$\Rightarrow \frac{d}{dt}\frac{1}{2}\int |u^\varepsilon|^2 \le C\left(\|u^\varepsilon\|^2_{L^2} + \|f\|^2_{L^2}\right)$$

$$\overset{\text{Grönwall}}{\Rightarrow} \sup_t \|u^\varepsilon\|^2_{L^2} \le C\left(\underbrace{\|g\|_{L^2}}_{\|g^\varepsilon\| \le \|g\|} + \int_0^T \|f\|^2_{L^2}\right)$$

Similar computations are omitted. $\square$

By $\boxed{\text{Lem 3.2.2}}$, It's not hard to verify the remained process.

4.2 Existence & Regularity of Linear Wave equations.

$\varphi: \underset{\underset{\mathbb{R}}{\cap}}{I} \times \mathbb{R}^d \to \mathbb{R}$ $\quad \begin{cases} \partial_\alpha (a^{\alpha\beta} \partial_\beta \varphi) = F & \text{in } I \times \mathbb{R}^d \\ (\varphi, \partial_t \varphi) = (\varphi_0, \varphi_1) & \text{on } \{t=0\} \times \mathbb{R}^d \end{cases}$

for t

α, β range from 0 to d, 0 refers to t.

Wave type equation

Let m denote the diagonal matrix $\mathrm{diag}(-1, 1, \cdots 1)_{d+1}$

Then we require $a = [a^{\alpha\beta}]$ to be sysmetric and satisfying

$$\sum_{\alpha,\beta} | a^{\alpha\beta} - m^{\alpha\beta} | < \frac{1}{10}.$$

$$|\partial\varphi|^2 := (\partial_t \varphi)^2 + (\partial_i \varphi)^2$$

$\boxed{\text{Thm 4.2.1}}$ Let φ be the solution to the wave type equation introduced previously. Then for some constant $C = C(d, T) > 0$, the following energy estimate holds.

$$\sup_t \|\partial\varphi\|_{L^2}^2 \le C \left(\|(\varphi_0, \varphi_1)\|^2_{\dot{H}_1 \times L^2} + \int_0^T \|F(t)\|_{L^2}^2 \, dt \right) \exp\left(C \int_0^T \|\partial a\|_{L^\infty}^2 \, dt \right)$$

Test by $\partial_t \varphi$: $\displaystyle\int_0^t \int_{\mathbb{R}^d} \partial_t \varphi \left(\partial_\alpha (a^{\alpha\beta} \partial_\beta \varphi) - F \right) = 0$

$(\partial_0 = \partial_t)$

① $$\int_0^t \int_{\mathbb{R}^d} \partial_t \varphi \, \partial_t (a^{00} \partial_t \varphi) = \int_0^t \int_{\mathbb{R}^d} \partial_t a^{00} (\partial_t \varphi)^2 + \frac{1}{2} \partial_t (\partial_t \varphi)^2 a^{00}$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} a^{00} (\partial_t \varphi)^2 \Big|_0^t + \int_0^t \int_{\mathbb{R}^d} \partial_t a^{00} (\partial_t \varphi)^2$$

② $$\int_0^t \int_{\mathbb{R}^d} \partial_t \varphi (\partial_i a^{ij} \partial_j \varphi) = - \int_0^t \int_{\mathbb{R}^d} \partial_t \partial_i \varphi \, a^{ij} \partial_j \varphi$$

$$= -\frac{1}{2} \cancel{\partial_t} \int_0^t \int_{\mathbb{R}^d} \partial_t (\partial_i \varphi \partial_j \varphi) a^{ij}$$

$$= -\frac{1}{2} \int_{\mathbb{R}^d} \partial_i \varphi \partial_j \varphi \, a^{ij} \Big|_0^t + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} (\partial_t a^{ij}) (\partial_i \varphi \partial_j \varphi)$$

③ $$\int_0^t \partial_t \varphi \, \partial_i (a^{i0} \partial_t \varphi) + \partial_t \varphi \, \partial_t (a^{i0} \partial_i \varphi)$$

$$= \iint \partial_t \varphi \left(\partial_i a^{i0} \partial_t \varphi + a^{i0} \partial_i \partial_t \varphi \right) + \partial_t \varphi \left(\partial_t a^{i0} \partial_i \varphi + a^{i0} \partial_t \partial_i \varphi \right)$$

$$= \iint \partial_i a^{i0} (\partial_t \varphi)^2 + a^{i0} \partial_i (\partial_t \varphi)^2 + \partial_t \varphi \, \partial_t a^{i0} \partial_i \varphi$$

$$= \iint \underbrace{\partial_i a^{i0} (\partial_t \varphi)^2 - \partial_i a^{i0} (\partial_t \varphi)^2}_{0} + (\partial_t a^{i0})(\partial_t \varphi)(\partial_i \varphi).$$

Combine ①②③ $\Rightarrow$

$$\left(\frac{1}{2}\int_{\mathbb{R}^d} a^{00}(\partial_t\varphi)^2 - \frac{1}{2}\int_{\mathbb{R}^d} a^{ij}\partial_i\varphi\partial_j\varphi\right)_t = \left(\frac{1}{2}\int_{\mathbb{R}^d}\left[a^{00}(\partial_t\varphi)^2 - a^{ij}\partial_i\varphi\partial_j\varphi\right]\right)_0$$

$$- \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \partial_t a^{00}(\partial_t\varphi)^2\,dx\,d\tau - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \partial_t a^{ij}\partial_i\varphi\partial_j\varphi\,dx\,d\tau$$

$$- \int_0^t\int_{\mathbb{R}^d} \cancel{\partial_i a^{i0}(\partial_t\varphi)^2 - \partial_i a^{i0}(\partial_t\varphi)^2}\ \partial_t a^{i0}\partial_t\varphi\partial_i\varphi + \int_0^t\int_{\mathbb{R}^d}\partial_t\varphi F$$

$$\Rightarrow \left|\frac{1}{2}\int_{\mathbb{R}^d}\left(a^{00}(\partial_t\varphi)^2 - a^{ij}\partial_i\varphi\partial_j\varphi\right)dx\right|_t \le \left|\frac{1}{2}\int\left(a^{00}(\partial_t\varphi)^2 - a^{ij}\partial_i\varphi\partial_j\varphi\right)dx\right|_{t=0}$$

$$+ C\int_0^t \|\partial a\|_{L^\infty}\|\partial\varphi\|_{L^2} + \int_0^t \|\partial\varphi\|_{L^2}\|F\|_{L^2}\,d\tau$$

Since $[a]$ is different from $\mathrm{diag}(-1, 1, \cdots 1)$ by a perturbation.

$$\Rightarrow \|\partial\varphi\|_{L^2}^2 \le C\left(\|\partial\varphi(0)\|_{L^2}^2 + \int_0^t \Big[\underbrace{\|\partial\varphi\|_{L^2}\|F\|_{L^2}}_{\mathrm{I}} + \|\partial a\|_{L^\infty}\|\partial\varphi\|\Big]\,d\tau\right)$$

$$\mathrm{I} \overset{\text{Young}}{\le} \delta\sup\|\partial\varphi\|^2 + \frac{C}{\delta}\left(\int\|F\|\right)^2$$

$$\cancel{\mathrm{I} \overset{\text{Young}}{\le} \int\left(\delta\|\partial\varphi\|^2 + \frac{1}{\delta}\|F\|^2\right) \overset{\text{Cauchy}}{\le} \underbrace{\delta\sup\|\partial\varphi\|^2}_{\text{absorb}} + \frac{C}{\delta}\left(\int\|F\|\right)^2}$$

$$\left(\int F\right)^2 \le \left(\int 1\right)\int F^2 = \int T F^2$$

$$\Rightarrow \sup\|\partial\varphi\|^2 \le C\left(\|\partial\varphi_0\|_{L^2}^2 + \int T\|F\|^2\right) + C\int\|\partial a\|_{L^\infty}\|\partial\varphi\|\,d\tau \qquad \square$$

We can obtain higher order estimate :

$\boxed{\text{Cor 4.2.2}}$

$$\sup_t\|(\varphi(t), \partial_t\varphi(t))\|_{H^k\times H^{k-1}}^2 \le C(d,k,T)\Bigg[\|(\varphi_0, \varphi_1)\|_{H^k\times H^{k-1}}^2$$

$$+ \int_0^T \|F\|_{H^{k-1}}^2 + \sum_{|\alpha|+|\beta|\le k-1}\|\partial\,\partial_x^\alpha a\ \partial\,\partial_x^\beta\varphi\|_{L^2}^2 + \|\partial_x^\alpha \partial a\,\partial\,\partial_x^\beta\varphi\|_{L^2}^2\,dt\Bigg]$$

$$\times\exp\left(c\int_0^T\|\partial a\|_{L^\infty}^2\,dt\right)$$

We will use Hahn-Banach thm to prove local existence.

It's hard to avoid some Fourier Theory so we just embrace it.

$$\boxed{\|u\|_{H^s(\mathbb{R}^d)} := \|\langle\xi\rangle^s \hat{u}\|_{L^2}}$$

$\boxed{\text{Lem 4.2.3}}$ $L^*\psi := \partial^\alpha(a^{\alpha\beta}\partial_\beta\psi)$, $\psi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$

$\forall m \in \mathbb{Z}$, $\exists\, C = C(m,T,a)$ s.t. $\|\psi(t)\|_{H^m(\mathbb{R}^d)} \le \int_t^T \|L^*\psi\|_{H^{m-1}}$ for $\forall t \in [0,T]$.

1: $m \geq 0$ $\quad \underbrace{L^*\psi = L^*\psi = G}_{\text{given}}$ with zero boundary

By $\boxed{\text{Cor 4.2.2}}$, $\|\psi\|^2_{L^\infty_t H^k_x} \le C\int_t^T \|L^*\psi\|$

(time reverse $\underset{\text{vanish}}{T} \to t$)

1. $m < 0$ induction $\quad \boxed{\|(1-\Delta)\Psi\|_{H^s} = \|\langle\xi\rangle^s\langle\xi\rangle^2\hat{\Psi}\|_{L^2} = \|\Psi\|_{H^{s+2}}}$

Let $\Psi = (1-\Delta)^{-1}\psi$

$$|L^*\psi - (1-\Delta)L^*\Psi| = |L^*(1-\Delta)\Psi - (1-\Delta)L^*\Psi|$$
$$\le C\sum_{1\le|\alpha|\le 3}|\partial_x^\alpha\Psi|$$

m_0+2 is Done.

$$\Rightarrow \|L^*\Psi\|_{H^{m_0+1}} = \|(1-\Delta)L^*\Psi\|_{H^{m_0-1}} = \|(1-\Delta)L^*\Psi - L^*\psi + L^*\psi\|_{H^{m_0-1}}$$
$$\le \|L^*\psi\|_{H^{m_0-1}} + C\|\sum_{1,2,3}\partial_x^\alpha\Psi\|_{H^{m_0-1}}$$
$$\le C\left(\|L^*\psi\|_{H^{m_0-1}} + \|\Psi\|_{H^{m_0+2}}\right)$$

Induction Hyperthesis:

$$\underline{\|\Psi(t)\|_{H^{m_0+2}}} \le C\int_t^T \|L^*\Psi\|_{H^{m_0+1}} \le C\int_t^T \|L^*\psi\|_{H^{m_0-1}} + \underline{\|\Psi\|_{H^{m_0+2}}}$$

$$\Rightarrow \|\Psi\|_{H^{m_0+2}} \le C\int_t^T \|L^*\psi\|_{H^{m_0-1}}$$
$$\|$$
$$\|\psi\|_{H^{m_0}}$$

$\square$

$\boxed{\text{Thm 4.2.4}}$ $F \in L^2(0,T; H^{k-1}(\mathbb{R}^d))$, then there exists a unique solution to (*) ~~satisfying~~ with

$$(\varphi, \partial_t\varphi) \in L^\infty(0,T; H^k(\mathbb{R}^d)) \times L^\infty(0,T; H^{k-1}(\mathbb{R}^d))$$

1. Uniqueness. $(\varphi_0, \varphi_1) = (0,0)$

$$\text{Map}: \underset{\substack{\uparrow \\ L^*(C_c^\infty((-\infty,T)\times\mathbb{R}^d))}}{L^*\psi} \mapsto \int_0^T\int_{\mathbb{R}^d} \psi F =: \langle \psi, F\rangle$$

(Well-definedness from energy estimate)

$$\int_0^T\int_{\mathbb{R}^d} \psi F \le \underbrace{\int_0^T \|F\|_{H^{k-1}}}_{C} \sup \|\psi\|_{H^{-k+1}} \le C\int_0^T \|L^*\psi\|_{H^{-k}}$$

$$\overset{\text{H-B}}{\Longrightarrow} \exists \varphi \in \left(L^1(0,T; H^{-k}(\mathbb{R}^d))\right)^* \underset{\uparrow}{=} L^\infty(-\infty, T; H^k(\mathbb{R}^d))$$

with $\varphi = 0$ for $t<0$ It's right though I don't know why

$\langle F, \psi\rangle = \langle \varphi, L^*\psi\rangle \Rightarrow \varphi$ is a solution in the sense of distribution.

For general case, $(\varphi_0, \varphi_1) \in H^k \times H^{k-1}$, u solves the equation

$$\begin{cases} L\eta = F - Lu \\ (u, \partial_t u)_{t=0} = (\varphi_0, \varphi_1) \\ (\eta, \partial_t\eta)_{t=0} = (0,0) \end{cases} \Rightarrow \varphi := \eta + u \text{ solves the desired equation.}$$

The existence of u can be guaranteed by Fourier method or Energy method.

$\Rightarrow \varphi$ has corresponding regularity.

$\partial_t\varphi$: If $(\varphi_0, \varphi_1) = (0,0)$. $F \in C_c^\infty$

$$a^{0\alpha}\partial_t\partial_\alpha\varphi = F - \underbrace{a^{ij}\partial_i\partial_j\varphi - \partial_\alpha a^{\alpha\beta}\partial_\beta\varphi}\in H^{k-2}$$

$$\Rightarrow \partial_t\varphi \in H^{k-2} \overset{\text{Bootstrap}}{\Longrightarrow} \partial_t\varphi \in H^{k-1}$$

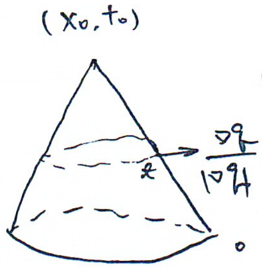
General, approximation seq. $\square$

4.3 Finite Propagation Speed

In this section, we consider the equation

$$\partial_t^2 \varphi - \underset{x}{a^{ij}}\partial_i\partial_j\varphi = 0$$

where $[a] = a^{ij}$ is symmetric and elliptic.



$$K_t = \{x : q(x) < t_0 - t\}$$

On ∂K_t : $q(x) = t_0 - t$

$$\xrightarrow{\frac{d}{dt}} \boxed{\nabla q(x(t))\ x'(t) = -1}$$

$\boxed{\text{Thm 4.3.1}}$ (Finite propagation speed) $\varphi \equiv \varphi_t \equiv 0$ on K_0 $\Rightarrow \varphi \equiv 0$ within K. Uniformly elliptic

$$e(t) := \frac{1}{2}\int_{K_t} \varphi_t^2 + \boxed{a^{ij}}\varphi_i\varphi_j\,dx$$

$$\frac{d}{dt}(e(t)) = \frac{1}{2}\int_{K_t}\partial_t(\varphi_t^2 + a^{ij}\varphi_i\varphi_j) + \frac{1}{2}\int_{\partial K_t}(\varphi_t^2 + a^{ij}\varphi_i\varphi_j)V_n$$

$$\underset{\text{Boundary Velocity}}{V_n} = x'\cdot n = \frac{x'\nabla q}{|\nabla q|} = \frac{-1}{|\nabla q|}$$

$$= \frac{1}{2}\int_{K_t} 2\varphi_t\varphi_{tt} + 2a^{ij}\varphi_i\varphi_{j,t} - \frac{1}{2}\int_{\partial K_t}\varphi_t^2 + a^{ij}\varphi_i\varphi_j\frac{1}{|\nabla q|}$$

$$= \underbrace{\int_{K_t}\varphi_t\varphi_{tt} + a^{ij}\partial_i\varphi\,\partial_t\partial_j\varphi}_{\mathrm{I}} \underbrace{- \frac{1}{2}\int_{\partial K_t}\varphi_t^2 + a^{ij}\varphi_i\varphi_j\frac{1}{|\nabla q|}}_{\mathrm{II}}$$

$$\mathrm{I} = \int_{K_t}\partial_t\varphi\,\partial_t^2\varphi + \int_{K_t} a^{ij}\partial_i\varphi\,\partial_t\partial_j\varphi$$

$$\overset{\text{IBP}}{=} \int_{K_t}\partial_t\varphi\,\partial_t^2\varphi + \int_{\partial K_t} a^{ij}\partial_i\varphi\,\partial_t\varphi - \int_{K_t}\partial_j(a^{ij}\partial_i\varphi)\partial_t\varphi$$

$$= \int_{K_t}\partial_t\varphi\left(\partial_t^2\varphi - \partial_j(a^{ij}\partial_i\varphi)\right) + \int_{\partial K_t} a^{ij}\partial_i\varphi\,\partial_t\varphi\,N_j$$

$$= \underbrace{-\int_{K_t}\underbrace{\partial_t\varphi}_{\partial_t\varphi^2}\,\underbrace{\partial_j a^{ij}}_{<\infty}\partial_i\varphi}_{C\,e(t)} + \int_{\partial K_t} a^{ij}\underline{\partial_i\varphi}\,\underline{N_j}\,\partial_t\varphi\,dS_x$$

$$\nabla\varphi\,\underset{\text{matrix}}{[a]}\,N \le \sqrt{a^{ij}\varphi_i\varphi_j}\sqrt{a^{ij}N_iN_j}$$

The second term:

$$\left|\int_{\partial K_t}\cdots\right| \leq \int_{\partial K_t}(a^{ij}\partial_i\varphi\,\partial_j\varphi)^{\frac{1}{2}}(a^{ij}N_iN_j)^{\frac{1}{2}}|\partial_t\varphi|\,dS_x$$

$\cancel{a^{ij}N_iN_j = \frac{\nabla q}{|\nabla q|}A\frac{\nabla q}{|\nabla q|}}$ We have $a^{ij}q_iq_j = 1$

$$= \int_{\partial K_t}\underline{(a^{ij}\partial_i\varphi\,\partial_j\varphi)^{\frac{1}{2}}}\,\underline{|\partial_t\varphi|}\frac{1}{|\nabla q|}\,dS_x$$

$$\underbrace{\leq \frac{1}{2}\int_{\partial K_t}(a^{ij}\partial_i\varphi\,\partial_j\varphi + \partial_t\varphi^2)\frac{1}{|\nabla q|}}_{\text{Cancel with II}}.$$

$\Rightarrow e'(t) \leq Ce(t)$ $\square$

More contents can be found in Habouri's book. ~~Convers~~ Conservation laws and Noether ~~laws~~ thm are also worthy of consideration.

Ch5. Sobolev Spaces: Fourier Theory

5.1 Fractional spaces

$\boxed{\text{Def } 5.1.1}$ (Non-homogeneous) $s \in \mathbb{R}$, $\langle \xi \rangle = \sqrt{1+|\xi|^2}$

$$H^s(\mathbb{R}^d) := \{ u \in \boxed{\mathcal{S}'(\mathbb{R}^d)} : \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^d) \}$$

tempered distribution

It's a Hilbert space with the inner product defined by

$$\langle u, v \rangle_{H^s} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \hat{u} \bar{\hat{v}} \, d\xi .$$

$\boxed{\text{prop } 5.1.2}$ (1) $C_c^\infty(\mathbb{R}^d) \overset{\text{dense}}{\subseteq} H^s(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d) \overset{\text{dense}}{\subseteq} H^s(\mathbb{R}^d)$

(2) $H^s(\mathbb{R}^d)$ coincides with $W^{s,2}(\mathbb{R}^d)$ when $s \in \mathbb{N}$.

(2) can be proved by Plancherel theorem.

$\boxed{\text{Def } 5.1.3}$ (Homogeneous) $\dot{H}^s(\mathbb{R}^d) := \{ u \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^d) \}$

NOTE: $\mathcal{S}'/\mathcal{P} \simeq \mathcal{S}'_h = \{ u \in \mathcal{S}'(\mathbb{R}^d) : (P(\xi)\hat{u}(\xi))(0) = 0, \ P \in \mathcal{P} \}$

$\mathcal{S}'/\mathcal{P}$ is not a closed subspace of $\mathcal{S}'$ in the weak*-topo.

$\boxed{\text{Def } 5.1.4}$ $f \in \mathcal{S}'$, $s \in \mathbb{R}$. $\widehat{P(\nabla) f}(\xi) := P(i\xi) \hat{f}(\xi)$. P is a polynomial

Similarly, given a locally integrable complex valued function m, we define <u>Fourier multiplier</u> by

$$\widehat{m(\nabla/i) f}(\xi) := m(\xi) \hat{f}$$

In particular, $\widehat{\langle \nabla \rangle f}(\xi) := \langle \xi \rangle \hat{f}(\xi)$, $\widehat{|\nabla| f} = |\xi| \hat{f}(\xi)$.

I.e.
$$\begin{cases} f \in H^s(\mathbb{R}^d) \Leftrightarrow \langle \nabla \rangle^s f \in L^2(\mathbb{R}^d) \\ f \in \dot{H}^s(\mathbb{R}^d) \Leftrightarrow |\nabla|^s f \in L^2(\mathbb{R}^d) \end{cases}$$

About Homogeneous Sobolev spaces, we have the following properties.

$\boxed{\text{prop } 5.1.5}$ (1) $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space iff $s < \frac{d}{2}$

(2) $s < \frac{d}{2}$, $\mathcal{S}_0(\mathbb{R}^d) := \{ u \in \mathcal{S}(\mathbb{R}^d) : \hat{u}(\xi) \text{ vanishes near } 0 \} \overset{\text{dense}}{\subseteq} \dot{H}^s(\mathbb{R}^d)$

(3) $(\dot{H}^s(\mathbb{R}^d))^* \cong \dot{H}^{-s}(\mathbb{R}^d)$.

(1) It suffices to show the completeness.

$|\xi|^s u_n(\xi)$ Cauchy in L^2 $\Rightarrow$ $|\xi|^s \hat{u}_n(\xi) \xrightarrow{L^2} f$

$f = |\xi|^s g$. It remains to show g is a tempered distribution.

$$\int_{B(0,1)} |g| = \int |\xi|^{s} g |\xi|^{-s} \leq \|\xi^s g\|_{L^2}\left(\int_{B(0,1)} |\xi|^{-2s}\right)^{\frac{1}{2}} < \infty$$

$\int_{B(0,1)^c}$: $|\xi| \simeq \langle \xi \rangle$

$u := \mathcal{F}^{-1}(g)$.

When $s \geq \frac{d}{2}$. we need the following lemma.

$\boxed{\text{Lem 5.1.6}}$ $s \geq \frac{d}{2}$ $(\dot{H}^s(\mathbb{R}^d), N)$ is a Banach space.

$$N : u \mapsto \|\hat{u}\|_{L^1(B(0,1))} + \|u\|_{\dot{H}^s(\mathbb{R}^d)}$$

proof: $\{u_n\}$ Cauchy $\Rightarrow$ $\begin{cases} \hat{u}_n \text{ Cauchy in } L^1(B(0,1)) \\ |\xi|^s \hat{u}_n \text{ Cauchy in } L^2(\mathbb{R}^d) \end{cases}$

$\Rightarrow \hat{u}_n \xrightarrow{L^1} f \quad B(0,1)$

$|\xi|^s \hat{u}_n \xrightarrow{L^2} g \quad \mathbb{R}^d$

Let $w = \begin{cases} f & B(0,1) \\ g/|\xi|^s & B(0,1)^c \end{cases}$

$$\int w\varphi = \int_{B(0,1)} f\varphi + \int_{B(0,1)^c} |\xi|^{-s} g \varphi$$

$$\leq \sup|\varphi| \, \|f\|_{L^1} + \|g\|_{L^2} \| |\xi|^{-s}\varphi\|_{L^2} < \infty$$

$\square$

If $\dot{H}^s(\mathbb{R}^d)$ endowed with $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d)}$ is also complete,

we must lead to the ineq $\|\hat{u}\|_{L^1(0,1)} \leq C\|u\|_{\dot{H}^s}$. $(*)$

Otherwise $\exists$ $\hat{u}_n$ s.t $\|\hat{u}_k\|_{L^1} > k\|u_k\|_{\dot{H}^s}$ where $\|u_k\|_N = 1$

$\Rightarrow \|u_k\|_{\dot{H}^s} \xrightarrow[O(\frac{1}{k})]{} 0$ This implies u_k is Cauchy in $\dot{H}^s$

$\Rightarrow u_k \to 0$ But $\|u_k\|_N = 1$ ↯

So the original ineq $(*)$ holds . But

$$\hat{u}_n := \sum_{k=1}^{n} \frac{2^{(s+\frac{d}{2})k}}{k} \chi_{2^{-k}\mathcal{A}} \qquad \mathcal{A} = \{ \tfrac{1}{4} < |\xi| < \tfrac{1}{3} \}$$

is a counter example!

(2) It suffices to show

"If $\forall \varphi \in \mathcal{S}_0$, $\langle u, \varphi \rangle_{\dot{H}^s} = 0 \Rightarrow u = 0$"

on the conclusion that $\dot{H}^s$ is a Hilbert space.

$$\langle u, \varphi \rangle_{\dot{H}^s} = \int \underbrace{|\xi|^{2s} \hat{u}}_{} \underbrace{\bar{\hat{\varphi}}}_{\mathcal{S}} \sim \int \hat{u}\, \underline{\mathcal{S}} \Rightarrow u = 0 \text{ on } \mathbb{R}^d \setminus \{0\}$$

$\Rightarrow \hat{u} = \sum c_\alpha \partial^\alpha \delta$ But $\|u\|_{\dot{H}^s} < \infty \Rightarrow u = 0.$ $\square$

5.2 Sobolev embedding theorems.

$\boxed{\text{Thm 5.2.1}}$ (Sobolev embedding) $0 \leq s < \frac{d}{2}$ $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$

for $2 \leq q < 2^* := \frac{2d}{d-2s}$ with the ineq $\|f\|_{L^q} \leq C(s, q, d) \|f\|_{H^s(\mathbb{R}^d)}$

prove for $f \in \mathcal{S}$ $1 \leq q' \leq 2$

$$\|f\|_{L^q} = \|(\hat{f})^\vee\|_{L^q} \overset{\text{H-Y (proved by R-T)}}{\leq} \|\hat{f}\|_{L^{q'}} = \|\langle \xi \rangle^{-s} \langle \xi \rangle^s \hat{f}\|_{L^{q'}}$$

$$\overset{\text{Hölder}}{\leq} \|\langle \xi \rangle^{-s}\|_{L^r} \|\langle \xi \rangle^s \hat{f}\|_{L^2} = \|\langle \xi \rangle^{-s}\|_{L^r} \|f\|_{H^s}$$

Here $\frac{1}{r} + \frac{1}{2} = \frac{1}{q'} = 1 - \frac{1}{q} \Rightarrow \frac{1}{r} = \frac{1}{2} - \frac{1}{q} < \frac{1}{2} - \frac{1}{2^*}$

$$= \frac{1}{2} - \frac{1}{2} + \frac{s}{d} = \boxed{\frac{s}{d}}$$

$\Rightarrow$ $d < rs$ $\Rightarrow \|\langle \xi \rangle^{-s}\|_{L^r} < \infty$ $\square$

$\boxed{\text{Rmk 5.2.2}}$ The embedding is not compact since $\mathbb{R}^d$ is not bounded.

$\boxed{\text{Thm 5.2.3}}$ (Critical embedding) $0 \leq s < \frac{d}{2}$. $H^s \hookrightarrow L^{2^*}$

$\dot{H}^s \hookrightarrow L^{2^*}$.

To prove it, we shall introduce HLS ineq first.

$\boxed{\text{Thm 5.2.4}}$ HLS ineq : $f \in L^p(\mathbb{R}^d)$ $0 < \gamma < d$, $1 < p < q < \infty$

$1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d} \Rightarrow \| |\cdot|^{\gamma} * f\|_{L^q} \leq C \|f\|_{L^p}$.

$$\| |\cdot|^{-\gamma} * f \|_{L^q} = \| \int y^{-\gamma} f(x-y)\, dy \|_{L^q_x}$$

$$\int y^{-\gamma} f(x-y) \overset{\text{truncation}}{=\!=} \underbrace{\int_{|y|<R} y^{-\gamma} f(x-y)}_{I_1} + \underbrace{\int_{|y|\geq R} y^{-\gamma} f(x-y)}_{I_2} = I_1 + I_2$$

$$|I_2| = \left| \int_{|y|\geq R} y^{-\gamma} f(x-y)\, dy \right| \leq \|f\|_{L^p} \| y^{-\gamma} \|_{L^{p'}} \cancel{< \infty} = R^{-\frac{d}{q}} \|f\|_{L^p}$$

$$\frac{1}{\gamma p'} = \frac{1}{\gamma}\left(1 - \frac{1}{p}\right) = \frac{1}{\gamma}\left(\frac{\gamma}{d} - \frac{1}{q}\right) < \frac{1}{d}$$

$$I_1 \overset{\text{dyadic}}{=\!=} \sum_{k=0}^{\infty} \int_{2^{-(k+1)}R \leq |y| < 2^{-k}R} y^{-\gamma} f(x-y)\, dy \leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)}R \leq |y| < 2^{-k}R} 2^{\gamma(k+1)} R^{-\gamma} |f(x-y)|\, dy$$

$$\leq \sum_{k=0}^{\infty} 2^{\gamma(k+1)} R^{-\gamma} (2^{-j}R)^d C_d Mf(x) \approx C R^{d-\gamma} Mf(x)$$

$$\gamma(k+1) - jd$$

$$\Rightarrow R = \frac{\|f\|_{L^p}^{\frac{p}{d}}}{Mf(x)} \Rightarrow I_1 + I_2 \leq C \|f\|_{L^p}^{1-\frac{p}{q}} (Mf)^{\frac{p}{q}}$$

$$LHS \leq \| I_1 + I_2 \|_{L^q} = C \|f\|_{L^p}^{1-\frac{p}{q}} \| (Mf)^{\frac{p}{q}} \|_{L^q} \leq C \|f\|_{L^p}. \qquad \square$$

$$\| (Mf) \|_{L^p}^{\frac{p}{q}}$$

$H^s \hookrightarrow L^{2^*}$ is right but its proof is not so direct, so we just prove the homogeneous case.

For $f \in \mathcal{S}$, let $g = (|\xi|^{d-\gamma} \hat{f})^{\vee} \Rightarrow \hat{f} = |\xi|^{-d+\gamma} \hat{g}$

$$\Rightarrow f = \left(|\xi|^{-d+\gamma}\right)^{\vee} * g \approx |x|^{-\gamma} * g$$

$$\Rightarrow \|f\|_{L^q} = \| |x|^{-\gamma} * g \|_{L^p} \overset{HLS}{\leq} C \|g\|_{L^p} \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d}$$

Let $p = 2$. $\quad d - \gamma = s \quad q = \dfrac{1}{\frac{\gamma}{d} - \frac{1}{2}} = \dfrac{2d}{2\gamma - d} = \dfrac{2d}{d-2s}$.

$$\|g\|_{L^2} \approx \|\hat{g}\|_{L^2} = \| |\xi|^{d-\gamma} \hat{f} \|_{L^2} = \|f\|_{\dot{H}^{d-\gamma}} \qquad \square$$

Case $s > \frac{d}{2}$

$\boxed{\text{Thm 5.2.5}}$ (1) $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ with $\|f\|_{L^\infty(\mathbb{R}^d)} \leq C\|f\|_{H^s(\mathbb{R}^d)}$ for $\forall f \in H^s$

(2) $\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}$

prove for $f, g \in \mathcal{S}$. when $s > \frac{d}{2}$, we have $\langle\xi\rangle^{-s} \in L^2(\mathbb{R}^d)$.

(1) $$|f(x)| = |\check{\hat{f}}(x)| = (2\pi)^{-\frac{d}{2}} \left|\int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi\right|$$
$$\leq C\left|\int \langle\xi\rangle^s \hat{f}(\xi) \langle\xi\rangle^{-s} d\xi\right| \leq C\|f\|_{H^s}\cdot\|\langle\xi\rangle^{-s}\|_{L^2}$$
$$\leq C\|f\|_{H^s}.$$

(2) $$\|fg\|_{H^s} = \|\langle\xi\rangle^s \widehat{fg}\|_{H^s} = C\|\langle\xi\rangle^s (\hat{f}*\hat{g})\|_{L^2}$$
$$= C\|\langle\xi\rangle^s \int \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta\|_{L^2_\xi}$$

$\langle\xi\rangle^s \leq C(\langle\xi-\eta\rangle^s + \langle\eta\rangle^s)$
$$\leq C\|\langle\xi-\eta\rangle^s \hat{f}(\xi-\eta)\hat{g}(\eta)\|_{L^2_\xi} + C\|\int \langle\eta\rangle^s \hat{f}(\xi-\eta)\hat{g}(\eta)\|_{L^2_\xi}$$
$$\leq C\|(\langle\xi\rangle^s\hat{f}) * \hat{g}\|_{L^2} + C\|\hat{f} * (\langle\ \rangle^s\hat{g})\|_{L^2}$$
$$\overset{\text{Young}}{\leq} C\|f\|_{H^s}\underbrace{\|\hat{g}\|_{L^1}}_{\|\langle\xi\rangle^s\hat{g}\langle\xi\rangle^{-s}\|_{L^1} \leq C\|g\|_{H^s}} + \cdots \leq \cdots$$

$\square$

$\boxed{\text{Thm 5.2.6}}$ (Morrey) $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{Z} \Rightarrow \dot{H}^s \subseteq C^{k,\rho}$, $k = [s-\frac{d}{2}]$, $\rho = \{s - \frac{d}{2}\}$

More precisely, we have
$$\sup_{|\alpha|=k}\sup_{x\neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|} \leq C_{d,s}\|f\|_{\dot{H}^s}.$$

The proof is chosen to be omitted.

$\boxed{\text{Def 5.2.7}}$ (BMO) $\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f - f_B|\, dx < \infty$

$$f_B = \frac{1}{|B|}\int_B f$$

Here $BMO = \{ f \in L^1_{loc}(\mathbb{R}^d) : \|f\|_{BMO} < \infty\}$.

$\boxed{\text{Thm 5.2.8}}$ $L^1_{loc}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d) \hookrightarrow BMO$

$\|f\|_{\dot{H}^{d/2}(\mathbb{R}^d)} \geq C \|f\|_{BMO}$. for $\forall f \in L^1_{loc}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$

$f_{l,A} = \left(\hat{\theta}\left(\frac{\cdot}{A}\right)\hat{f}\right)^{\vee}$ $\quad f_{h,A} = f - f_{l,A}$ $\quad \hat{f}_{l,A} = \hat{\theta}\left(\frac{\cdot}{A}\right)\hat{f}$

$\underline{C_c^\infty \quad \hat{\theta} = 1 \text{ near } \xi = 0}$

$f_{l,A}$ is setted to localize the frequency variable ξ near A

$$\frac{1}{|B|}\int_B |f - f_B|\,dx \leq \frac{1}{|B|}\int_B |f_{l,A} - (f_{l,A})_B|\,dx + \frac{1}{|B|}\int_B |f_{h,A} - (f_{h,A})_B|\,dx$$

$$\leq \frac{1}{|B|}\|f_{l,A} - (f_{l,A})_B\|_{L^2(B)}|B|^{\frac{1}{2}} + \cdots$$

$$\leq \|f_{l,A} - (f_{l,A})_B\|_{L^2(B, \frac{dx}{|B|})} + \frac{2}{|B|^{\frac{1}{2}}}\|f_{h,A}\|_{L^2(B)}$$

$$\|f_{h,A}\|_{L^2} \approx \|\hat{f}_{h,A}\|_{L^2} \leq \left\|\left(\frac{\xi}{A}\right)^{\frac{d}{2}}\hat{f}_{h,A}\right\|_{L^2} \leq A^{-\frac{d}{2}}\|f_{h,A}\|_{\dot{H}^{\frac{d}{2}}}$$

$$\|f_{l,A} - (f_{l,A})_B\|_{L^2(B, \frac{dx}{|B|})} \cancel{\underset{\text{Poincaré}}{\leq} CR\|\nabla f_{l,A}\|} \leq CR\|\nabla f_{l,A}\|_{L^\infty}$$

$$\frac{1}{B}\int (f(x) - f(y))\,dy \leq 2R\|\nabla f\|_{L^\infty}$$

$$\leq CR\int_{\mathbb{R}^d}|\xi||\hat{f}| = CR\underbrace{\int_{\mathbb{R}^d}|\xi|^{1-\frac{d}{2}}}_{\approx A}|\xi|^{\frac{d}{2}}|\hat{f}_{l,A}|$$

$$\leq CAR\|f\|_{\dot{H}^s}$$

Let $A = R^{-1}$ $\square$.

$\boxed{\text{Thm 5.2.9}}$ (Compact embedding) $t < s$. multiplication by a function in $\mathcal{S}(\mathbb{R}^d)$ is compact ~~embedding~~ operator from $H^s(\mathbb{R}^d)$ to $H^t(\mathbb{R}^d)$

$$\{f_n\} \subseteq H^s(\mathbb{R}^d) \quad \|f_n\|_{H^s} \leq 1 \overset{\text{Eberlein-Šmulian}}{\Longrightarrow} f_{n_k} \rightharpoonup f \text{ in } H^s$$

$\|f\|_{H^s} \leq 1$

Let $g_n = f - f_n$ (relabel)

It remains to show $\varphi g_n \to 0$ in H^t.

(It is clear that $\sup_n \|\varphi g_n\| < \infty$)

$$\int_{\mathbb{R}^d} \langle\xi\rangle^{2t} |\widehat{\varphi g_n}(\xi)|^2 \, d\xi = \int_{|\xi|\le R} \langle\xi\rangle^{2t} |\widehat{\varphi g_n}(\xi)|^2 d\xi + \int_{|\xi|>R} \langle\xi\rangle^{2(t-s)} \langle\xi\rangle^{2s} |\widehat{\varphi g_n}|^2 d\xi$$

$$\le \underbrace{\int_{|\xi|\le R} \langle\xi\rangle^{2t} |\widehat{\varphi g_n}(\xi)|^2 d\xi}_{I} + \frac{C\boxed{\|\varphi g_n\|_{H^s}}}{(1+R^2)^{s-t}} \Leftarrow \text{uniformly bounded}$$

$$I = \int_{|\xi|\le R} \langle\xi\rangle^{2t} |\hat{\varphi} * \hat{g}_n(\xi)|^2 \approx \int \langle\xi\rangle^{2t} \left| \int \hat{\varphi}(\xi-\eta) \hat{g}(\eta)\, d\eta \right|^2$$

$$\approx \int \langle\xi\rangle^{2t} \left| \int \underbrace{\langle\eta\rangle^{-2s} \hat{\varphi}(\xi-\eta)}\langle\eta\rangle^{2s} \hat{g}(\eta) \right|^2$$

$$\approx \int \langle\xi\rangle^{2t} \left| (g_n , \underbrace{(\varphi(\xi-\cdot) \langle\cdot\rangle^{-2s})^{\vee}}_{\varphi})_{H^s} \right|^2 \xrightarrow{?} 0$$

To take the limit reasonably, we shall ~~see~~ show $\sup_n |\hat{\varphi} * \hat{g}_n| < \infty$

$$|\hat{\varphi} * \hat{g}_n| \le \|g_n\|_{H^s}^2 \|\varphi(\xi-\cdot)\langle\cdot\rangle^{-2s}\|_{L^2}^2$$

$$\int_{\mathbb{R}^d} \langle\eta\rangle^{-2s} |\hat{\varphi}(\xi-\eta)|^2 d\eta \quad \overset{|\xi|\le R}{\le} C\int_{|\eta|\le 2R} \langle\eta\rangle^{2|s|} + C\int_{|\xi|>2R} \langle\xi\rangle^{2|s|} \langle\xi\rangle^{-2N_0}$$

$$N_0 = \frac{d}{2} + |s| + 1$$

$$< \infty$$

□

$\boxed{\text{Thm 5.2.10}}$ (Trace) Let $s > \frac{1}{2}$, the restriction map γ defined by

$$\gamma : \begin{cases} \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{d-1}) \\ f \mapsto \gamma(f) : (x_1, \cdots x_d) \mapsto f(0, x_2, \cdots x_d) \end{cases}$$

can be extended from $H^s(\mathbb{R}^d) \xrightarrow{\text{onto}} H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$

A more useful version of trace thm would be stated in the following.

$$H^s(\mathbb{R}^d_+) := \{ u \in L^2(\mathbb{R}^d) : \langle\xi'\rangle^{s-k} \partial_d^k \hat{u}(\xi', x_d) \in L^2(\mathbb{R}^d) \}$$

Here the Fourier transform is defined $\quad [s] \ge k \ge 0 \in \mathbb{Z}$

for $x' = (x_1, \cdots x_{d-1})$.

$\boxed{\text{Cor 5.2.11}}$ (trace ineq.) $s > \frac{1}{2}$ $\|Trf\|_{H^{s-\frac{1}{2}}(\partial\mathbb{R}^d_+)} \leq C\|f\|_{H^s(\mathbb{R}^d_+)}$

proof of (5.2.11): $f \in C^\infty(\mathbb{R}^d_+)$

$$\|Trf\|^2_{H^{s-1}(\partial\mathbb{R}^d_+)} = \int_{\partial\mathbb{R}^d_+} \langle\xi'\rangle^{2s-1} \underbrace{|\hat{f}(\xi',0)|^2}\, d\xi'$$

$$= -2\mathrm{Re} \int_0^\infty\int_{\mathbb{R}^d} \langle\xi'\rangle^{2s-1}\, \partial_d \hat{f}(\xi',x_d)\, \overline{\hat{f}(\xi',x_d)}$$

Fourier transform Not relies on x_d $= -2\mathrm{Re} \int_0^\infty \int_{\mathbb{R}^d} \partial_d \langle\nabla'\rangle^{s-1} f \quad \langle\nabla'\rangle^s f$

$$= -2 \int_0^\infty \int_{\mathbb{R}^d} \partial_d \langle\nabla'\rangle^{s-1} f \quad \langle\nabla'\rangle^s f\, dx\, dx_d$$

$$\leq 2\|f\|_{H^s(\mathbb{R}^d_+)}$$

$\square$

proof of (5.2.10)

existence: BLT thm It suffices to show $\|\gamma(f)\|_{H^{s-\frac{1}{2}}} \leq C\|f\|_{H^s}$ for $\forall f \in \mathcal{S}$

$$f(0,x') = C\int_{\mathbb{R}^d} e^{ix'\cdot\xi'}\hat{f}(\xi_1,\xi')\, d\xi_1\, d\xi'$$

$$= C\int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'}\left(\int_{\mathbb{R}} \hat{f}(\xi_1,\xi')\, d\xi_1\right) d\xi'$$

$$\Rightarrow \widehat{\gamma(f)}(\xi') = C\int_{\mathbb{R}} \hat{f}(\xi_1,\xi')\, d\xi_1$$

$$\leq C\Big(\underbrace{\int_{\mathbb{R}} (1+\xi_1^2+|\xi'|^2)^{-s}\, d\xi_1}_{<\infty}\Big)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \langle\xi\rangle^{2s} \hat{f}(\xi)\, d\xi_1\right)^{\frac{1}{2}}$$

$s > \frac{1}{2}$

$\approx C_s \langle\xi'\rangle^{1-2s}$

$$\|\gamma f\|^2_{H^{s-\frac{1}{2}}} \approx \int \cancel{\langle\xi'\rangle^{2s-1}}\langle\xi\rangle^{2s}\hat{f}\, d\xi_1\, d\xi \quad \leq C_s\|f\|^2_{H^s}$$

onto: right inverse to γ C_c^∞, $\chi(0)=1$

$$Rv := (2\pi)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'}\chi(x_1\langle\xi'\rangle)\hat{v}(\xi')\, d\xi'$$

$\|Rv\|_{H^s} \overset{C}{\leq} \|v\|_{H^{s-\frac{1}{2}}}$

$\Rightarrow \gamma R v = v$

$\square$

Sth can be added: Littlewood-Paley theory.