Solution of PDE 2 Lecture Notes.

$\boxed{1.1.1}$ (1) Let $f^*(x) = \int_0^x f'(t)\,dt$ ~~F(0)~~ where f' is the weak derivative of f.

$$\Rightarrow F = f - f^* \qquad \cancel{F' \overset{LDT}{=} f} \qquad \cancel{\int F\,\partial\varphi = (-1)\int F'\varphi}$$

$$\cancel{\overset{LDT}{\Leftarrow} (-1)\int (f'-f')\varphi = 0 \quad \text{for } \forall \varphi \in C_c^\infty}$$

$$\Rightarrow \int F\,\partial\varphi = \int f\,\partial\varphi - \int f^*\partial\varphi$$

$$= (-1)\int f'\varphi - (-1)\int f\varphi = 0$$

Claim: Weak derivative equals to zero $\Rightarrow$ const

proof: $f_m \overset{\in C^\infty}{\longrightarrow} f$ in $W^{1,p}(0,1)$ $\Rightarrow$ $f_m' \to 0$ $L^p(0,1)$

$$\Rightarrow |f_n(y)-f_n(x)| \le \int_x^y |f_n'| \le \|f_n\|_1 \to 0$$

$$\overset{\text{sub seq}}{\Rightarrow} f_{n_k}(y) - f_{n_k}(x) \overset{a.e.}{=} 0 \qquad \checkmark$$

(2) $$|f(y)-f(x)| \overset{(1)}{=} \left|\int_x^y f'(t)\,dt\right| \le \|f'\|_{L^1(x,y)}$$

$$\overset{\text{Hölder}}{\le} \|f'\|_{L^p(x,y)}\,|x-y|^{1-\frac{1}{p}} \le \text{RHS} \qquad \square$$

$\boxed{1.1.2}$ $$\infty > \|u\|_{L^p(U)} \ge \left\|\frac{1}{2^k}|x_k-x|^{-a}\right\|_{L^p(U)} \Rightarrow ap < d$$

$$\infty > \|u'\|_{L^p(U)} \ge \left\|\frac{1}{2^k}|x_k-x|^{-(a+1)}\right\|_{L^p(U)} \Rightarrow (a+1)p < d$$

$$\Rightarrow a < \frac{d-p}{p}$$

It remains to show that $a < \frac{d-p}{p}$ is ~~suff~~ sufficient.

Note that x^p is convex, and then we have Jensen ineq.

$$\Rightarrow \int \left(\sum_{k=1}^\infty \frac{1}{2^k}|x-x_k|^{-(a+1)}\right)^p \overset{\text{Jensen}}{\le} \int \sum_{k=1}^\infty |x-x_k|^{\overset{-(a+1)p}{\cancel{-ap}}}\cdot\frac{1}{2^k}$$

$$\overset{\text{Fubini}}{=} \sum_{k=1}^\infty \frac{1}{2^k}\underbrace{\int_0^1 |x_k-x|^{-(a+1)p}\,dx}_{<\infty} < \infty \qquad \square$$

$\boxed{1.1.3}$ $\zeta \in C_c^\infty(U) \Rightarrow$ It's bounded.

Thus it's clearly ~~that~~ $\zeta u \in W^{k,p}(U)$.

It remains ~~to~~ show Leibniz's ~~rule~~ formula holds still.

$$\cancel{\int \zeta u\, \partial^\alpha \varphi}$$

$$\cancel{\int \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\zeta\,\partial^{\alpha-\beta}u\,\varphi = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_U \partial^\beta\zeta\,\partial^{\alpha-\beta}u\,\varphi}$$

$$\cancel{= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\alpha-\beta|}\int_U u\,\partial^{\alpha-\beta}(\partial^\beta\zeta\cdot\varphi)}$$

Well, we'll prove it by induction.

$|\alpha|=1$. $$\int_U \zeta u\, \partial^\alpha\phi = \int_U u\left(\partial^\alpha(\zeta\phi) - \partial^\alpha\zeta\cdot\phi\right)$$

$$= (-1)\int_U \partial^\alpha u\,\zeta\phi - \int u\,\partial^\alpha\zeta\,\phi$$

$$\Rightarrow \partial^\alpha(\zeta u) = \partial^\alpha u\cdot\zeta + u\,\partial^\alpha\zeta.$$

$|\alpha|=k$. and the formula holds for $\forall l<k$, $|\alpha|=l$

suppose: $\alpha=\beta+\gamma$ where $|\gamma|=1$.

$$\Rightarrow \int \zeta u\cdot\partial^\alpha\varphi = \int \zeta u\,\partial^\beta(\partial^\gamma\varphi) = (-1)^{|\beta|}\int \underline{\partial^\beta(\zeta u)}\,\partial^\gamma\varphi$$

$$\overset{\text{Induction}}{=} (-1)^{|\beta|}\int\sum_{l\le\beta}\binom{|\beta|}{|l|}\partial^l\zeta\cdot\partial^{\beta-l}u\cdot\partial^\gamma\varphi$$

$$= (-1)^{|\beta+\gamma|}\int\sum_{|l|\le|\beta|}\binom{|\beta|}{|l|}\left[\partial^{l+\gamma}\zeta\,\partial^{\beta-l}u + \partial^l\zeta\,\partial^{\beta-l+\gamma}u\right]\varphi.$$

$$= (-1)^{|\alpha|}\int\sum_{|l|\le|\beta|}\left[\binom{|\beta|}{|l|+1}+\binom{|\beta|}{|l|}\right]\partial^{l+\gamma}\zeta\,\partial^{\beta-l+\gamma}u\,\varphi$$

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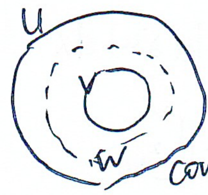
Here I made a mistake since $\binom{*}{*}$ should refer to multinomial coefficient.

$$= (-1)^\alpha\int\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\zeta\,\partial^{\alpha-\beta}u\cdot\varphi$$

□

1.2.1



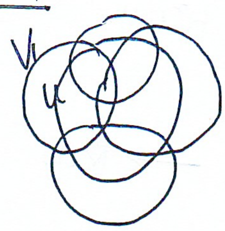
Let $V \subset\subset W \subset\subset U$, $\delta = \min\left(\mathrm{dist}(\partial V, \partial W), \mathrm{dist}(\partial W, \partial U)\right)$ 3

consider $\chi_W * \eta_\varepsilon$ where $\varepsilon < \delta$

$\Rightarrow$ $\chi_W * \eta_\varepsilon$ is ~~the~~ function we need. □

(C^∞ Urysohn lem)

1.2.2



~~$W_i \subset\subset V_i$ $\varphi_i = 1$ on $\overline{W_i}$, supp $\varphi_i \subset V_i$~~

~~$\zeta_1 = \varphi_1$, $\zeta_2 = \varphi_2(1-\varphi_1)$~~

Let $W_i = \overline{U \cap V_i} \Rightarrow U \subseteq \bigcup_{i=1}^{N} W_i$

Since $\bigcup_{i=1}^{N} W_i \supseteq U \cap \left(\bigcup_{i=1}^{N} V_i\right) = U$.

Since $\mathbb{R}^d$ is T4 space, ~~we can supp~~ there exists $\zeta_i' \in C^\infty(U)$ s.t.

supp $\zeta_i' \subseteq V_i$ & $\zeta_i' = 1$ on W_i.

$\Rightarrow \sum_{i=1}^{N} \zeta_i' > 0$ on U $\Rightarrow$ Let $\zeta_i = \dfrac{\zeta_i'}{\sum_{i=1}^{N} \zeta_i'}$ □

(partition of unity)

1.2.3 $\int_U |F(f)|^p dx = \int_U |F(f) - F(0)| dx \overset{\text{Lagrange}}{\le} M \int_U |f|^p dx < \infty \Rightarrow F(f) \in L^p(U)$

For a fixed test function $\varphi \in C_c^\infty(U)$, suppose its supp $\varphi \subseteq V \subset\subset U$.

~~In V, $f_m \to f$ in $W^{1,p}(V)$ and $f_{m'} \to f$ a.e. + L^p~~

~~$\int_U F(f)\partial_i \varphi \overset{\text{can verify}}{=} \lim_m \int_U F(f_m)\partial_i \varphi = \lim_{m'} (-1) \int_{\underset{\text{compact}}{V}} F'(f_{m'})\partial_i f_{m'} \varphi$ a.e. $\Rightarrow$ a.u.~~

~~$\overset{\text{Generalized}}{\underset{\text{DCT}}{=}} (-1)\int_U F'(f)\partial_i f \varphi$.~~ □

~~But also~~ We can see why $F(0) = 0$ is not necessary when U is of finite $\mathcal{L}$-measure by the process of proof.

$\boxed{1.2.4}$ $\cancel{\ }\ \|u\|_{L^2(U)} \cdot \|\partial^2 u\|_{L^2(U)} \overset{\text{Hölder}}{\geq} \|u\partial^2 u\|_{L^1(U)} = \int_U |u\partial^2 u|\, dx$

$$\geq \left|\int u\partial^2 u\, dx\right| \underset{\text{by parts}}{\overset{\text{Integral}}{=}} \left|\int (u')^2\, dx\right| = \|\nabla u\|^2_{L^2(U)}$$

If $\partial^2 u$ here refers to Hessian matrix instead of Δ.

then we have $\|\nabla u\|_{L^2(U)} \leq \sqrt{d}\, \|u\|_{L^2(U)} \cdot \|\partial^2 u\|_{L^2(U)}$ ($\sqrt{d} \to C$.)

$u \in H^1_0(U) \Rightarrow \exists \{v_i\} \subseteq C^\infty_c(U)$ $\quad v_i \to u$ in $H^1(U)$

$u \in H^2(U) \Rightarrow \exists \{w_j\} \subseteq C^\infty(\bar{U})$ $\quad w_j \to u$ in $H^2(U)$

$\cancel{\partial v_i \partial w_j \to \partial}$

$|\nabla v_i||\nabla w_j| \to |\nabla u|^2$ in L^1 $\quad$ (Indeed $\nabla v_i \cdot \nabla w_i \to |\nabla u|^2$ in L^1)

$$\left|\int_U \nabla v_i \cdot \nabla w_i\, dx\right| = \left|-\int_U v_i \Delta w_i\, dx\right|$$

$$\overset{\text{Hölder}}{\leq} \|v_i\|_{L^2(U)} \cdot \|\Delta w_i\|_{L^2(U)} \leq \sqrt{d}\, \|v_i\|_{L^2(U)} \|\partial^2 w_i\|_{L^2(U)}$$

Take limit: $\|u\|_{L^2(U)} \leq \sqrt{d}\, \|u\|_{L^2(U)} \cdot \|\partial^2 u\|_{L^2(U)}$ $\square$

$\boxed{1.2.3}$ Fix: $\int_U F(f)\partial_i \varphi\, dx \overset{\text{verify}}{=} \lim_\varepsilon \int_U F(f_\varepsilon)\partial_i \varphi\, dx$

$$= \lim_\varepsilon (-1) \int F'(f_\varepsilon)\partial_i f_\varepsilon \varphi\, dx$$

$$\leq \overline{\lim_\varepsilon} \int_U \underbrace{|F'(f_\varepsilon)|}_{\text{Bound}} \underbrace{|\partial_i f_\varepsilon - \partial_i f|}_{L^1} \cdot |\varphi|\, dx + \int_U \underbrace{|F'(f_\varepsilon) - F'(f)|\, |\partial_i f| |\varphi|}_{\substack{\text{DCT} \\ \leq 2M|\partial_i f||\varphi|}}\, dx$$

$\to 0$ as $\varepsilon \to 0$. $\square$

$\boxed{1.2.5}$ $\|\nabla u\|_{L^p(U)} = \cancel{\| \nabla u \cdot \nabla u \cdot |\nabla u|^{p-2} \|^{\frac{1}{p}}_{L^1(U)}}$

$$\cancel{\overset{\text{Hölder}}{\leq} \left(\|\nabla u\|_{L^p(U)} \cdot \|\nabla u\|_{L^p(U)} \cdot \| |\nabla u|^{p-2} \|_{L^{\frac{p}{p-2}}(U)} \right)^{\frac{1}{p}}}$$

$$= \|\nabla u\|^{\frac{2}{p}}_{L^p(U)}.$$

$$\cancel{= \sum_i \int_U \partial_i u \cdot \partial_i u\, |\nabla u|^{p-2}\, dx}$$

$$\cancel{= \sum_i \int_U u\, \partial_i \left(\partial_i u\, |\nabla u|^{p-2}\right) dx}$$

$\boxed{1.2.5}$ (1) $\|\partial_i u\|^p_{L^p(U)} = \int_U \partial_i u \; \partial_i u \cdot |\partial_i u|^{p-2} \, dx$

$$= -\int_U \partial u \; \partial_i \left(\partial_i u \, |\partial_i u|^{p-2} \right) dx$$

$$= -(p-1) \int_U u \, |\partial_i u|^{p-2} \, \partial_{ii} u$$

$$\leq (p-1) \|u\|_{L^p(U)} \|\partial_i u\|^{p-2}_{L^{\frac{p}{p-2}}} \cdot \|\partial_{ii} u\|_{L^p}$$

$$= C \|u\|_{L^p(U)} \|\partial_i u\|^{p-2}_{L^p(U)} \|\partial_{ii} u\|_{L^p(U)}$$

$$\Rightarrow \|\partial_i u\|_{L^p(U)} \leq C \|u\|^{\frac{1}{2}}_{L^p(U)} \cdot \|\partial_{ii} u\|^{\frac{1}{2}}_{L^p(U)}$$

$$\Rightarrow \|\nabla u\|_{L^p} \leq C \|u\|^{\frac{1}{2}}_{L^p(U)} \|\partial^2 u\|^{\frac{1}{2}}_{L^p(U)}. \quad \square$$

(2) $\|\partial_i u\|^{2p}_{L^{2p}(U)} = \int_U |\partial_i u|^{2p} \, dx$

$$\approx \int_U u \, |\partial_i u|^{2p-2} \, \partial_{ii} u$$

$$\overset{\text{Holder}}{\leq} C \|u\|_{L^\infty(U)} \|\partial_i u\|^{2p-2}_{L^{\frac{2p}{2p-2}}(U)} \cdot \|\partial_{ii} u\|_{L^p(U)}$$

$\left(\infty, \frac{2p}{2p-2}, p\right)$

$$\|\partial_i u\|^{2p-2}_{L^{\frac{2p}{2p-2}}} = \left(\int_u |\partial_i u|^{2p-2 \cdot \frac{2p}{2p-2}} \right)^{\frac{p-1}{p}} = \left(\int_U |\partial_i u|^{2p} \right)^{\frac{1}{2p} \cdot (2p-2)} = \|\partial_i u\|^{2p-2}_{L^{2p}(U)}$$

$$\Rightarrow \|\partial_i u\|_{L^{2p}(U)} \leq C \|u\|^{\frac{1}{2}}_{L^\infty(U)} \cdot \|\partial_{ii} u\|^{\frac{1}{2}}_{L^p(U)}$$

different

$$\Rightarrow \|\nabla u\|_{L^{2p}(U)} \leq C \|u\|^{\frac{1}{2}}_{L^\infty(\partial U)} \cdot \|D^2 u\|^{\frac{1}{2}}_{L^p(U)}. \quad \square$$

$\boxed{1.2.6}$ $f_\varepsilon = \eta_\varepsilon * f$. suppose $V \subset\subset U$.

$\nabla f_\varepsilon = \eta_\varepsilon * \underline{\underline{\nabla f}}$ Let ε be small sufficiently

$\Rightarrow \nabla f_\varepsilon = 0$ ~~a.e.~~ in V. Since f_ε is a smooth function

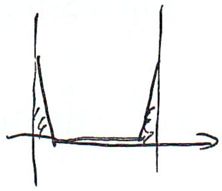
$\Rightarrow f_\varepsilon = C_\varepsilon$ in V. Since $f_\varepsilon \longrightarrow f$ in $W^{1,p}(V)$.

We have f equaling to a constant by $\{f_\varepsilon\}$'s subsequence.

Connectedness argument tells to conclusion. $\square$

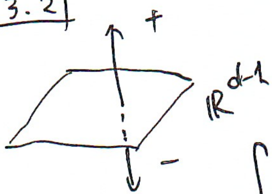
$\boxed{1.3.1}$ Let $d=1$. $U=(0,1)$.

$$f_n = \begin{cases} 1-(n+1)x & x\in[0,\frac{1}{n+1}] \\ (n+1)x-n & x\in[\frac{n}{n+1},1] \\ 0 & \text{otherwise} \end{cases}$$



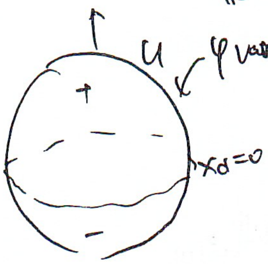
$\Rightarrow \|f_n\|_{L^p(U)} \longrightarrow 0 \quad tr\, f = 1 \quad$ then "tr" is not continuous $\square$

$\boxed{1.3.2}$ It suffice to show the synthesized derivative $\frac{\partial f}{\partial x_d}$ is $L^2(\mathbb{R}^d)$.



$$\int_{\mathbb{R}^d} u\,\partial_i\varphi = \int_{U^+} u\,\partial_i\varphi + \int_{U^-} u\,\partial_i\varphi$$

u, φ vanishes



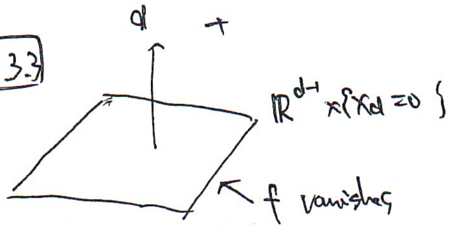
$$= \int_{U^+}[\partial_i(u\varphi) - \partial_i u\,\varphi] + \int_{U^-}[\partial_i(u\varphi) - \partial_i u\,\varphi]$$

$$= \int_{\partial U^+} - \int_{U^+} \partial_i u\,\varphi - \int_{U^-}\partial_i u\,\varphi$$

$$= -\int_U v_i\varphi$$

in which $v_i = \begin{cases} \partial_i u & \mathbb{R}^d_+ \\ \partial_i u & \mathbb{R}^d_- \end{cases} \Rightarrow v_i \in L^2$ $\square$

$\boxed{1.3.3}$



~~For a given $\varphi\in C_c^\infty(\mathbb{R}^d_+)$. Suppose $\operatorname{supp}\varphi \subset\subset U$.~~

~~For $1\le i\le d-1$, $\int_{\mathbb{R}^d_+} f\,\partial_i\varphi = -\int_U \partial_i f\,\varphi$~~

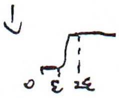
It suffices to show $\partial_i f$ is of trace zero for $1\le i\le d-1$

$\Leftrightarrow \exists$ C_c^∞ functions $\tilde{f}_m \longrightarrow \partial_i f$ in H^1

$f\in H^2(\mathbb{R}^d_+) \Rightarrow f_n \xrightarrow{\in C^\infty(\mathbb{R}^d_+)} f$ in $H^2(\mathbb{R}^d_+)$. Since $tr\, f = 0$.

using C^∞ Urysohn, there exists χ_n s.t. $\chi_n \longrightarrow$ a.e. $\chi_{\mathbb{R}^d_+}$

& $\operatorname{dist}(\operatorname{supp}\chi_n, \partial\mathbb{R}^d_+) > \frac{1}{n}$. Consider $\underline{f_n\chi_n \xrightarrow{H^2} f\ ?}$ $\checkmark$



standard bump $\square$

1.4.1 Assume U is bounded.

(1) $\|f\|_{L^P(U)} \leq \|f\|_{W^{1,P}(U)} \quad \forall f$

(2) $\{f_m\}$ is bounded in $W^{1,P}(U)$

$1 \leq p < d$ ✓ GNS + A-A

$d < p < \infty$ Morrey $\Rightarrow$ Uniformly bounded

equicontinuous : $|f_k^*(x) - f_k^*(y)| \leq \overset{\text{Morrey}}{\underline{\underline{M}}} |x-y|^{\alpha}$ ✓

A-A $\|f_{k_j}^* - f\|_{L^P} = \|f_{k_j}^* - f^*\|_{L^P} \to 0$ ✓

$p=\infty$ Lipschitz condition $\to$ quasi convex ✓

$p=d$. $W^{1,d} \hookrightarrow\hookrightarrow W^{1,P} \overset{R\text{-}K.}{\hookrightarrow\hookrightarrow} L^d$ $\square$

$p < d$

This step I used the KRF thm (A-A thm of L^P version from Haim's book Page 111) uniform translation comes from

$\sup_k \|u_k(\cdot+h) - u_k\|_{L^q(U)} \leq |h| \, \|\nabla u\|_{L^q(U)} \lesssim |h|$

1.4.2 $\int |\ln \ln(1+\frac{1}{|x|})|^d dx \leq \int \sqrt{\ln(1+\frac{1}{|x|})}^d$

$\leq \int \frac{1}{|x|^{\frac{d}{2}}} dx < \infty$

$\int \left(\frac{1}{\ln(1+\frac{1}{r})} \cdot \frac{1}{1+\frac{1}{r}} \cdot \frac{1}{r^2}\right)^d r^{d-1} dr$

$= \int_0^1 \frac{1}{r(\ln(1+\frac{1}{r}))^d} dx$ ~~$\leq \int_0^1 \frac{1}{r(\ln r)^d}$~~ $< \infty$.

~~$\int_0^\varepsilon \frac{r^{-1}}{(\ln\frac{1}{r})^d} = \int_{\infty}^{T} \frac{1}{e^{-t}}$~~ ~~$t = \ln\frac{1}{r},\ e^{-t} = r$~~ $\square$

1.4.3 $\|u\|^2_{L^2(U)} = \|u\|^2_{L^2(U\setminus Z)} = \|u - (u)_U + (u)_U\|^2_{L^2(U\setminus Z)}$

$\leq \underbrace{\|u-(u)_U\|^2_{L^2(U\setminus Z)}}_{\Downarrow} + \underbrace{\|(u)_U\|^2_{L^2(U\setminus Z)}}_{\Downarrow\ \text{Jensen}}$ $\left(\|(u)_U\|_{L^2(U\setminus Z)} = \cancel{\int \frac{1}{\int_U}\int}\right)$ Jensen $U\setminus Z$

$\leq \|\nabla u\|^2_{L^2(U)} + (1-\alpha)\|u\|^2_{L^2(U)} \Rightarrow \|u\|^2_{L^2(U)} \leq \frac{C}{\alpha}\|u\|^2_{L^2}$. $\square$

$\boxed{1.4.4}$ ~~$\nabla(|x|^{-1}) = \frac{x}{|x|^3}$ $\Delta(|x|^{-1})$~~

~~$\nabla\cdot\left(\frac{x}{|x|^2}\right) = \frac{d-2}{|x|^2}$ $\Delta(|x|^{-1}) = -\frac{d-3}{|x|^3}$~~

~~$\Rightarrow \int_{B_r} \frac{u^2}{|x|^2}\,dx = \frac{1}{d-2}\int_{B_r} u^2 \cdot \nabla\cdot\left(\frac{x}{|x|^2}\right) dx$~~

~~$= \frac{1}{d-2}\int$~~

$\nabla(|x|^{-1}) = -\frac{\vec{x}}{|x|^2}$, $\vec{n} = \frac{\vec{x}}{|x|}$

$$LHS = \int_{B_r} u^2 \vec{n}\cdot\nabla(|x|^{-1}) \overset{IBP}{=\!=} \int_{B_r} \frac{1}{|x|}\nabla\cdot(u^2\vec{n})\,dx - \frac{1}{r}\int_{\partial B_r} u^2\,dx$$

$$\nabla\cdot(u^2\vec{n}) = 2u\cdot\nabla u\cdot\vec{n} + u^2\frac{d-1}{|x|}$$

$$= \int_{B_r} \cancel{2u\,\nabla u\cdot\vec{n}\,dx +} |x|^{-1}\left(2u\,\nabla u\cdot\vec{n} + \underbrace{(d-1)u^2|x|^{-1}}\right) - \frac{1}{r}\int_{\partial B_r} u^2\,dS$$

$$\Rightarrow (d-2)I = \underbrace{\frac{1}{r}\int_{\partial B_r} u^2\,dS} - \int_{B_r} 2u\frac{\nabla u\cdot x}{|x|^2}\,dx$$

$$= \frac{1}{r^2}\int_{B_r}\nabla(u^2\cdot x)\,dx - \int_{B_r} 2u\frac{\nabla u\cdot x}{|x|^2}\,dx$$

$$= \frac{1}{r^2}\int_{B_r}(\underbrace{2u\,\nabla u\cdot x} + du^2)\,dx - \int_{B_r} 2u\frac{\nabla u\cdot x}{|x|^2}\,dx$$

$$= \underbrace{\frac{d}{r^2}\int_{B_r} u^2\,dx} + \int_{B_r} 2u\,\nabla u\cdot x\underbrace{\left(\frac{1}{r^2} - \frac{1}{|x|^2}\right)}_{L'}$$

$$\int_{B_r} 2u\,\nabla u\cdot x\left(\frac{1}{r^2} - \frac{1}{x^2}\right)dx \le \frac{2}{r^2}\int_{B_r} |u||\nabla u|\frac{|r^2 - x^2|}{|x|}$$

$$\le 2\int_{B_r} |u||\nabla u||x|^{-1} \le 2\left(\int\frac{u^2}{|x|^2}\right)^{\frac{1}{2}}\left(\int(\nabla u)^2\right)^{\frac{1}{2}}$$

That is to say $\underbrace{(d-2)I} \le \frac{d}{r^2}\int_{B_r} u^2\,dx + \underbrace{2\sqrt{I}\cdot\left(\int(\nabla u)^2\right)^{\frac{1}{2}}}$

$$\le \frac{d}{r^2}\int_{B_r} u^2 + \underbrace{\tfrac{1}{2}I} + 4\int|\nabla u|^2$$

$$\Rightarrow I \le \max\left\{\frac{d}{d-\frac{5}{2}}, \underset{\approx 4}{\frac{4}{d-2}}\right\}\left(\int_{B_r}\frac{u^2}{r^2} + |\nabla u|^2\,dx\right). \qquad \square$$

$\boxed{1.4.5}$ (1) It suffices to show

$$\cancel{0 = \int_{\mathbb{R}^d} u^2 \nabla\cdot F + \int_{\mathbb{R}^d} \nabla(u^2)\cdot F \, dx}$$

$$\cancel{= \int_{\mathbb{R}^d} \nabla\cdot(u^2 F)\, dx}$$

Ω_ε, ε

$$\int_{\Omega_\varepsilon} u^2 \nabla\cdot F = \underbrace{\int_{\partial\Omega_\varepsilon} u^2 F\cdot \hat{n}\, dS}_{\lesssim \sup_{\varepsilon} |F| \cdot \varepsilon^{d-1}} - \int_{\Omega_\varepsilon} \nabla u^2 \cdot F$$

✓

(2) Using 1.4.4. □

$\boxed{1.4.6}$

$$\int_U u_\varepsilon \Delta u_\varepsilon \, dx = \int_{\partial U} \underbrace{u_\varepsilon}_{=0 \text{ on } \partial U} \nabla u_\varepsilon \cdot \hat{n}\, dS - \int_U |\nabla u_\varepsilon|^2 dx$$

$$= -\int_U (\nabla u_\varepsilon)^2 dx$$

$$\Rightarrow \int_U |\nabla u_\varepsilon|^2 dx = -\int_U u_\varepsilon \Delta u_\varepsilon \, dx = -\frac{1}{\varepsilon}\int_U u_\varepsilon (u_\varepsilon - f)\, 1_B \, dx$$

$$\Rightarrow \underbrace{\int_U |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_U u_\varepsilon^2 \, dx}_{\geq 0} = \frac{1}{\varepsilon}\int_U u_\varepsilon f \, 1_B \, dx$$

$$\leq \frac{1}{\varepsilon} \|u_\varepsilon\|_{L^2(U)} \|f\|_{L^2(B)}$$

$$\leq \frac{A}{\varepsilon} \|u_\varepsilon\|^2_{L^2(U)} + \frac{1}{\varepsilon 2A} \|f\|^2_{L^2(B)}$$

To obtain the uniform estimate. A should be like $\frac{c}{\varepsilon}$, but then we lose the control of $\|u_\varepsilon\|^2_{L^2(U)}$

The choice of multiplier is not so good.

$$-\int_U \Delta u_\varepsilon \cdot (u_\varepsilon - f) + \frac{1}{\varepsilon}\int_U (u_\varepsilon - f)^2 1_B \, dx = 0$$

IBP: $\int_U \nabla u_\varepsilon \cdot \nabla(u_\varepsilon - f) + \frac{1}{\varepsilon}\int_U (u_\varepsilon - f)^2 1_B = 0$

$$\Rightarrow \|\nabla u\|^2_{L^2(U)} \int_U (\nabla u)^2 dx \leq \int_U \nabla u \cdot \nabla f \, dx \leq \|\nabla u\|_{L^2(U)} \|f\|_{L^2(U)}$$

$\Rightarrow \|\nabla u_\varepsilon\|_{L^2(U)}$ is uniformly bounded. □

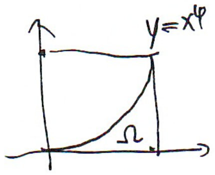
$\boxed{1.4.6}$ (2) By (1), we have

$$\frac{1}{\varepsilon}\|u_\varepsilon - f\|^2_{L^2(B)} + \beta\|\nabla u\|^2_{L^2(U)} = \int_U \nabla u \cdot f \, dx$$

$$\le \underbrace{\|\nabla u_\varepsilon\|_{L^2(U)} \cdot \|f\|_{L^2(U)}}_{\text{Bounded uniformly in } \varepsilon.}$$

$$\Rightarrow \overline{\lim_{\varepsilon \to 0}} \|u_\varepsilon - f\|^2_{L^2(B)} \le 0$$ □

$\boxed{1.4.7}$



$$\int_\Omega \frac{1}{x^2}\,dxdy = \int_0^1 \frac{dx}{x^2}\int_0^{x^4} dy = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\int_\Omega \frac{1}{x^4} = 1 \quad \Rightarrow \quad \frac{1}{x} \in H^1(\Omega)$$

$$\int_0^1 \frac{1}{x} = \infty \quad \Rightarrow \quad \frac{1}{x} \notin L^5(\Omega)$$

$\Omega \subseteq \mathbb{R}^2$ $\quad f(x) \in W^{1,2}$ $\quad p = d$. No contradiction. □

$\boxed{1.4.8}$. $W^{2,2} \hookrightarrow\hookrightarrow W^{1,2}$

$\|f\|_{H^1(U)} \le \|f\|_{H^2(U)}$ ✓

It remains to show that any bounded sequence of $\{f_m\}$ in $H^2(U)$ has a convergent subsequence in $H^1(U)$.

Note $H^2 \xrightarrow{\text{can}} H^1 \hookrightarrow\hookrightarrow L^2$

That is to say, A Bounded sequence of H^2 has a convergent subsequence in L^2. Then we can treat its derivatives d times. Then we obtain a convergent subsequence in H^1.

The ineq. should be proced by contradiction:

$\exists \varepsilon_0, \forall k > 0, \exists u_k \in H^2$ s.t

$$\|\nabla u\|_{L^2(U)} \ge \varepsilon_0 \|u_k\|_{H^2(U)} + k\|u_k\|_{L^2(U)}$$

We can assume $\|u_k\|_{H^2} = 1 \Rightarrow u_{k_n} \xrightarrow{H^1} u \in H^1$

$$\infty > \|\nabla u_{k_n}\|_{L^2(U)} \ge \varepsilon_0 + \underbrace{k\|u_k\|_{L^2(U)}}_{\nearrow \infty} \quad ↯$$ □

$\boxed{1.4.9}$ Case of Unit ball: By Poincáre Inequality

$$\Rightarrow \| f - (f)_{x,1} \| \lesssim_{d,p} C \| \nabla f \|_{L^p(B(x,1))}.$$

Case of $B(x,r)$

$$\fint_{B(x,r)} \Big| f(x) - \fint f(t) \Big|^p dt = r^d \int_{B(0,1)} \Big| f(x+ry) - \fint f(x+ry) \Big|^p dy$$

$$\int_{B(x,r)} |\nabla u|^p = r^d \int_{B(0,1)} |\nabla u(x+ry)\, r|^p \, dy$$

$$\Rightarrow \| f - (f)_{(x,r)} \|_{L^p(u)} \lesssim_{d,p} \| \nabla f \|_{L^p(u)} \qquad \square$$

$\boxed{1.4.10}$

$$\fint_{B(x,r)} |f - (f)_{x,r}| \, dy = \frac{C}{r^d} \int |f - (f)_{x,r}| \, dy$$

$$\leq \frac{C}{r^d} \| f - (f)_{x,r} \|_{L^d(B(x,r))} \| 1 \|_{L^{d'}(B(x,r))}$$

$$= \frac{C}{r^d} \cdot \left(\mathcal{L}^{d'}(B(x,r)) \right)^{\frac{1}{d'}} \cdot r \| \nabla u \|_{L^d(\mathbb{R}^d)}$$

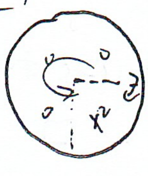
$$= \frac{C}{r^{d+2}}$$

$$\fint \| f - (f)_{x,r} \| \underset{d}{\overset{(1.4.10)}{\lesssim}} r \| \nabla f \|_{L^1(B)} \lesssim_d r \| \nabla f \|_{L^d(B)} \| 1 \|_{L^{d'}(B)}$$

$$\lesssim r \, r^{d(1-\frac{1}{d})} \| \nabla f \|_{L^d(B)}$$

$$\Rightarrow \fint |f - (f)_{x,r}| \, dy \lesssim_d \| \nabla f \|_{L^d}. \qquad \square$$

$\boxed{1.4.11}$ ~~u(x,y) =~~ It's clearly that $u \in W^{1,\infty}(U)$.

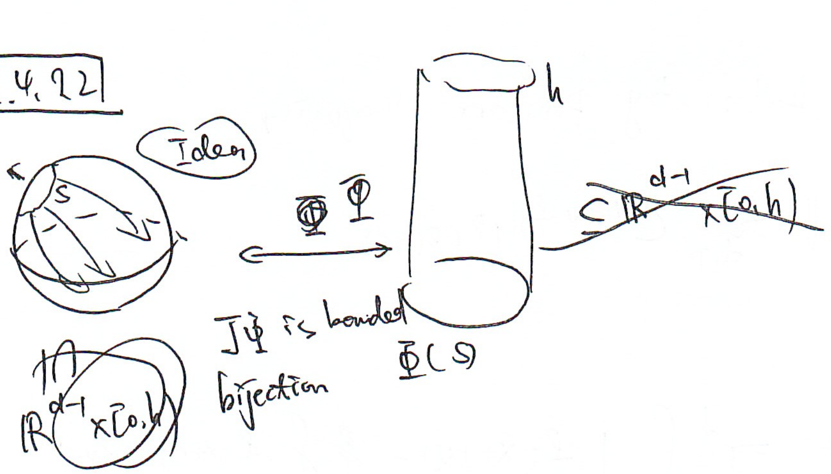


← Not Lipschitz.

$\square$

$\boxed{1.4.22}$

Idea



$\hat{u}(y,t) = u(\Phi(y,t))$ $\quad \hat{u}(y,0)=0$

$$|\hat{u}(y,t)-\hat{u}(y,0)| \lesssim \int_0^t |\nabla u^{(y,s)}|\,ds \le t^{\frac{1}{p'}}\left(\int_0^t |\nabla u|^p\right)^{\frac{1}{p}}$$

$$\|u\|_{L^p(U)} = \int |\hat{u}|^p\,dy\,dt \lesssim \int \left(t^{p-1}\int_0^t |\nabla u|\right)dy\,dt$$

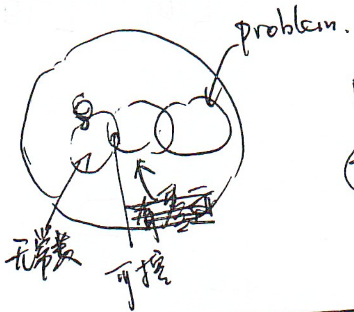
$$\le \int_U dy \underbrace{\int_0^h t^{p-1}\left(\int_0^t |\partial_t u|^p ds\right)dt}_{\le \frac{h^p}{p}\int_0^h |\partial_t u|^p ds}$$

$$\le C\int |\nabla u|^p$$

~~Using~~ 注. 这里有一个神奇的补丁技术

因为肯定用单位分解，边界全在 S 或 U 内部的是没有问题的

i.e. 不引入常数项. 问题在于那些边界与 $\partial U\setminus S$ 有交的区域



用三角不等式在重叠部分上做处理即可

有点冗长.

感觉还是紧性论证比较好 $\|u\|_{L^p(U)}=1$, $\|\nabla u\|_{L^p(U)}\to 0$

$W^{1,p}(U) \hookrightarrow\hookrightarrow L^p(U)$ $\Rightarrow$ $\|\nabla u\|=0$ $\Rightarrow u=c\neq 0$.

但 $u|_S=0$ $\square$

$\boxed{1.4.13}$

$$-\int_U u \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) = \lambda \int_U |u|^{p-2} u^2$$

IBP
$$\Rightarrow \lambda = \frac{\int |\nabla u|^p\,dx}{\int |u|^p\,dx} \overset{?}{\gtrsim} C\mathcal{L}^d(U)^{-\frac{p}{d}}$$

$\boxed{p<d}$ GNS + Hölder $\int |\nabla u|^p \overset{GNS}{\geq} C\left(\int |u|^{p^*}\right)^{\frac{p^*}{p}} \geq C\left(\| |u|^{p^*}\|_{\frac{p}{p^*}} \|1\|_{(\frac{p}{p^*})'}\right)^{\frac{p^*}{p}}$

$$\|\nabla u\|_p \geq C_{p,U}\|u\|_{p^*} \geq \frac{C\|u\|_p}{\|1\|_d} = C\|u\|_p\cdot\left(\mathcal{L}^d(U)\right)^{-\frac{1}{d}}$$

$$\Rightarrow \lambda \geq C\mathcal{L}^d(U)^{-\frac{p}{d}}$$

$p \geq d$ ($p>d$) Since u is of zero trace, we can

Morrey ineq + Poincaré ineq $\Rightarrow |u(x)| \leq C\|\nabla u\|_{L^p(U)} \sim 1$

$$\Rightarrow \int |u|^p \leq \left(\mathcal{L}^d(U)\right)^p$$

$p>d=1$. $|u| \overset{\text{Morrey}}{\leq} C\|\nabla u\|_{L^p(U)}$

$$\int |u|^p dx \leq \left(C\|\nabla u\|_{L^p(U)}\right)^p \mathcal{L}^d(U)$$

$\Rightarrow \lambda \geq$

$p \geq d=1$. It suffices to show $\|\nabla u\|_{L^p(U)} \mathcal{L}(U) \gtrsim \|u\|_{L^p(U)}$.

It follows from $\boxed{\text{Ex. 1.4.9}}$.

$p \geq d \geq 2$:

$p>d$ zero extension + isoperimetric ineq

$$\|u\|_{L^\infty} \lesssim r^{1-\frac{d}{p}}\|\nabla u\|_{L^p(U)}$$

$$\Rightarrow \|u\|_{L^p(U)} \lesssim \underset{\approx \operatorname{diam} U}{r^{1-\frac{d}{p}}}\|\nabla u\|_{L^p(U)}\left(\mathcal{L}^d(U)\right)^{\frac{1}{p}}$$

这一步会因为 u 的形状变得很糟糕，但是如果承认 Faber-Krahn ineq: $\lambda_1(\Omega) \geq \lambda_1(B)$ with $\mathcal{L}(\Omega)=\mathcal{L}(B)$ 我们的估计就几乎完了

$p=d$

$\boxed{\text{Ex 1.5.1}}$ $\|fg\|_{H^k(U)} \le C\|fM\|_{H^k(U)} = CM\|f\|_{H^k(U)} \le C^2\|f\|_{H^k(U)}\|g\|_{H^k(U)}$. $\square$

$\boxed{\text{Ex 2.1.1}}$ (1) $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U) \Rightarrow \exists\, v_{n_k} \to v$ in $L^2(U)$

$\|v_{n_k}-v\|_{H^{-1}(U)} = \sup_u \left|\int (v_{n_k}-v)u\right| \le \sup_u \|v_{n_k}-v\|_{L^2(U)} \cdot \|u\|_{L^2(U)} \le \|v_{n_k}-v\|_{L^2(U)}$

$\to 0$. in $H^{-1}(U)$

H_0^1 is reflexive $\Rightarrow v_k \rightharpoonup v$, $v \in H_0^1$ $\square$

(2) prove it by contradiction. $\exists \varepsilon_0 > 0$, $\forall k > 0$. $\exists\, v_k \in H_0^1$ s.t.

$$\begin{cases} \|v_k\|_{L^2(U)} > \varepsilon_0 + k\|v_k\|_{H^{-1}(U)} \\ \|v_k\|_{H^{-1}(U)} \ge \|v_k\|_{L^2(U)} \end{cases} \Rightarrow \lightning$$ $\square$

$\boxed{\text{Ex 2.1.2}}$ $H^{-1}(\mathbb{R}^d) = (H^1(\mathbb{R}^d))'$

On the one hand, $(H^1(\mathbb{R}^d))' \subseteq (H_0^1(\mathbb{R}^d))' = H^{-1}(\mathbb{R}^d)$

On the ~~another~~ other hand, $\begin{cases} C_c^\infty(\mathbb{R}^d) \overset{\text{dense}}{\subseteq} H^1(\mathbb{R}^d) \\ C_c^\infty(\mathbb{R}^d) \subseteq H_0^1(\mathbb{R}^d) \end{cases}$, then we can use BLT then. $\square$

$\boxed{\text{Ex 2.2.1}}$ $|B[u,v]| = \left|\int a^{ij}\partial_i u\, \partial_j v + cuv\right|$

$\le \|a^{ij}\|_{L^\infty(U)}\|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} + \|c\|_{L^\infty(U)}\|u\|_{L^2(U)}\|v\|_{L^2(U)}$

$\le C\|u\|_{H_0^1(U)} \cdot \|v\|_{H_0^1(U)}$

$B[u,u] \ge \theta\|Du\|^2_{L^2(U)} - \mu\|u\|_{L^2(U)}$

$\ge \frac{\theta}{2}\|Du\|^2_{L^2(U)} + \left(\underbrace{\frac{C\theta}{2}}_{\text{from poincáre}} - \mu\right)\|u\|^2_{L^2(U)}$ $\boxed{\frac{C\theta}{2} > \mu}$ $\square$

$\boxed{\text{EX 2.2.2}}$ Define $B[u,v] \overset{\text{IBP}}{\underset{\text{using condition}}{=}} \int \Delta u \Delta v = \int f v$ where $v \in H_0^2(U)$.

(1) $|B[u,v]| \le \|\Delta u\|_{L^2(U)} \|\Delta v\|_{L^2(U)} \le \|u\|_{H_0^2(U)} \|v\|_{H_0^2(U)}$

(2) $B[u,u] = \int (\Delta u)^2 = \int \partial_i^2 \partial_j^2 u \underset{\text{Poincaré}}{\ge} \beta \|u\|_{H_0^2(U)}$

By Lax-Milgram thm. $\square$

$\boxed{\text{EX 2.2.3}}$ Since we do not have zero trace, we cannot use Poincaré ineq, so that we cannot use Lax-Milgram thm.

$(\Rightarrow)$ $\int \nabla u \cdot \nabla v = \int f v$ for $\forall v \in H^1(U)$

take $v \equiv 1 \Rightarrow 0 = \int f$.

$(\Leftarrow)$ ~~Uniqueness~~ : Consider $f \equiv 0$. $\Rightarrow \int \nabla u \cdot \nabla v = 0$ for $\forall v \in H^1$

take $v = u \Rightarrow \int |\nabla u|^2 = 0$ Since U is connected $\Rightarrow u$ is a constant

Consider boundary condition $\Rightarrow u = 0$ ∂u is of measure zero

Existence: $\|\cdot\| := \left(\int_U |\nabla u|^2\right)^{\frac{1}{2}}$ space is called V. $V = H^1(U)/\mathbb{R}$

(1) $B[u,v] \overset{C\text{-}S}{=} \|\nabla u\|_{L^2(U)}^2 \cdot \|\nabla v\|_{L^2(U)} = \|u\| \cdot \|v\|$ V is a Hilbert Space.

(2) $B[u,u] = \int |\nabla u|^2 = \|u\|$

$\int f = 0$ provides the $\boxed{\text{well-definedness}}$ of the space

Then we can use Lax-Milgram thm. $\square$

$\boxed{\text{EX 2.2.4}}$ First, we should transform the equation into a weak sense.

$$\int_U f v = -\int \Delta u \cdot v = \int_U \nabla u \cdot \nabla v - \int_{\partial U} v \, \underline{\nabla u \cdot n}$$

$$= \int_U \nabla u \cdot \nabla v + \int_{\partial U} u v$$

$$B[u,v] := \underbrace{\int_{\partial U} \underline{uv} \, dS_x}_{\text{trace thm}} + \int_U \underline{\nabla u \cdot \nabla v \, dx} \le C \|u\|_{H^1(U)} \|v\|_{H^1(U)}$$

$$B[u,u] = \int_{\partial U} u^2 \, dS_x + \int_U \nabla u^2 \, dx \overset{\exists c\,?}{\geq} c\|u\|^2_{H^1(U)}$$

Claim: Such c exists.

proof: If not. $\forall \frac{1}{k}$ $\exists u_k$ s.t.

$$\underbrace{\int_{\partial U} u_k^2}_{\text{controlled by } H^1} + \int_U \nabla u_k^2 < \frac{1}{k}\|u\|_{H^1(U)}$$

normalization: $\|u_k\|_{H^1(U)} = 1$ $\quad + \; H^1(U) \hookrightarrow\hookrightarrow L^2(U)$

$$\Rightarrow u_{n_k} \longrightarrow u \text{ in } L^2(U)$$

$$\Rightarrow \|\nabla u_{n_k}\| \longrightarrow 0 \text{ in } L^2(U)$$

By completeness, $\nabla u_{n_k} \to \underset{\text{A notation}}{\tilde{u}} \overset{?}{=} 0$ a.e.

i.e. $\exists \{\underset{\text{new index}}{u_k}\}$ $\quad u_k \to u$ in H^1 with $\nabla u = 0$ a.e.

$\Rightarrow u = \text{const}$ a.e.

~~the~~ value ~~of the trace of~~ u relies on the value ~~near~~ of u near ∂U

$\Rightarrow \underbrace{u = \text{const a.e.}}_{\text{contradiction!}}$ But on the other hand ~~to~~ $u \neq 0$. from the integral, it would ~~not~~ be zero

$\square$

Ex 2.25

$$\int_U f v \, dx = -\int_U \Delta u \, v = -\int_{\partial U} \underset{\text{smooth}}{v} \, \nabla u \cdot \vec{n} + \int \nabla u \cdot \nabla v$$

$\boxed{0 \text{ near } T_1}$

$$= -\int_{T_2} v \, \nabla u \cdot \vec{n} + \int_U \nabla u \cdot \nabla v.$$

$H = \{ v \mid \operatorname{tr} v = 0 \text{ of } T_1 \}$ $\quad = \int_U \nabla u \cdot \nabla v.$

test set

$\square$

Using $\boxed{\text{Ex 1.4.12}}$. Done!

Ex 2.26 $B[u,v] = \cancel{\int_U \nabla u \cdot \nabla v \, dx} \int_U a^{ij} \partial_i u \, \partial_j v \, dx$

If u is the weak solution of $-\partial_j(a^{ij}\partial_i u) = 0$

$$\Rightarrow \int a^{ij} \partial_i u \, \partial_j v = 0 \quad \text{for } \forall v \in H_0^1(U)$$

$\boxed{\text{EX } 2.26}$ $w=\phi(u)$

$$B[w,v] = \int a^{ij}\partial_i(\phi(u))\,\partial_j v\,dx$$
$$= \int a^{ij}\phi'(u)\,\partial_i u\,\partial_j v\,dx$$

we know $\phi''\ge 0$

$$= -\int \partial_j\left(a^{ij}\phi'(u)\,\partial_i u\right) v\,dx$$
$$= -\int \Big(\partial_j a^{ij}\,\phi'(u)\,\partial_i u\, v + \underbrace{a^{ij}\phi''(u)\,\partial_i u\,\partial_j u}_{\ge 0}\, v + a^{ij}\phi'(u)\,\partial_{ij} u\, v\Big)\,dx$$

let $\psi = \phi'(u)v$

$$\Rightarrow B[u,\psi]=0 = \int a^{ij}\partial_i u\,\partial_j(\phi'(u)v)$$
$$= \underbrace{\int a^{ij}\partial_i u\,\partial_j v\,\phi'(u)}_{B[\phi(u),v]} + \overbrace{\int a^{ij}\partial_i u\,\underbrace{\phi''(u)}_{\ge 0}\,\partial_j u\,\underbrace{v}_{\ge 0}}^{\ge 0}$$

$$\Rightarrow B[\phi(u),v]\le 0$$ □

$\boxed{\text{EX } 2.27}$ (1) $u_n \rightharpoonup u$ in $H^1(U)$ & $l=\liminf I[u_n]$

Using Banach-Steinhaus thm, we know $\{u_n\}$ is bounded in $H^1(U)$.

Then we can just apply ~~compactness~~ embedding thm.

(2) We may assume $u_{k_n}\to u$ a.e. and the assumption would not break the conclusion of (1)

By Egoroff's thm, $\exists\, E_\varepsilon$ that $\mathcal{L}^d(U\setminus E_\varepsilon)<\varepsilon$

We can assume $E_\varepsilon \subseteq E_{\varepsilon'}$ for $\forall\, 0<\varepsilon'<\varepsilon$

Let $F_\varepsilon = \{u \mid |u(x)| + |Du(x)| \le \frac{1}{\varepsilon}\}$ $\mathcal{L}^d(F_\varepsilon)\to\mathcal{L}^d(U)$ as $\varepsilon\to 0$

Let $G_\varepsilon = E_\varepsilon \cap F_\varepsilon$ and then we shall see $\exists\,\varepsilon_0$ sufficiently small s.t. $\forall\,\varepsilon<\varepsilon_0$ $\mathcal{L}^d(U\setminus G_\varepsilon)<\varepsilon$.

(3) $\liminf I[u_n] \ge I[u]$

$$l = \lim I[u_{n_k}] = \lim \int_U \frac{1}{2} a^{ij}\partial_i u_{n_k}\partial_j u_{n_k}$$

Ex 2.2.7

$a^{ij}\xi_i\xi_j$ is a convex function.

① $\phi(\xi) := a^{ij}\xi_i\xi_j$

Since $b\phi(\xi)+(1-b)\phi(\varsigma) = b\,a^{ij}\xi_i\xi_j+(1-b)a^{ij}\varsigma_i\varsigma_j$

$$\phi(b\xi+(1-b)\varsigma) = a^{ij}(b\xi_i+(1-b)\varsigma_i)(b\xi_j+(1-b)\varsigma_j)$$
$$= a^{ij}(b^2\xi_i\xi_j+(1-b)^2\varsigma_i\varsigma_j) + b(1-b)\varsigma_i\xi_j + b(1-b)\xi_i\varsigma_j)$$

Compute their difference

$$a^{ij}\left(\cancel{(b-b^2)}\, b(1-b)\xi_i\xi_j + b(1-b)\varsigma_i\varsigma_j - b(1-b)\varsigma_i\xi_j - b(1-b)\xi_i\varsigma_j\right)$$
$$= a^{ij}[\underbrace{(1-b)b}_{>0}](\xi_i-\varsigma_i)(\xi_j-\varsigma_j) \quad \checkmark \quad \Rightarrow \phi(\xi)\geq\phi(\eta)+\nabla\phi(\eta)\cdot(\xi-\eta)$$

② $I[u]\leq\liminf I[u_n]$

$$I[u_n] = \int\frac{1}{2}a^{ij}\partial_i u_n\partial_j u_n \geq \int\frac{1}{2}a^{ij}\partial_i u\partial_j u + \int\nabla\phi(\nabla u)\cdot(\nabla u_n-\nabla u)$$

where $(\nabla\phi(u))_k = \left(\frac{1}{2}a^{kj}\partial_j u+\frac{1}{2}a^{ik}\partial_i u\right)$

$$= I[u] + \int a^{kj}\partial_j u(\partial_k u_n - \partial_k u)$$

$L[v] = \int\frac{1}{2}a^{kj}\partial_j u\,\underline{\underline{\partial_k v}}$ is a linear functional on $H^1(U)$

$\Rightarrow u_n\rightharpoonup u \Rightarrow L[u_n]\longrightarrow L[u]$

$\Rightarrow \liminf I[u_n] \geq I[u]+2I[u]-2I[u] = I[u]$.

Rmk: I initially thought the ineq. is incorrect, but I made a mistake indeed since "weak convergence" is always about linear test while here I is actually non-linear same time.

A example is that $u_n(x)=\frac{\sin(2\pi nx)}{n}$ $\quad I[u]=\int_0^1|u_n'(x)|^2$.

KH

We need a result from Mazur, which states that, C is a convex ~~set~~ subspace $\Rightarrow$ C is weakly closed

If we have some geometric insights, we shall see that the theorem is a direct application of ~~Ban~~ Hahn-Banach theorem.

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$\boxed{\text{EX 2.2.7}}$ (4) $m = \inf_{u\in\mathcal{A}} I[u] \Rightarrow \exists\, u_n$ s.t. $I[u_n] \to m$

~~Fix $w\in\mathcal{A}$ $\Rightarrow u_n - w \in H^1_0(U)$ closed subspace +~~

$I[u_n] = \int \frac{1}{2} a^{ij} \partial_i u \partial_j u \geq \frac{\theta}{2}\int_U |\nabla u|^2\, dx$

$\Rightarrow \sup_n \|\nabla u_n\|_{L^2(U)} < \infty$

Fix $w \in \mathcal{A}$

$\Rightarrow \|u_n\|_{L^2(U)} \leq \|u_n - w\|_{L^2(U)} + \|w\|_{L^2(U)}$

$\overset{\text{Poincaré}}{\leq} \|\nabla u_n - \nabla w\|_{L^2(U)} + \|w\|_{L^2(U)} \overset{\text{uniformly}}{<\infty}$

$\Rightarrow \sup_n \|u_n\|_{H^1(U)} < \infty \Rightarrow u_{k_n} \rightharpoonup u$

Since H^1_0 is convex & closed in H^1 norm

By Mazur's thm $\Rightarrow u \in \mathcal{A}$

by (3) $I[u] \leq \liminf I[u] = m$; $u\in\mathcal{A}$ $I[u] \geq m$.

(5) From (3), we ~~at~~ have already seen that ϕ is strictly convex

So if u_1, u_2 are two different minimizers, we can obtain a new minimizer by $\frac{u_1+u_2}{2}$.

(6) ~~IEQ~~ Consider $I[u+\tau v]$ where $\tau\in\mathbb{R}$, $v \in C_c^\infty(U)$.

We note that we can regard it as a function of τ and it attains its maximum at $\tau = 0$, i.e. $i'(\tau) = (I[u+\tau v])' = 0$ at $\tau=0$

$$i'(\tau) = \frac{d}{d\tau}\left(\int_U \frac{1}{2} a^{ij} \partial_i(u+\tau v)\partial_j(u+\tau v)\, dx\right)$$

$$= \frac{d}{d\tau}\left(\int_U \frac{1}{2} a^{ij} \partial_i u \partial_j u + \tau(\partial_i v \partial_j u + \partial_i u \partial_j v) + \tau^2 \partial_i v \partial_j v\right)$$

$$= \int \frac{1}{2} a^{ij}(\partial_i v \partial_j u + \partial_i u \partial_j v) \quad \text{at } \tau = 0$$

$$= \int a^{ij} \partial_i u \partial_j v = 0 \quad \text{for } \forall v \in \boxed{C_c^\infty(U)} \checkmark$$

$\square$

$\boxed{\text{EX 2.3.1}}$ If not, $\exists f_k, u_k$ s.t.

$$\|u_k\|_{L^2(U)} > k\|f_k\|_{L^2(U)}$$

suppose $\|u_k\|_{L^2(U)} = 1 \Rightarrow f_k \to 0$ in $L^2(U)$

Energy estimate $\beta\|u\|_{H_0^1(U)} \le \underset{\underset{(f,u)_{L^2(U)}}{\parallel}}{B[u,u]} + \gamma\|u\|_{L^2(U)} \lesssim \|f\|_{L^2(U)} + \|u\|_{L^2(U)}$

$\Rightarrow \{u_k\}$ is bounded in $H_0^1(U)$

$\Rightarrow \begin{cases} u_{k_n} \rightharpoonup u \in H_0^1(U) \\ u_{k_n} \to u \text{ in } L^2 \text{ embedding} \end{cases} \Rightarrow \begin{cases} u \text{ is weak solution of } \begin{cases} Lu = \lambda u & u \\ u = 0 & \partial U \end{cases} \\ \lightning \text{ since } \|u\|_{L^2(U)} = 1 \end{cases}$ $\square$

$\boxed{\text{EX 2.4.1}}$ ~~$Lu = -\partial_j(a^{ij}\partial_i u)$ $\xrightarrow{\text{By Lax-Milgram}}$ $L^{-1} : L^2(U) \to H_0^1(U)$~~

~~By the meantime, $L^{-1} : L^2(U) \to H_0^1(U) \hookrightarrow\hookrightarrow L^2(U)$ $\Rightarrow L^{-1}$ is a compact operator on $L^2(U)$. By ~~Prop~~ Courant-Fisher minimax principle~~

~~$\mu_n^+ = \inf\limits_{E_{n-1}} \sup\limits_{x \in E_{n-1}^\perp} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$ $A = L^{-1}$ $\mu_1 \ge \cdots \ge 0$~~

~~$(n-1)$-dimensional~~

~~E_{n-1} is any closed subspace of L^2.~~

~~An observation is that $\langle L^{-1}u, u\rangle_{L^2(U)} = B[v, v]$ where $u = Lv$~~

Since L is symmetric, we can choose a sequence of eigenfunctions $\{w_k\}$ to be an orthonormal basis of $L^2(U)$ where $w_k \in H_0^1(U)$.

① Let $S_{k-1} = \text{span}\{w_1, \cdots w_{k-1}\}$ and it's of dimension $k-1$.

$u \in S_{k-1}^\perp \Rightarrow u = \sum\limits_{i=k}^{\infty} d^i w_i$ & $\sum d^{i^2} = 1$.

$B[u,u] = (Lu, u)_{L^2(U)} = \sum\limits_{i=k}^{\infty} \lambda_i d^{i^2} \ge \lambda_k.$

Note that if $u = w_i \Rightarrow (Lu, u)_{L^2(U)} = \lambda_k$.

$\Rightarrow \max\min (Lu, u) \ge \lambda_k$.

② If S is a $(n-1)$-dimensional space $\Rightarrow$ $S^\perp$'s minimal index $\le k$.

$\Rightarrow \min\limits_{S^\perp} B[u,u] \le \lambda_k$. $\square$

$\boxed{\text{EX 2.4.2}}$ We shall apply a theorem in Evans's book ~~Page~~ 6.5 thm 3 to ensure our process is reasonable.

Maybe we need some smooth conditions, ~~but~~ and life is fine.

① u is a solution to $Lu = \lambda_1 u$

$$\Rightarrow \quad \lambda_1 = \inf_x \frac{Lu}{u} \quad \Rightarrow \lambda_1 \le \sup_u \inf_x \frac{Lu}{u}.$$

② Claim $\lambda_1 \ge \sup_{u} \inf_x \frac{Lu}{u}$

Let ψ be ~~the~~ corresponding eigenfunction ~~to~~ λ_1 for L^*

$$\Rightarrow \cancel{Lu,\psi} \,(Lu - \lambda_1 u, \psi) = 0$$

Note that $\psi > 0 \Rightarrow \exists x_0$ s.t. $Lu - \lambda_1 u \le 0$ where $u > 0$

$$\Rightarrow \lambda \ge \frac{Lu(x_0)}{u(x_0)} \Rightarrow \lambda \ge \inf \frac{Lu(x_0)}{u(x_0)} \quad \text{for } \forall u \in C^\infty(\bar{U}),\ u > 0$$

$\square$

$\boxed{\text{EX 2.4.3}}$ $-A$ is symmetric $\xRightarrow{\text{Principal eigenvalue}}$ $\lambda(\tau) = \int_{U(\tau)} |\nabla w|^2 dx$

$$\lambda'(\tau) = \underline{\int_{U(\tau)} \frac{\partial}{\partial \tau} (|\nabla w|^2)\, dx} + \int_{\partial U(\tau)} |\nabla w|^2 v \cdot N \, dS_x \quad \text{(From Evans's C.4)}$$

$$\int_{U(\tau)} \frac{\partial}{\partial \tau} (|\nabla w|^2) dx = 2\int_{U(\tau)} \nabla w \cdot \nabla\left(\frac{\partial w}{\partial \tau}\right) dx$$

$w = 0 \Rightarrow \nabla w$ is paralleled to N, $\nabla w|_{\partial U} = \nabla w \cdot N$

Note the $\int_{U(\tau)} w(x,\tau)^2\, dx = 1$

$$\xRightarrow{\frac{d}{d\tau}} 0 = \underbrace{\int_{U(\tau)} 2 w(x,\tau) \cdot \partial_\tau w(x,\tau)\, dx}_{=0} + \int_{\partial U(\tau)} \underset{0}{w(x,\tau)^2}\, v \cdot n\, dS_x$$

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$$= -\int_{U(\tau)} \underset{\lambda w}{\boxed{\Delta w}} \frac{\partial w}{\partial \tau} dx + \cancel{\int_{\partial U} \frac{\partial w}{\partial \tau} (\nabla w \cdot n)\, dS_x}$$

$$= \cancel{\int_{U(\tau)} \lambda_1 w \frac{\partial w}{\partial \tau} + \int_{\partial U} \frac{\partial w}{\partial \tau} (\nabla w \cdot n)\, dS_x}$$

$$= 2\int_{\partial U(\tau)} \underset{0}{w}\, \nabla\left(\frac{\partial w}{\partial \tau}\right) n \, dS_x - 2\int_{U(\tau)} w \frac{d}{d\tau} (\Delta w)\, dx$$

$$= 2\int_{U(\tau)} w \frac{d}{d\tau} (\underset{\lambda_1 w}{-\Delta w})\, dx = 2\lambda_1' \underbrace{\int_{U(\tau)} w^2}_{=1} + 2\int_U w\, \partial_\tau w = 2\lambda'$$

$\square$