$\boxed{\text{EX2.4.4}}$ (1) $\sup_n \|u_n\|_{H_0^1} < \infty \Rightarrow u_{n_k} \rightharpoonup u \in H_0^1(U)$

Compactness $u_{n_k} \to u$ in L^2, a.e.

Egoroff $u_{n_k} \overset{a.un.}{\rightrightarrows} u$ in U_ε

$\phi(u) = |\nabla u|^2 = (\partial_i u)^2$

$\phi'(u) = 2 \ldots$

$I[u_{n_k}] = \int |\nabla u_{n_k}|^2 dx \geq \int_{U_\varepsilon} |\nabla u_{n_k}|^2$

$\overset{\text{convex}}{\geq} \int_{U_\varepsilon} \cancel{\nabla u_{n_k} - \nabla u} \, |\nabla u|^2 + \underbrace{2 \nabla u \cdot \nabla(u_k - u)}_{\to 0} \, dx$

$\overset{\text{weak}}{\Rightarrow} \int_{U_\varepsilon} |\nabla u|^2 \qquad \varepsilon \to 0^+$

$\Rightarrow m \geq I[u]$.

(2) In (1). $u \in H_0^1$

(3) $\frac{\partial j}{\partial \sigma} = \int 2(u + \tau v + \sigma w) w \quad . (\tau, \sigma) = (0,0)$

$\Rightarrow \left.\frac{\partial j}{\partial \sigma}\right|_{(\tau,\sigma)=(0,0)} = 2\int u w \neq 0$

$\Rightarrow$ Implicity thm $\exists \phi \in C^1$ s.t. $\sigma = \phi(\tau)$ with $\phi(0) = 0$.

~~$0 = \left.\frac{\partial j}{\partial \tau}\right|_{\tau=0} = 2\int uv +$~~

$0 = j(\tau, \phi(\tau)) \Rightarrow 0 = \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial \sigma} \phi'(\tau)$

$\Rightarrow \phi'(0) = -\frac{\frac{\partial j}{\partial \tau}}{\frac{\partial j}{\partial \sigma}} = -\frac{\int uv}{\int uw}$ ✓

(4) $w(\tau) = \tau v + \phi(\tau) w$

~~$i(\tau) = I[u + w(\tau)] = I[u] \Rightarrow \boxed{i'(\tau) = 0}$~~

~~$\Rightarrow 2\int (u + \tau v + \phi(\tau) w)\left(v - \frac{\int uv}{\int uw} w\right) = 0$~~

$i(\tau) = I[u + w(\tau)]$. We know $i'(0) = 0$ from (1)(2).

$\Rightarrow 0 = i'(0) = \frac{d}{d\tau} \int_U |\nabla(u + w(\tau))|^2 = \int_U \nabla u \nabla v + \phi'(0) \nabla u \nabla w$

$$\int_U \nabla u \nabla v = \underbrace{\left(-\frac{\int \nabla u \nabla w}{\int u w}\right)}_{\lambda_1} \int uv$$

(5) Note that $\lambda = \dfrac{\int |\nabla u|^2}{\int u^2}$ & Courant principle. $\square$

$\boxed{\text{EX 2.5.1}}$ Note $|c(u)| \le M|u|$ for $\forall x$ from gradient estimate

$$\underset{\displaystyle \int \nabla u \cdot \nabla v.}{\int_U -\Delta u\, v \overset{\|}{}} = \int_U \tilde{f} v = \int_U (f - c(u)) v$$

Let $v = D_k^{-h} D_k^h u$

$$\Rightarrow \int_U |D_k^h \nabla u|^2 \le \int_U (|f| + M|u|)|v| = \int_U (|f| + M|u|) |D_k^{-h} D_k^h v|$$

$$\le C_\varepsilon \left(\int_U |f|^2 + |u|^2 \right) + \underbrace{\varepsilon \int |D_k^h \nabla u|^2}_{\text{Absorbed}}$$

$\square$

$h \to 0^+$ Done.

$\boxed{\text{EX 2.5.2}}$ ~~$\partial_1^2 \left(\sqrt{\ln \sqrt{x_1^2 + x_2^2}}\, (x_1^2 - x_2^2) \right)$~~

~~$= \partial_1 \left(2x_1 \sqrt{\ln \sqrt{x_1^2+x_2^2}} + (x_1^2 - x_2^2) \dfrac{1}{2 \sqrt{\frac{1}{2}\ln (x_1^2+x_2^2)}} \cdot \dfrac{x_1}{(x_1^2+x_2^2)} \right)$~~

Tedious computation.

$\boxed{\text{EX 2.5.3}}$ $f(x) = \begin{cases} 0 & [0, \frac{1}{2}] \\ 1 & (\frac{1}{2}, 1]. \end{cases}$ $\square$

$\boxed{\text{EX 2.5.4}}$ See the process in my note. $\square$

$\boxed{\text{EX 2.6.1}}$ consider $v = |\nabla u|^2 + \lambda u^2$

$$L|\nabla u|^2 = -a^{kl} \partial_k \partial_l (\partial_i u)^2$$

$$= -a^{kl} \partial_k (2 \partial_i u\, \partial_l \partial_i u)$$

$$= -a^{kl} (2 \partial_k \partial_i u\, \partial_l \partial_i u + 2 \partial_i u\, \partial_k \partial_l \partial_i u)$$

$$= -2a^{kl} \partial_k \partial_i u\, \partial_l \partial_i u - \sum_i 2 \partial_i u \left(a^{kl} \partial_k \partial_l \cancel{\partial_i}\, u \right)$$

$$= \underbrace{-2a^{kl}\partial_k\partial_i u \, \partial_l \partial_i u}_{-} + \underbrace{2\sum_i \partial_i u \left(\partial_i a^{kl} \partial_k \partial_l u\right)}_{?}$$

$$Lu^2 = -a^{kl}\partial_k\partial_l(u^2) = -a^{kl}\partial_k(2u\,\partial_l u)$$

$$= -a^{kl}(\underbrace{2\partial_k u \partial_l u}_{-} + \underbrace{2u\,\partial_k\partial_l u}_{?=0})$$

$$= -2a^{kl}\partial_k u\, \partial_l u$$

For $L(|\nabla u|^2)$, we shall establish an estimate for the second term.

$$2\sum \partial_i u \,\partial_i a^{kl}\partial_k\partial_l u \le C|\nabla u||\partial^2 u| \le \varepsilon|\partial^2 u|^2 + C_\varepsilon|\nabla u|^2$$

$$\Rightarrow L(|\nabla u|^2 + \lambda u^2) \le -C|\partial^2 u|^2 + \varepsilon|\partial^2 u|^2 + C_\varepsilon|\nabla u|^2 - \lambda\, \theta|\nabla u|^2 \le 0$$

when λ is sufficiently large.

$$\Rightarrow \max_{\partial u} |\nabla u|^2 + \lambda u^2 = \max_u (|\nabla u| + \lambda u^2)$$

$\square$

~~$\Rightarrow \max_{\partial u} \frac{1}{\lambda}|\nabla u|^2 + u^2 \ge \max_u \frac{1}{\lambda}|\nabla u|^2 + u^2 \quad \lambda \to \infty$~~

~~$\Rightarrow \max_{\partial u} |\nabla u|^2 + u^2 \ge \max u^2$~~

$\boxed{\text{Ex 2.6.2}}$ Since $u|_{\partial u} = 0 \Rightarrow \nabla u$ is pararelled to N

$\|f\|_{L^\infty} = M$

Consider $L(\underbrace{Mw - u}_{h}) \ge M - f \ge 0$

$h|_{\partial u} \ge 0 \Rightarrow x^0$ is minimal point

$$\Rightarrow \frac{\partial h}{\partial N} < 0 \Rightarrow M\frac{\partial w}{\partial N} < \frac{\partial u}{\partial N}$$

$$L(\underbrace{Mw + u}_{\bar h}) \ge M + f \ge 0$$

$\bar h|_{\partial u} \ge 0 \quad \cdots \quad M\frac{\partial w}{\partial N} < -\frac{\partial u}{\partial N}$

Note that $L(-w) \le -1 \Rightarrow \frac{\partial(-w)}{\partial N} > 0 \Rightarrow \left|\frac{\partial u}{\partial N}\right| \le M\left|\frac{\partial w}{\partial N}\right|$

$\square$

Ex 2.6.3

$w = \frac{u}{v}$

$\cancel{\tilde{L}w}$ $\quad \partial_i\partial_j w = \partial_j\left(\frac{\partial_j u\, v - u\,\partial_j v}{v^2}\right)$

$$= \frac{\left[\left(\partial_i\partial_j u\, v + \boxed{\partial_j u\,\partial_i v} - \boxed{\partial_i u\,\partial_j v} - u\,\partial_i\partial_j v\right)v^2 - \left(\partial_j u\, v - u\,\partial_j v\right) 2v\,\partial_j v\right]}{v^4}$$

$$\Rightarrow -a^{ij}\partial_i\partial_j w = \frac{-a^{ij}\left[\left(\partial_i\partial_j u\, v - u\,\partial_i\partial_j v\right)v^2 - \left(\partial_j u\, v - u\,\partial_j v\right)2v\,\partial_i v\right]}{v^4}$$

$$= Lu\,\frac{1}{v} - u\,Lv\,\frac{1}{v} - \frac{2}{v^2}\left(-a^{ij}\partial_i v\,\partial_j u\right) + \frac{2}{v^2}\left(-a^{ij}\partial_i v\,\partial_j v\right)$$

$\partial_i w = \frac{\partial_i u\, v - u\,\partial_i v}{v^2}$

$w = \frac{u}{v}$

$$-a^{ij}\partial_i\partial_j w = -a^{ij}\partial_i\left(\frac{\partial_j u\, v - u\,\partial_j v}{v^2}\right)$$

$$= -a^{ij}\left(\frac{\left(\partial_i\partial_j u\, v + \partial_j u\,\partial_i v - \partial_i u\,\partial_j v - u\,\partial_i\partial_j v\right)v^2 - \left(\partial_j u\, v - u\,\partial_j v\right)2v\,\partial_i v}{v^4}\right)$$

$$= -a^{ij}\partial_i\partial_j u\,\frac{1}{v} + a^{ij}\partial_i\partial_j v\,\frac{u}{v^2} + 2a^{ij}\partial_j u\,\partial_i v\,\frac{1}{v^2} - 2a^{ij}u\,\partial_i v\,\partial_j v\,\frac{1}{v^3}$$

$$\partial_i w\left(b^i - a^{ij}\partial_j v\,\frac{2}{v}\right) = \frac{\partial_i u\, v - u\,\partial_i v}{v^2}\left(b^i - a^{ij}\partial_j v\,\frac{2}{v}\right)$$

$$= b^i\partial_i u\,\frac{1}{v} - b^i\partial_i v\,\frac{u}{v^2} - a^{ij}\partial_i u\,\partial_j v\,\frac{2}{v^2} + 2a^{ij}\partial_i v\,\partial_j v\,\frac{u}{v^3}$$

$$\Rightarrow \tilde{L}w = -a^{ij}\partial_i\partial_j w + \left(b^i - a^{ij}\partial_j v\,\frac{2}{v}\right)\partial_i w$$

$$= \left(-a^{ij}\partial_i\partial_j u + b^i\partial_i u\right)\frac{1}{v} - \left(-a^{ij}\partial_i\partial_j v + b^i\partial_i v\right)\frac{u}{v^2}$$

$$= (Lu - cu)\frac{1}{v} - (Lv - cv)\frac{u}{v^2} = \underbrace{\frac{Lu}{v}}_{\le 0} - Lv\,\frac{u}{v^2} \le 0 \text{ on } \Omega$$

$$\Omega = \{u > 0\}$$

$$\Rightarrow 0 < \sup_{\Omega} w = \sup_{\partial\Omega}\frac{u}{v} = 0 \quad ↯$$

□

$\boxed{\text{Ex 2.6.4}}$

define $u(0) := \frac{1}{|\partial B_\varepsilon|}\int_{\partial B_\varepsilon} u(x)\,dx$

$\boxed{d \ge 3}$ $\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} u(x)\,dx \approx \frac{1}{|B_\varepsilon|}\int_0^\varepsilon \int_{\partial B_r} \frac{1}{|r|^{2-d}}\,dS_x\,dr$

$\approx \frac{1}{\varepsilon^d}\int_0^\varepsilon r^{d-1+(2-d)}$

$\approx$

$\int_{B_r(0)} \Delta u\,dx = \int_{\partial B_r} \nabla u \cdot n\,dS_x \approx \int (2-d)\,r^{1-d}\,|S_r|$

$= (2-d)\,\omega_{d-1}$

$\Rightarrow \Delta u(0) = -(d-2)\,\omega_{d-1}\,\delta_0$

$u(0) := \lim_{r \to 0} \frac{1}{|B_r|}\int_{B_r} u\,dx$

$\frac{1}{|B_r|}\int_{B_r} u\,dx = \frac{1}{|B_r|}\int_0^r dt \int_{\partial B_t} o(|t|^{2-d})\,dS_x$

$= \frac{1}{|B_r|}\int_0^r \omega_{d-1}\, o(|t|)$

$= \frac{\frac{\omega_{d-1}}{2}\, o(r^2)}{\omega_d\, r^d}$

We needn't define it precisely.

Fundamental Solution

$\Downarrow$

$$\begin{cases} -\Delta v = 0 & B_r \\ v = u & \partial B_r \end{cases} \xRightarrow[\text{regularity}]{\text{existence}} -\Delta(u - v + \varepsilon\Phi) = 0 \text{ in } \check{B}$$

$\Rightarrow u - v + \varepsilon\Phi \le 0 \qquad \varepsilon \to 0$

$\Rightarrow u \le v$

Similarly, $v \le u$ $\square$

$\boxed{\text{Ex 2.6.5}}$ (1) $\frac{\partial x_j^*}{\partial_i x} = \partial_i\left(\frac{x_j}{|x|^2}\right) = \frac{\delta_{ij}|x|^2 - x_j(2x_i)}{|x|^4}$

$\nabla(x^*) = \left(\partial_j x_i^*\right)_{ij} = \left(\frac{\delta_{ij}}{|x|^2}\right)_{ij} - 2\left(\frac{x_i x_j}{|x|^4}\right)_{ij}$

$= \frac{1}{|x|^2} I - \frac{2}{|x|^4}\left(\frac{x_i x_j}{|x|^4}\right)_{ij}$

$\cancel{\nabla(x^*)[\nabla(x^*)]^T} =$

(1) $\nabla(x^*) = \left(\frac{\partial x_j^*}{\partial x_i}\right)_{ij} = \frac{1}{|x|^2} I - \frac{2}{|x|^4}(x_i x_j)_{ij} = \frac{1}{|x|^2} I - \frac{2}{|x|^4} x^T x$

$(x x^T = |x|^2)$ ↑ row

$$(\nabla x^*)(\nabla x^*)^T = \left(\frac{1}{|x|^2} I - \frac{2}{|x|^4} x^T x\right)^2$$
$$= \frac{1}{|x|^4} I - \frac{4}{|x|^6} x^T x + \frac{4}{|x|^8} x^T \underset{\sim}{x x^T} x = \frac{I}{|x|^4}.$$

(2) $\cancel{\Delta x^* = \partial_i^2 \left[(\partial_1^2 + \cdots + \partial_d^2) x_i^* \right]_i}$

$\cancel{= \nabla \cdot (\nabla x^*) = \left(\frac{1}{|x^*|^2} x^* x^{*T}\right) \nabla(\nabla x^*) = \frac{x^*}{|x^*|^2} \underbrace{(\nabla x^*)^T (\nabla x^*)}}$

$\cancel{\nabla = (\partial_1 \cdots \partial_d)}$ $\cancel{= \frac{x}{|x|^2} |x|^2 |x|^{-4} I_d = \frac{x}{|x|^2}}$?

computation by ~~brute~~.

$$\partial_i^2 x_j^* = \partial_i\left(\frac{\delta_{ij}}{|x|^2} - \frac{2 x_i x_j}{|x|^4}\right) = -\frac{\delta_{ij}}{|x|^4}(2x_i) - \left[\frac{(2x_j + 2x_i\delta_{ij})|x|^4 - 2x_i x_j \cdot 2|x|^2 \cdot 2x_i}{|x|^8}\right]$$
$$= -\frac{\delta_{ij}}{|x|^4}(2x_i) - \frac{2x_j + 2x_i\delta_{ij}}{|x|^4} + \frac{8 x_i^2 x_j |x|^2}{|x|^8}$$

$$\Delta x_j^* = -\frac{2x_j}{|x|^4} - \frac{2d x_j + 2x_j}{|x|^4} + \frac{8|x|^4 \cdot x_j}{|x|^8}$$
$$= \frac{(4-2d) x_j}{|x|^4}$$

$$\Rightarrow \Delta x^* = 2(2-d)\frac{x}{|x|^4}$$

(3) $\Delta(\mathcal{K}u(x)) = \Delta\left(u(x^*)|x^*|^{d-2}\right) = \partial_i^2\left[u(x^*)|x^*|^{d-2}\right]$

$$= \underbrace{\partial_i^2(u(x^*))}\ + 2\underline{\partial_i(u(x^*))}\,\partial_i\left(|x^*|^{d-2}\right) + \partial_i^2\left(|x^*|^{d-2}\right) (\ )$$

$\cancel{\Rightarrow \Delta(\mathcal{K}u(x)) = 2\nabla(u(x^*)) \cdot \nabla\left(|x|^{2-d}\right) + \Delta\left(|x|^{2-d}\right) u)}$ $\underset{0}{\|}$

$\cancel{= 2\nabla(u(x^*)) \cdot \left(-d\frac{x}{|x|^{d+2}}\right)}$

$$= \left[\frac{1}{|x|^4}\underline{\Delta_{x^*} u} + \frac{2(2-d)}{|x|^6}(\nabla_{x^*} u \cdot x)\right]|x|^{2-d} + 2\nabla_{x^*} u \cdot (\nabla x^*)(2-d)\frac{x}{|x|^d}$$

$$= |x|^{-d-2}\overset{0}{\Delta_{x^*} u} = 0$$

(4) $\Delta u = 0$ when $|x| \geq 1$ $\Rightarrow \Delta(\mathcal{K}u(x)) = 0$ when $0 < |x| \leq 1$

□

$\boxed{\text{EX2.6.6}}$ ~~Note that $\Delta \ln|x| = 0$ in the punctured plane.~~

$$\cancel{\begin{cases} \Delta(u - \ln|x|) = 0 & \text{on } \mathbb{R}^n \setminus B_1 \\ u - \ln|x| = 0 & \text{on } \partial B_1 \\ \lim_{|x|\to\infty} \frac{u}{\ln|x|} = 0 \end{cases} \xRightarrow{\text{Max}} u \le \ln|x|}$$

(1) By $\boxed{\text{Ex 2.6.4}}$ & $\boxed{\text{Ex 2.6.5}}$, we can see that $Ru(x)$ is a harmonic function in the unit disk ~~with ze~~.

u is of zero boundary condition. $\Rightarrow u \equiv 0$.

(2) $u = \frac{1}{|x|} - 1$ $\boxed{d=3}$

(3) ~~$Ru(x) = \frac{u\left(\frac{x}{|x|^2}\right)}{|x|^{d-2}}$~~ ~~$\frac{u(x^*)}{|x^*|^{2-d}} \to 0$ as $x^* \to 0$~~ $\square$

~~$u(x^*) \to 0$ as $x^* \to 0$~~

$$\frac{Ru(x)}{|x|^{2-d}} = u\left(\frac{x}{|x|^2}\right) \to 0 \quad \text{as } |x| \to 0$$

$\Rightarrow Ru(x) \equiv 0$ in the unit disk $\Rightarrow u$ $\square$

$\boxed{\text{EX2.6.7}}$ $u(x) = \sqrt{-\ln|x|}\,(x_1^2 - x_2^2)$

v is a C^2 solution to $\Delta u = (\quad)$

$\Rightarrow w = u - v$ is a solution to $\Delta w = 0$ in $\mathring{B}_R$

Bdd $\Rightarrow \Delta w = 0$ (removable singularity, similar to Complex analysis)

$\Rightarrow w = 0$ ↯ $\square$

$\boxed{\text{EX 2.6.8}}$ (1) $\Delta(|\nabla u|^2) = \partial_i^2 (\partial_j u)^2 =$ ~~$\partial_i(2\partial_j u\, \partial_i u)$~~ $\partial_i(2\partial_j u\, \partial_i\partial_j u)$

~~$= 2\partial_j\partial_i u + 2\partial_i^2 u$~~

~~$= 2\underbrace{\partial_i\partial_j u}_{\text{elliptic}}$~~ $= 2(\partial_i\partial_j u)^2 \ge 0$

$\Rightarrow \overset{-A}{\overset{\shortparallel}{L}}(|\nabla u|^2) \le 0 \Rightarrow |\nabla u|^2$ attains its maximum at the boundary

(2)

(2) $$\Delta(\varphi|\nabla u|^2) = \Delta\varphi\,|\nabla u|^2 + 2\nabla\varphi\cdot\nabla(|\nabla u|^2) + \varphi\,\Delta(|\nabla u|^2)$$

$\varphi \in C_c^2(B_1)$ $$= \Delta\varphi\,|\nabla u|^2 + 4\partial_i\varphi\,\partial_i\partial_j u\,\partial_j u + 2\varphi(\partial_i\partial_j u)^2$$

$$2\partial_i\varphi\,\partial_i\partial_j u\,\partial_j u + \varphi(\partial_i\partial_j u)^2 \overset{\text{Young}}{\geq} -\varepsilon|\partial_i\varphi\,\partial_i\partial_j u|^2 - C_\varepsilon|\nabla u|^2 + \varphi(\partial_i\partial_j u)^2$$

$$\geq \left[\varphi - \varepsilon\|\nabla\varphi\|_{L^\infty}^2\right]|\partial_i\partial_j u|^2 - C_\varepsilon|\nabla u|^2$$

choose φ to make sure $\varphi - \varepsilon\|\nabla\varphi\|_{L^\infty}^2 \geq 0$ in B_1

$$\Rightarrow \exists C' \text{ s.t. } \Delta(\varphi|\nabla u|^2) \geq -C'|\nabla u|^2$$

$$\Rightarrow L[\varphi|\nabla u|^2] \leq C'|\nabla u|^2$$

Note that $Lu^2 = -\Delta(u^2) = -\partial_i^2 u^2 = -\partial_i(2u\,\partial_i u) = -2(\partial_i u)^2$

$$\Rightarrow L(\varphi|\nabla u|^2 + \lambda u^2) \leq 0$$

$$\Rightarrow \max_{B_1} \varphi|\nabla u|^2 + \lambda u^2 \leq \max_{\partial B_1} \varphi|\nabla u|^2 + \lambda u^2 = \max_{\partial B_1} \lambda u^2.$$

~~Let $\varphi \in C_c^2$ $\Rightarrow$ $\max_{B_{\frac12}} \varphi|\nabla u|^2 \leq \lambda \min_{B_{\frac12}} u^2 \leq C \max_{\partial B_1} u^2$~~

$\operatorname{supp}\varphi \supseteq B_{\frac12}$

$$\max_{B_{\frac12}} |\nabla u|^2 \leq \lambda \max_{B_1} u^2.$$

□.

$\boxed{\text{Ex3.1.1}}$ See my note.

$\boxed{\text{Ex3.1.2}}$ See my note.

$\boxed{\text{EX3.2.1}}$ ~~$\int d_j^m \nabla w_j \nabla w_k = \int f w_k$ $\quad j=1,\dots,m$~~

~~$\Rightarrow d_j^m = \dfrac{\int f w_k}{\int \nabla w_j \nabla w_k}$~~

~~To show the conclusion, it suffices to show $\sup_m \sum d_k^{m\,2} < \infty$~~

$\boxed{\text{EX 3.2.1}}$

$$\int \nabla u_m \cdot \nabla w_k = \int f w_k$$

$$\Rightarrow \int |\nabla u_m|^2 = \int f u_m$$

$$\overset{\text{Poincáre}}{\Rightarrow} \quad \|u_m\|_{H_0^1}^2 \le C \|f\| \|u_m\|_{H_0^1}$$

$$\Rightarrow \quad \|u_m\|_{H_0^1} < \infty$$

$$\overset{\text{Banach-Alaoglu}}{\Rightarrow} \quad u_{m_k} \rightharpoonup u \text{ in } H_0^1$$

$$\Rightarrow \int \nabla u_{m_k} \nabla w_k = \int f w_k$$

$$\Rightarrow \int \nabla u \, \nabla w_k = \int f w_k \qquad \square$$

$\boxed{\text{EX 3.2.2}}$ Take u as the multiplier and compute the energy integral.

$$\underset{\frac{d}{dt}\left(\frac{1}{2}u^2\right)}{\overset{\|}{u \partial_t u}} - u \Delta u = 0$$

$$\Rightarrow \frac{d}{dt} \int_U \frac{1}{2} u^2 + \int |\nabla u|^2 = 0$$

$$\overset{\int_0^t}{\Rightarrow} \quad \int \frac{1}{2} u^2(t,\cdot)\, dx - \int \frac{1}{2} g^2 + \int_0^t \int_U \underline{\underline{|\nabla u|^2}} = 0$$

$\boxed{\int |\nabla u|^2 \ge \lambda_1 \int u^2}$ principal eigenvalue

$$\Rightarrow y = \int u^2$$

$$y' = -2 \int |\nabla u|^2 \le -2\lambda_1 y$$

$$\Rightarrow y \le e^{-2\lambda_1 t} \|g\|_{L^2}^2$$

$$\Rightarrow \|u\|_{L^2} \le e^{-\lambda_1 t} \|g\|_{L^2} \qquad \square$$

$\boxed{\text{EX 3.3.1}}$ See my note. $\square$

$\boxed{EX\ 3.3.2}$ To simplify the process, we only need to estimate the term $\cancel{\|u_m(t)\|^2_{H^2(U)}}$ $L^\infty_t H^2_x$

$B[u_m, w_k] = (f - u_m', w_k)$ where $\{w_k\}$ is the family of eigenfunctions of L

$\Rightarrow B[u_m, Lu_m] = (f - u_m', Lu_m)$ $\quad Lu_m \in \mathrm{Span}\{w_1, \cdots w_m\}$

$\Rightarrow B[u_m, Lu_m] = (Lu_m, Lu_m) = \|Lu_m\|^2_{L^2(U)}$

~~Elliptic~~ $\cancel{\leq}$

$$\overset{\text{Elliptic}}{\Rightarrow} \|u_m\|_{H^2(U)} \leq C\left(\|Lu_m\|^2_{L^2(U)} + \|u_m\|^2_{L^2}\right)$$
$$\leq C\left(\|f\|^2_{L^2} + \|u_m'\|^2_{L^2} + \|u_m\|^2_{L^2}\right) \qquad \square$$

$\boxed{EX\ 3.4.1}$ consider $v = e^{\gamma t} u$

$$\Rightarrow \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0$$

$$\overset{\text{weak}}{\Rightarrow} \max_{\bar{U}_T} |v| = \max_{\bar{U}} |g| \leq A$$

$$\Rightarrow |u| \leq A e^{-\gamma t}. \qquad \square$$

~~$\boxed{EX\ 3.4.2}$~~ (2) $v = e^{-\gamma t} u \Rightarrow$

$$\partial_t v - \Delta v + \underbrace{(c+\gamma)}_{\gamma \text{ large } \geq 0} u = 0 \Rightarrow$$

$$\min_{\bar{U}_T} v \geq \min_{\Gamma_T} v = 0 \Rightarrow u \geq 0 \qquad \square$$

$c_i = (g, w_i)$

$\boxed{EX\ 3.4.2}$ $u(t,x) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} \cancel{c_k(t)} w_k(x) = c_1 e^{-\lambda_1 t} \underbrace{w_1(x)}_{\text{smooth}} + v(t,x)$

$$\partial_t\left(c_1 e^{-\lambda_1 t} w_1(x) + v(t,x)\right) - \Delta\left(c_1 e^{-\lambda_1 t} w_1(x) + v(t,x)\right) = 0$$

$$\Rightarrow (\partial_t v - \Delta v) - \underline{\lambda_1 c_1 e^{-\lambda_1 t} w_1(x) - c_1 e^{-\lambda_1 t} \Delta w_1(x)} = 0$$

$$\Rightarrow \begin{cases} \partial_t v - \Delta v = 2\lambda_1 c_1 e^{-\lambda_1 t} w_1(x) \\ v = g - c_1 w \end{cases}$$

~~$\|v(t/2)\|_{L^\infty} \leq \| S(t/2) v$~~

Claim:
$$\|V(t,\cdot)\|_{L^\infty(U)} \le C(t/2)^{-d/4} e^{-\lambda_2 t/2} \|g - c_1 w_1\|_{L^2(U)}$$

proof: ① $\|V(s,\cdot)\|_{L^2(U)} = \| S(s)(g - c_1 w_1)\|_{L^2(U)}$

$$\le e^{-\lambda_2 s} \|g - c_1 w_1\|_{L^2(U)}$$

Here the heat eq has a source term but it matters a little.

$s = t/2$

$$\Rightarrow \|V(t/2,\cdot)\|_{L^2(U)} \le e^{-\lambda_2 t/2} \|g - c w_1\| L^2(U)$$

② $\|S(\tau)\|_{L^2 \to L^\infty} \le C\tau^{-d/4}$

$-\frac{d}{4}$ is the best coefficient, ~~without~~ but is not eas here using spectral method we can obtain a weak bound t^{-k} $k \ge \frac{d}{4}$ easier

③ $V(t) = S(t/2) V(t/2)$

$$\Rightarrow \|V(t,\cdot)\|_{L^\infty(U)} \le \|S(t/2)\|_{L^2 \to L^\infty} \|V(t/2)\|_{L^2}$$

$$\le c(t/2)^{-d/4} e^{-\lambda_2 t/2} \|g - c_1 w_1\|_{L^2(U)}$$

Maybe the time cut is not just suitable but the tech is all the same.

$\Rightarrow$ u attains exp decay at infty. □

$\boxed{\text{EX } 3.4.3}$
$\boxed{\text{EX } 3.4.4}$
$\boxed{\text{EX } 4.1.1}$
} Omit

$\boxed{\text{EX 4.1.2}}$ Multiply u_t to the equation.

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\int (\partial_t u)^2 + |\nabla u|^2\right] + \underset{\ge 0}{D}\underset{\le 0}{\int}(\partial_t u)^2 = 0$$

By Gronwall ineq. we can ~~con~~ get the desired conclusion. □

$\boxed{\text{EX 4.2.1}}$ $Eq_r = \partial_t^2\varphi - a^{ij}\partial_i\partial_j\varphi = 0$

$$\varphi := A(t,x)\exp\left(i P(t,x)\,\varepsilon^{-1}\right)$$

$$\partial_t\varphi = A_t \exp(ip\varepsilon^{-1}) + A\exp(ip(t,x)\varepsilon^{-1})\, i\varepsilon^{-1}p_t$$

$$\partial_t^2\varphi = A_{tt}\exp(ip\varepsilon^{-1}) + 2A_t\exp(ip\varepsilon^{-1})\, i\varepsilon^{-1}p_t + A\left(\exp(ip\varepsilon^{-1})\left(\underline{-\varepsilon^{-2}p_t^2}\right) + \exp(ip\varepsilon^{-1})\, i\varepsilon^{-1}p_{tt}\right)$$

$$\partial_i\varphi = (A_i + Ai\varepsilon^{-1}p_i)\exp(ip\varepsilon^{-1})$$

$$\partial_i\partial_j\varphi = \left(A_{ij} + Ai\varepsilon^{-1}p_{ij} + 2iA_ip_j\varepsilon^{-1} - \underline{Ap_ip_j}\,\varepsilon^{-2}\right)\exp(ip\varepsilon^{-1})$$

classify by the order of ε. $\Rightarrow$ $-Ap_t^2 + a^{ij}Ap_ip_j = A\left(-p_t^2 + a^{ij}p_ip_j\right) = 0$ □

$\boxed{\text{EX 4.2.2}}$ Multiply φ_t

$$\Rightarrow \varphi_t\varphi_{tt} - \varphi_t\Delta\varphi + \varphi_t f(\varphi) = 0$$

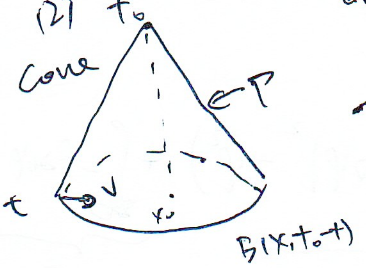
$$\overset{\text{int}}{\Rightarrow} \frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}^d}(\varphi_t)^2 + |\nabla\varphi|^2\right] + \int \underset{\substack{\| \\ F'}}{f(\varphi)} = \oint 0 \quad (\varphi \to 0 \text{ as } |x|\to\infty)$$

Let $F(u) = \int_0^u f(s)\,ds$

$\Rightarrow E(t) = \frac{1}{2}\int \varphi_t^2 + |\nabla\varphi|^2 + F(\varphi)\,dx$ is conversed.

~~$\frac{d}{dt}e(t) = \int_B \partial_t\varphi\,\partial_t^2\varphi + \nabla\varphi\cdot\nabla(\partial_t\varphi) + f(\varphi)\,dx + \int_{\partial B}\left[\frac{1}{2}\left((\partial_t\varphi)^2 + |\nabla\varphi|^2\right) + F(\varphi)\right](-v)\,dS_x$~~

(2) cone



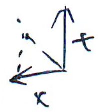
$$e(t) = \int_B \frac12\left(|\partial_t\varphi|^2 + |\nabla\varphi|^2\right) + F(\varphi)$$

$$e'(t) = \int_B \frac{d}{dt}\left[\frac12|\partial\varphi|^2 + \frac12|\nabla\varphi|^2 + F(\varphi)\right] - \int_{\partial B} \frac12|\partial_t\varphi|^2 + \frac12|\nabla\varphi|^2 + F(\varphi)$$

$$= \int_{\partial B} \partial_t\varphi\, \partial_\nu\varphi - \frac12(\partial_t\varphi)^2 - \frac12|\nabla\varphi|^2 - F(\varphi)$$

$$\Rightarrow \underset{0}{\underset{\shortparallel}{e(t_0)}} - e(0) = \int_T (\quad)$$

$$\Rightarrow e(0) = C\int_\Gamma \frac12(\partial_t\varphi)^2 + \frac12|\nabla\varphi|^2 + F(\varphi) - \partial_t\varphi\, \partial_\nu\varphi \; d\sigma_x$$

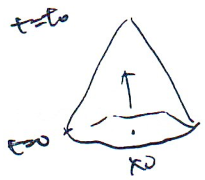


$$\frac12\left|(\partial_t\varphi)v - \nabla\varphi\right|^2 = \frac12(\partial_t\varphi)^2 + \frac12|\nabla\varphi|^2 - \partial_t\varphi\, v\cdot\nabla\varphi$$

from the change of measure

$$\Rightarrow e(0) = \frac{1}{\sqrt2}\int_\Gamma \cdots$$

(3)



$$e(0) = \cancel{\int_{B(x_0,t_0)} f(\varphi)}\, dx = 0$$

$$e(t) - e(0) = \int_{\text{}} (\quad) \le \int_\Gamma = e(0) = 0.$$

$$\le \int_{B(x_0,t_0-t)} \frac12(\partial_t\varphi)^2 + |\nabla\varphi|^2$$

$$e(t,x) \ge 0.$$

(4) $$E(t) = \int_{B(x_0,t_0-t)} e(t,x)\,dx$$

$$E'(t) = \int_{B(x_0,t_0-t)} e'\,dx - \int_{\partial B(x_0,t_0-t)} e\; dS$$

$$\cancel{E(t) - E(0) =}$$

$$\Rightarrow E'(t) = \underbrace{\int_{\partial B}\left(-\frac12\left|(\partial_t\varphi)v - \nabla\varphi\right|^2 - \varphi^2\right)}_{\le 0} + \underbrace{\int_B -f\partial_t\varphi + 2\varphi\partial_t\varphi}_{\le CE}$$

By Grönwall $\square$

$\boxed{\text{EX 5.1.1}}$ $\{u_n\}$ Cauchy in H^s $\Rightarrow$ $\{\langle\xi\rangle^s \hat{u}_n\}$ Cauchy in L^2

$\langle\xi\rangle^s \hat{u} \longrightarrow f$ in L^2

It suffices to show $(\langle\xi\rangle^{-s} f)^\vee \in \mathcal{S}'$

To show $\langle\xi\rangle^{-s} f \in \mathcal{S}'$. $\left|\int \langle\xi\rangle^{-s} f \varphi\right|$

$\leq \int \langle\xi\rangle^{|s|} |f||\varphi| \leq \|f\|_{L^2} \|\langle\xi\rangle^{|s|} \langle\xi\rangle^{[s]+d+1} c\|_{L^2} < \infty$. $\square$

$\boxed{\text{EX 5.1.2}}$ (1) $(\phi,\varphi) = \int \phi\varphi \overset{\text{Plancherel}}{=} c\int \hat{\phi}\bar{\hat{\varphi}}$

$= c\int \underbrace{|\xi|^{-s}\hat{\phi}}\ \underbrace{|\xi|^{s}\bar{\hat{\varphi}}}$

$\leq c\|\phi\|_{\dot{H}^{-s}} \|\varphi\|_{\dot{H}^s}$

$\Rightarrow$ BLT.

(2) $\dot{H}^s$ is a Hilbert space $\Rightarrow$ ~~$\langle L,\phi\rangle \overset{\exists! u}{=} B[u,\phi]$~~

~~$|\langle L,\phi\rangle| = \left|\int u\phi\right| \leq \|u\|_{\dot{H}^s} \|\phi\|_{H}$~~

$\langle L,\phi\rangle \overset{\exists! v}{=} \langle v,\phi\rangle_{\dot{H}^s} = \int |\xi|^{2s} \hat{v}\bar{\hat{\phi}}$

$= \int \underbrace{|\xi|^{2s}\hat{v}}_{\substack{\| \\ \hat{u}}} \bar{\hat{\phi}} = \int \hat{u}\bar{\hat{\phi}} = \int u\phi$

~~Unique~~ u $\exists$ ~~$0 = \int u_1\phi = \int (u_1-u_2)\phi$~~

$\|u\|^2_{\dot{H}^s} = \int |\xi|^{-2s}|\hat{u}|^2 = \int |\xi|^{2s}|\hat{v}|^2 = \|v\|^2_{\dot{H}^s} < \infty$

$\Rightarrow \|L\|_{(\dot{H}^s)^*} = \|v\|_{\dot{H}^s} = \|u\|_{\dot{H}^{-s}}$

Uniqueness: $B[u_1-u_2,\phi] = 0$ $\forall \phi \in \dot{H}^s$

$\Rightarrow$ ~~$\widehat{u_1-u_2} = 0$ a.e.~~

Riesz $\Rightarrow u_1 = u_2$ $\square$

$\boxed{\text{EX 5.1.5}}$

L^2_{loc}: $u = u_{low} + u_{high}$

$$\begin{cases} \hat{u}_{low}(\xi) = \chi_{\{|\xi| \le 1\}} \hat{u} \\ \hat{u}_{high} = \hat{u} - \hat{u}_{low} \end{cases}$$

$$\int_{\mathbb{R}^2} |u_{high}|^2 = \int |\hat{u}_{high}|^2 \le \int |\xi|^{2s} |\hat{u}_h|^2 < \infty \Rightarrow u_h \in L^2$$

$$\int |u_l|^2 = \int |\hat{u}_l|^2$$

near 0, $\gtrsim |t|^2 \lesssim |t|^2$ converge.

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(y)|^2}{|y|^{d+2s}} \, dx \, dy \overset{\text{change variable}}{=} \int |\hat{u}(\xi)|^2 d\xi \int |\xi|^{2s} \frac{|e^{i\vec{\xi}\cdot\vec{t}} - 1|^2}{|t|^{d+2s}} \, dt$$

$$\le C \|u\|^2_{\dot{H}^s}$$

$\varphi \equiv 1$ on K $\quad u_n \in \mathcal{S}$

$$\int |\varphi(u_n - u_m)|^2 = \int |\hat{\varphi} * (\hat{u}_n - \hat{u}_m)|^2$$

H: $\hat{\varphi} \quad \|\hat{u}_m - \hat{u}_n\|_{L^2} \le \| |\xi|^s (\hat{u}_m - \hat{u}_n) \|_{L^2} \to 0$

L: $\|$

See 《Fourier Analysis and partial differential equations》 P28.

$\boxed{\text{EX 5.1.3}}$ & $\boxed{\text{EX 5.1.4}}$ are trivial $\square$

$\boxed{\text{EX 5.2.1}}$ $|\xi|^2 \leq 2\left(|\xi-\eta|^2 + |\eta|^2\right)$

$$\Rightarrow \left(1+|\xi|^2\right) \leq 2\left(1+|\xi-\eta|^2\right)\left(1+|\eta|^2\right) \quad \square$$

$\boxed{\text{EX 5.2.2}}$

$$\cancel{\|\varphi f\|_{H^s}^2 = \int \langle\xi\rangle^{2s}|\widehat{(\varphi f)}|^2 = \int \langle\xi\rangle^{2s}|(\hat\varphi * \hat f)|^2}$$

$$\cancel{= \int \langle\xi\rangle^{2s}\left(\int \hat\varphi(t)\,\hat f(\xi_1 - t)\,dt\right)^2}$$

$$\|\varphi f\|_{H^s} = \|\langle\xi\rangle^s(\hat\varphi * \hat f)(\xi)\|_{L^2} = \left\|\int \langle\xi\rangle^s \hat\varphi(\xi-t)\hat f(t)\,dt\right\|_{L^2_\xi}$$

$$\leq \left\|\int 2^{\frac{s}{2}}\underbrace{\langle\xi-\eta\rangle^{s}}\langle\eta\rangle^s \underbrace{\hat\varphi(\xi-\eta)}\hat f(\eta)\,d\eta\right\|_{L^2_\xi}$$

$$\overset{\text{Minkowski}}{\leq} C \underbrace{\int |\langle\xi-\eta\rangle^{a}\hat\varphi(\xi-\eta)|}_{<\infty} \cdot \|f\|_{H^s}$$

if $\varphi \in \mathcal{S}$, φ can absorb higher order term. $\square$

$\boxed{\text{EX 5.2.3}}$ By Morrey's ineq and note that $\|f\|_{\dot H^s} \leq \|f\|_{H^s}$ for $s>0$ $\square$

$\boxed{\text{EX 5.2.4}}$ $f_n \to f$ in $H_s \overset{\in C_0}{\subseteq} C_0$ $f = g$?

$f_n \to g$ in C_0

If Not $|f-g| > \varepsilon/2$ in a Ball B_δ

$\Rightarrow |f_m - f| > \varepsilon/2$ in ...

But $\|f_m - f\|_{H^s} \to 0$ ↯

$\Rightarrow$ ι is a closed operator $\Rightarrow$ $H^s \hookrightarrow C_0$ continuously

$$\cancel{\text{for } \delta^{\wedge}\in\mathcal{S}\quad \|\langle\xi\rangle^{-s}\hat\delta\|_{L^2} = \langle\xi\rangle^{-s}}$$

$|\delta f| < \infty$ $\Rightarrow$ $\delta \in (H^s)^*$ $\Rightarrow \int \langle\xi\rangle^{-2s} < \infty$

$\Rightarrow 2s > d$ $\square$

$\boxed{\text{EX 5.2.5}}$ $\| \langle \xi \rangle^s \hat{\varphi}_n \|_{L^2} = \| \langle \xi \rangle^s e^{-i\xi a_n} \underbrace{\hat{\varphi}(\cancel{x-a_n})}_{\xi} \|_{L^2_\xi} < \infty$

$$\cancel{\| \langle \xi \rangle^t \hat{\varphi}_m - \langle \xi \rangle^t \hat{\varphi}_n \|_{L^2} = \| \langle \xi \rangle^t \hat{\varphi}(\xi) [e^{-i\xi a_m} - e^{-i\xi a_n}] \|_{L^2_\xi} = \| \langle \xi \rangle^t \hat{\varphi}(\xi) [e^{-i\xi(a_m - a_n)} - 1] \|_{L^2_\xi} \xrightarrow{\text{If}} 0}$$

$\varphi_{n_k} \to \varphi$ H^t implies $\varphi_{n_k} \to \varphi$ $\mathscr{D}'$

But the spts are disjoint for $k \to \infty \Rightarrow \varphi = 0$ in Distribution sense. But it's impossible. $\square$

$\boxed{\text{EX 5.2.6}}$ See Folland Page 307

$\boxed{\text{EX 5.2.7}}$ $\cancel{\square u = \delta \overset{\text{Fourier}}{\Longrightarrow} \partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 1}$

~~If $f \in L^1_{loc} \subseteq \mathscr{S}'$~~

It's worthy to note that the original conclusion is wrong. When $d = 1, 2$, we can actually write down the explicit form of the fundamental solution, which lists following:

$$\begin{cases} d=1: \frac{1}{2} H(t - |x|) \\ d=2 \quad \frac{1}{2\pi} H(t) \frac{1}{\sqrt{t^2 - |x|^2}} \end{cases}$$

For $d \geq 3$, we shall divide the question into two distinct cases.

$\boxed{d \text{ is ODD}}$ $E(t,x) = C_n \partial_t^{\frac{n-3}{2}} \left(\frac{\delta(t - |x|)}{|x|} \right)$

$\boxed{d \text{ is even}}$ $E(t,x) = C_n \partial^{\frac{n-2}{2}} \left((t^2 - |x|^2)_+^{-\frac{1}{2}} \right) H(t)$

Where $H(t)$ denotes the Heaviside function. $\square$

$\boxed{\text{EX 5.2.8}}$ $L^p \hookrightarrow \dot{H}^s$ $s = d(\frac{1}{2} - \frac{1}{p}) < \frac{d}{2}$ $1 < p \leq 2$

$-s = d(\frac{1}{p} - \frac{1}{2}) > 0$

$$\dot{H}^s \simeq (\dot{H}^{-s})^* \hookleftarrow (L^{2^*})^* \simeq L^p$$

$$\frac{2d}{d+2s} = \frac{2}{1+1-\frac{2}{p}} = \frac{1}{1-\frac{1}{p}} = q \qquad \square$$

$\dot{H}^{-s} \hookrightarrow L^{2^*} =$

$\boxed{\text{EX 5.2.9}}$ $\|\varphi_\varepsilon\|_{L^{2^*}} = \|\varphi\|_{L^{2^*}}$

$$\|\varphi_\varepsilon\|_{\dot H^s}^2 = \| |\xi|^s \hat{\varphi}_\varepsilon \|_{L^2}^2 = \| |\xi|^s (\widehat{e^{i\frac{x_1}{\varepsilon}}} * \hat{\varphi}(\xi)) \|_{L^2}^2$$

$$\cancel{\widehat{e^{i\frac{x_1}{\varepsilon}}}(\xi) = \int_{\mathbb{R}^d} e^{i\frac{x_1}{\varepsilon}} e^{-i(x_1\xi_1 + \cdots)} dx_1 \cdots dx_d = \left[\int e^{ix_1(\varepsilon^{-1} - \xi_1)} dx_1\right] \prod_{i=2}^d \int_{-\infty}^{\infty} e^{-ix_i\xi_i} dx_i}$$

$$= \| |\xi|^s \hat{\varphi}(\xi - \tfrac{1}{\varepsilon} e_1) \|_{H^s}^2 = \int_{\mathbb{R}^d} |\eta + \varepsilon^{-1} e_1|^{2s} |\hat{\varphi}(\eta)|^2 d\eta$$

$$= \varepsilon^{-2s} \underbrace{\int_{\mathbb{R}^d} |\varepsilon\eta + e_1|^{2s} |\hat{\varphi}(\eta)|^2 d\eta}_{\text{Bounded}} \longrightarrow O(\varepsilon^{-2s}) \qquad \square$$

$\boxed{\text{EX 5.2.10}}$ $\theta_A(x) = A^d \theta(Ax)$

Besov (Homogeneous) $\|f\|_{\dot B^{-\sigma}_{\infty,\infty}} := \sup_A A^{-\sigma} \|\theta_A * f\|_{L^\infty}$.

It suffices to show

$$A^{-\sigma} \|\theta_A * f\|_{L^\infty} \leq \frac{C}{\sqrt{\frac{d}{2} - s}} \|f\|_{\dot H^s}$$

$$\cancel{LHS = A^{-\sigma} \left| \int_{\mathbb{R}^d} A^d \theta(A(\xi - y)) f(y) dy \right| = A^{d-\sigma} \left| \int_{\mathbb{R}^d} \theta(A(\xi - y)) f(y) dy \right| = A^{d-\sigma} \left| \int_{\mathbb{R}^d} \hat{\theta}(-\xi/A) e^{-i\xi\xi} \hat{f}(\xi) d\xi \right| \leq A^{d-\sigma} \| |\xi|^{-s} \hat{\theta}(-\xi/A) e^{-i\xi\xi} \|_{L^2} \| f \|_{\dot H^s}}$$

$$\cancel{A^{d-\sigma} \| |\xi|^{-s} \hat{\theta}(-\xi/A) \|_{L^2} = A^{\frac{d}{2}+s} \left(\int |\xi|^{-2s} |\hat{\theta}(-\xi/A)|^2 d\xi \right)^{\frac{1}{2}}}$$

$=$

$\boxed{-\sigma = s - \frac{d}{2}}$

$\theta_A = \theta(Ax)$

LHS $\overset{\sim}{\approx} A^{\frac{d}{2}+s}$ $|\theta_A * f|$

$$|\theta_A * f|^2 \lesssim \left(\int A^{-2d}\, \hat{\theta}(\xi/A)\, |\xi|^{-2s}\, d\xi\right)\left(\|f\|_{\dot{H}^s}^2\right)$$

$$= A^{-(d+2s)}\, \||\xi|^{-s}\hat{\theta}\|_{L^2}^2\, \|f\|_{\dot{H}^s}^2$$

It suffices to treat with $\||\xi|^{-s}\hat{\theta}\|_{L^2}^2 = I$

$$\phi \quad I \approx c\int_{|\eta|\leq 1} |\eta|^{-2s} \approx c\int_0^1 |\eta|^{-2s+d-1} \approx \frac{c}{d-2s}$$

$\square$

$\boxed{\text{EX } 1.2.11}$

$$\cancel{\left|A^{d-\sigma}\int \theta_A(\xi-\eta)\,\varphi(\eta)\, e^{i\eta_1/\varepsilon}\, d\eta\right|}$$

$$\cancel{= A^{d-\sigma}\left|\int A^{-d} e^{-i\xi\zeta}\,\hat{\theta}\left(\tfrac{\zeta}{A}\right)\,\varphi(\zeta-\varepsilon^{-1}e_1)\, d\zeta\right|}$$

$$\cancel{= A^{-\sigma}\left|\int \hat{\theta}\left(\tfrac{\zeta}{A}\right)\hat{\varphi}(\zeta-\varepsilon^{-1}e_1)\, d\zeta\right|}$$

$\boxed{\text{suppose } \operatorname{supp} \hat{\varphi} \subseteq B_A}$

$$\cancel{= A^{-\sigma}\left|\int \hat{\theta}\left(\frac{\zeta+\varepsilon^{-1}e}{A}\right)\hat{\varphi}(\zeta)\, d\zeta\right|}$$

$e_1\varepsilon^{-1}$

$\cancel{=1 \quad \max \frac{e_1}{A\varepsilon}}$

I: $A\varepsilon > 1$. $\eta \in \operatorname{supp}\hat{\varphi}$ $\cancel{|\eta + e_1/\varepsilon| \geq \frac{1}{\varepsilon} - A > 0}$ $|\theta| < A$

~~(A equivalent definition of homogeneous Besov space relate to HL operator Δ_j)~~

$$\cancel{\|\varphi\|_{\dot{B}^{\sigma}_{\infty\infty}} = \sup_{j\in\mathbb{Z}} 2^{jd}\|\Delta_j\varphi\|_{L^\infty} \lesssim \sup_{2^j\lesssim A} 2^{-j\sigma} \approx A^{-\sigma} < \varepsilon^{\sigma}}$$

$$A^d\|\theta(A\cdot)*\phi_\varepsilon\|_{L^\infty} \overset{\text{Hölder}}{\leq} \|\theta\|_{L^1}\|\phi_\varepsilon\|_{L^\infty} = \|\theta\|_{L^1}\|\phi\|_{L^\infty}$$

If $A\varepsilon \gtrsim 1$

$$A^{d-\sigma}\|\theta(A\cdot)*\phi_\varepsilon\|_{L^\infty} \leq \varepsilon^{\sigma}\|\theta\|_{L^1}\|\phi\|_{L^\infty}$$

If $A\varepsilon < 1$ $\quad (-i\varepsilon\partial_1)^d \exp\left(\frac{ix_1}{\varepsilon}\right) = \exp(ix_1\varepsilon^{-1})$

$$A^d(\theta(A\cdot)*\phi_\varepsilon)(x) = A^d\int \theta(A(x-y))\,\phi(y)\,(-i\varepsilon\partial_1)^d \exp(iy_1\varepsilon^{-1})\, dy$$

$$\overset{\text{IBP}}{=} (i\varepsilon A)^d\int \partial_1^d(\theta(A(x-y)))\,\phi(y)\, dy$$

$$\overset{\text{Leibniz}}{=} (i\varepsilon A)^d \int \sum_{k\le d} \binom{d}{k} A^k (-\partial_1)^k \theta(A\cdot) * \left(e^{iy_1\varepsilon^{-1}} \partial_1^{d-k} \phi\right)(x).$$

$$A^k \left\| (-\partial_1^k \theta)(A\cdot) * e^{iy_1\varepsilon^{-1}} \partial_1^{d-k} \phi \right\|_{L^\infty} \overset{\text{Hölder}}{\le} \|\partial_1^k \theta\|_{L^{\frac{d}{k}}} \|\partial^{d-k}\phi\|_{L^{(\frac{d}{k})'}}$$

$$\Rightarrow (\ ) \le C A^{\sigma} (A\varepsilon)^d \le C A^{\sigma} \varepsilon^{\sigma}$$

□

$\boxed{\text{EX 5.2.12}}$ Suppose $\|u\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}} = 1$ $\quad u = u_{\ell,A} + u_{h,A}$ $\quad p = 2^* = \frac{2d}{d-2s}$

From the definition of $\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}$ $\Rightarrow \|u_{\ell,A}\|_{L^\infty} \le A^{\frac{d}{2}-s}$

$$\{|u| > \lambda\} \subseteq \{|u_{\ell,A}| > \tfrac{\lambda}{2}\} \cup \{|u_{h,A}| > \tfrac{\lambda}{2}\}$$

$$A = A_\lambda = \left(\frac{\lambda}{2}\right)^{\frac{p}{d}} \Rightarrow m\{|u_{\ell,A}| > \tfrac{\lambda}{2}\} = 0$$

$$\|u\|_{L^p}^p = p\int_0^\infty \lambda^{p-1} \underline{\mu\{|u_{h,A_\lambda}| > \tfrac{\lambda}{2}\}}$$

$$\overset{\text{chebyshev}}{\le} p\int_0^\infty \lambda^{p-1}\, 4\, \frac{\|u_{h,A_\lambda}\|_{L^2}^2}{\lambda^2}\, d\lambda = 4p\int_0^\infty \lambda^{p-3} \|u_{h,A_\lambda}\|_{L^2}^2\, d\lambda$$

$$\le Cp \int_0^\infty \lambda^{p-3} \int_{|\xi| \ge CA_\lambda} |\hat{u}(\xi)|^2\, d\xi\, d\lambda$$

$$\Downarrow$$

$$\lambda \le 2\left(\frac{|\xi|}{c}\right)^{\frac{d}{p}} = C_\xi$$

$$\overset{\text{Fubini}}{=} Cp \int_{\mathbb{R}^d} \left(\int^{C_\xi} \lambda^{p-3}\right) |\hat{u}(\xi)|^2$$

$$= C\frac{p}{p-2} \int |\xi|^{\frac{d(p-2)}{p}} |\hat{u}(\xi)|^2 \Rightarrow \left(\frac{p}{p-2}\right)^{\frac{1}{p}} \|u\|_{\dot{H}^s} \text{ as } u \text{ bd.}$$

□

$\boxed{\text{EX 5.2.13}}$ See << Fourier Analysis and PDEs >> P47 □