

# Functional Analysis and Spectral Theory

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## Closed and closable, symmetric and self-adjoint

Let  $H$  always denote Hilbert space and operators are densely defined.

- Unbounded operator  $A$ :  $D(A)$  is densely defined,  $\sup_{x \in H \setminus \{0\}} \frac{\|Ax\|}{\|x\|} = \infty$ .
- Closed operator  $A$ :  $(x, Ax)$  is closed in the graph norm.
- Closable operator  $A$ : There is a closed operator  $B$  such that  $A \subseteq B$ .
- Symmetric operator  $A$ :  $\langle Ax, y \rangle = \langle x, Ay \rangle$  holds for every  $x, y \in D(A)$ .
- Self-adjoint operator  $A$ : Symmetric and  $D(A) = D(A^*)$ .

### Definition ( $D(A^*)$ )

$D(A^*) = \{y \in H : \langle Ax, y \rangle \leq C_y \|x\|, \forall x \in D(A)\}$ , and then  $A^*$  is defined by

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \quad \text{where } x \in D(A) \text{ } y \in D(A^*).$$

## Closed and closable, symmetric and self-adjoint

The following proposition tells their relations and properties.

### Proposition

- (1).  $A^*$  is a closed operator.
- (2).  $A$  is closable  $\Leftrightarrow D(A^*)$  is dense in  $H$ ;
- (3). if  $A$  is closable, then  $A^{**} = \bar{A}$ .
- (4). Symmetric operators are closable operators.

### Theorem

- (1). If  $A$  is symmetric, then  $A \subseteq A^{**} = \bar{A} \subseteq A^*$ .
- (2). If  $A$  is symmetric closed, then  $A = A^{**} \subseteq A^*$ .
- (3). If  $A$  is self-adjoint, then  $A = A^{**} = A^*$ .

### Remark

If  $A$  is not closable, we cannot get any spectral information, so closeness is not a technical condition.

# Resolvent Set and Spectrum

Recall the definition of spectrum.

- Resolvent set:  $\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ has bounded inverse}\}$ .
- point spectrum:  $\sigma_p(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda I) \neq \{0\}\}$
- Continuous spectrum:  
 $\sigma_c(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda I) = \{0\} \text{ and } \text{Ran}(A - \lambda I) \neq H \text{ but } \overline{\text{Ran}(A - \lambda I)} = H\}$
- Residual spectrum:  $\sigma_r(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda I) = \{0\} \text{ and } \overline{\text{Ran}(A - \lambda I)} \neq H\}$ .

Proposition (Spectrum of self-adjoint operator  $A$ )

- (1).  $\sigma(A) \subseteq \mathbb{R}$ ;
- (2).  $\sigma_r(A) = \emptyset$ ;
- (3). *Eigenfunctions of different eigenvalues are orthogonal.*

## An example

### Example

$-\Delta$  is a self-adjoint operator on  $H^2(\mathbb{R}^d)$  and its spectrum  $\sigma(-\Delta) = [0, \infty)$ .

### Proof.

$\langle -\Delta f, g \rangle = \langle |\xi|^2 \hat{f}, \hat{g} \rangle = \langle \hat{f}, |\xi|^2 \hat{g} \rangle$  holds for every  $f \in H^2(\mathbb{R}^d)$  and  $g \in D((-\Delta)^*)$ . Then  $D(-\Delta) = D((-\Delta)^*)$ .

Consider  $\lambda \in \mathbb{R}$ , and  $(-\Delta - \lambda)u = f$ . Then we have  $\hat{u} = \frac{\hat{f}}{|\xi|^2 - \lambda}$ . If  $\lambda < 0$ , then it is clear

$|\xi|^2 \hat{u} \in L^2(\mathbb{R}^d)$  so that  $\lambda \in \rho(-\Delta)$ . If  $\lambda \geq 0$ , let  $|\xi_0|^2 = \lambda$ ,  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and  $\hat{u}_n(\xi) = n^{\frac{d}{2}} \varphi(n(\xi - \xi_0))$ . Then we see  $\|u_n\|_2^2 = \|\varphi\|_2^2$  and  $\|(-\Delta - \lambda)u_n\|_2^2 \lesssim \frac{1}{n} \|\varphi\|_2^2$ , which implies that  $0 \leq \lambda \in \sigma(-\Delta)$ .  $\square$

## Basic result

For an bounded operator  $A$ , we have the followings:

- $\rho(A)$  is open;
- $\sigma(A)$  is closed;
- $R_A(\lambda) = (A - \lambda I)^{-1}$  is an analytic function with respect to  $\lambda \in \rho(A)$ ;
- If  $\lambda, \mu \in \rho(A)$ , then  $R_A(\lambda)R_A(\mu) = R_A(\mu)R_A(\lambda)$ ;
- $R_A(\lambda) - R_A(\mu) = (\lambda - \mu)R_A(\lambda)R_A(\mu)$ ;
- For  $\forall z \in \rho(A) \cap \rho(B)$ , we have  $R_A(z) - R_B(z) = R_A(z)(B - A)R_B(z)$ ;
- (Spectral Radius Formula)  $r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$ ;
- If  $A$  is self-adjoint operator, then  $\|A^*A\|_{op} = \|A\|_{op}^2$ .

## Continuous calculus

### Theorem (Bounded and continuous)

Let  $A$  be a bounded self-adjoint operator. Then there is a unique mapping  $\phi_A : \mathcal{C}(\sigma(A)) \rightarrow L(H)$  satisfying the following properties.

- (1).  $\phi_A$  is an algebraic  $*$ -homomorphism.
- (2).  $\phi_A$  is isometric, that is to say,  $\|\phi_A(f)\|_{op} = \|f\|_{L^\infty(\sigma(A))}$ .
- (3). If  $f(x) = x$ , then  $\phi_A(f) = A$ .
- (4). (Spectral mapping theorem)  $\sigma(f(A)) = f(\sigma(A))$ .
- (5). If  $f(x) \geq 0$  for  $\forall x \in \sigma(A)$ , then  $f(A) \geq 0$  and  $\langle f(A)x, x \rangle \geq 0$ .
- (6). If  $A\psi = \lambda\psi$ , then  $f(A)\psi = f(\lambda)\psi$ .

Idea: We can prove polynomial case first and use Stone-Weierstrass theorem to lift the result.

## Polynomial case

We have the following two lemmas.

### Lemma

Let  $p(x)$  be a polynomial  $p(x) = \sum_{i=0}^n a_i x^i$ . Then we have  $\sigma(P(A)) = P(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$ .

Idea of proof: One side can be proved by  $p(x) - p(\lambda) = (x - \lambda)q(\lambda)$  and the other side by  $p(x) - z = \prod_{i=1}^n (x - \lambda_i(z))$ .

### Lemma

$\|p(A)\|_{op} = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = \|p\|_{L^\infty(\sigma(A))}$ .

Idea of proof: Observe that  $p(A)$  is also self-adjoint, and then compute directly.

# Mapping theorem and applications

## Proof of spectral mapping theorem.

To show  $f(\sigma(A)) \subseteq \sigma(f(A))$ , let  $\lambda \in \sigma(A)$ , localization  $\chi_\varepsilon = \chi\left(\frac{x-\lambda}{\varepsilon}\right) \in C_c^\infty$  with  $\|\chi\|_{L^\infty} = 1$ . Then there is a function  $\psi_\varepsilon$  with  $\|\psi_\varepsilon\| = 1$  such that

$$\|\chi_\varepsilon(A)\psi_\varepsilon\| = \|\chi_\varepsilon\|_{op} = \|\chi_\varepsilon\|_{L^\infty(\sigma(A))} = 1.$$

$$\Rightarrow \lim_{\varepsilon \rightarrow 0} \|(f(A) - f(\lambda))\chi_\varepsilon(A)\psi_\varepsilon\| \leq \lim_{\varepsilon \rightarrow 0} \left( \sup_{\mu \in \sigma(A)} |f(\mu) - f(\lambda)| \right) \|\chi_\varepsilon\|_{L^\infty(\sigma(A))} = 0.$$

It implies that  $f(\lambda) \in \sigma(f(A))$ . Conversely, we show that  $\sigma(f(A)) \subseteq f(\sigma(A))$ . Let  $z \in \sigma(f(A))$ . If  $z \notin f(\sigma(A))$ , then  $\frac{1}{f(\lambda) - z}$  is a continuous function with respect to  $\lambda \in \sigma(A)$ . Then it implies that  $z \in \rho(f(A))$ , but it leads to contradiction. □

We have two useful applications for bounded self-adjoint operators:

- If  $A \geq 0$ , then  $\sigma(A) \subseteq [0, \infty)$  and there is self-adjoint  $B \geq 0$  such that  $B^2 = A$ .
- $\|R_\lambda(A)\|_{op} = \frac{1}{\text{dist}(\lambda, \sigma(A))}$ .

# Borel calculus

## Theorem (Bounded and bounded Borel)

Let  $A$  be a bounded self-adjoint operator. Then there is a unique  $\phi_A : \mathcal{B}_b(\sigma(A)) \rightarrow L(H)$ ,  $f \mapsto f(A)$ , satisfying the following properties:

- (1).  $\phi_A$  is an algebraic  $*$ -homomorphism.
- (2).  $\phi_A$  is norm continuous:  $\|\phi_A(f)\|_{op} \leq \|f\|_{\mathcal{B}_b}$ .
- (3). Let  $f(x) = x$ , and then  $\phi_A(f) = A$ .
- (4). Suppose  $f_n(x) \rightarrow f(x)$  for each  $x$  and  $\sup_n \|f_n\|_{\mathcal{B}_b}$  is bounded. Then  $\phi_A(f_n) \rightarrow \phi_A(f)$  strongly.

Moreover  $\phi_A$  has the properties:

- (5). If  $A\psi = \lambda\psi$ , then  $\phi_A(f)\psi = f(\lambda)\psi$ .
- (6). If  $f \geq 0$ , then  $\phi_A(f) \geq 0$ .
- (7). If  $BA = AB$ , then  $\phi_A(f)B = B\phi_A(f)$ .

## Remark

The spectral mapping theorem the same as the last does not hold yet.

## Scalar spectral measure

Idea of this generalization: Through integral to lower the requirements of regularity.

Sketch of proof.

We can define the bilinear form

$$\langle x, f(A)y \rangle = \int_{\sigma(A)} f(z) d\mu_{x,y}(z)$$

where  $\mu_{x,y}$  is defined originally from continuous functions. Using approximation, we can prove most of them. For (4), let  $f_n$  converge to  $f$  for each  $x$ . To show strong convergence, it is equivalent to show

$$\int_{\sigma(A)} |f_n - f|^2 d\mu_{x,x} \rightarrow 0.$$

We already know that  $f_n$  is uniformly bounded and converges pointwise, and then we can use dominated convergence theorem with respect to the measure  $\mu_{x,x}$ . □

# Applications

## Definition (p.v.m.)

The family  $\{E(\cdot) : \mathcal{B}(\mathbb{R}) \rightarrow P(H)\}$  spectral projections of a bounded self-adjoint operator, which has the following properties:

(1).  $E(\emptyset) = 0$ ,  $E((-a, a)) = I$  for some  $a$ .

(2).  $E(B_1 \cap B_2) = \mathbf{1}_{B_1} \mathbf{1}_{B_2}$ .

(3). Suppose  $\{B_i\}_{i=1}^{\infty}$  is disjoint pairwise, then  $E(\cup B_i) = s - \sum_{i=1}^{\infty} E(B_i)$ .

is called projection-valued measure(p.v.m.).

One can verify that  $E(B) := \chi_B(A)$  is a p.v.m..

## Proposition (p.v.m. v.s. scalar spectral measures)

For each  $x, y \in H$ , the measure  $\mu_{x,y}$  constructed from the Borel functional calculus satisfies

$$\mu_{x,y}(B) = \langle x, E(B)y \rangle, \quad B \in \mathcal{B}(\sigma(A)).$$

## Summary of Borel calculus in p.v.m.

Through p.v.m., we can see the following corresponding relation.

### Theorem

$$A \mapsto \{E_\Omega\} = \{\chi_\Omega(A)\} \mapsto A = \int \lambda dE_\Omega.$$

Let  $E_\lambda = E((-\infty, \lambda])$ ,  $\text{supp}E := \mathbb{R} \setminus \bigcup\{U : U \text{ is open and } E(U) = 0.\}$ .

### Proposition

$\text{supp}E = \sigma(A)$  where  $A = \int \lambda dE_\lambda$ .

### Sketch of proof.

If  $\lambda_0 \in \sigma(A) \setminus \text{supp}E$ , then we can define  $g(\lambda)$  on a vanishing set of  $E$ , and then  $B = \int g(\lambda)dE_\lambda$ . Then we can calculus and then get  $(A - \lambda_0 I)B = I$ . For the other side, we need to integrate a positive function on a set outside  $\sigma(A)$ . □

# Unbounded Borel calculus

$$\Phi(f) = \int_{\Omega} f dE, \quad E_{x,y} = \langle x, E(B)y \rangle,$$

$$D(\Phi(f)) = D_f = \{y \in H : \int_{\Omega} |f|^2 dE_{y,y} < \infty\}.$$

## Theorem (Unbounded and Borel, simplified version)

- (1).  $D_f$  is a dense linear subspace.
- (2).  $\forall x \in H, y \in D_f, \int_{\Omega} |f| d|E_{x,y}| \leq \|x\| \left( \int_{\Omega} |f|^2 dE_{y,y} \right)^{\frac{1}{2}}$ .
- (3). There is a unique closed operator  $\Phi(f)$  on  $D_f$  satisfying

$$\langle x, \Phi(f)y \rangle = \int_{\Omega} f(\lambda) dE_{x,y}(\lambda) \quad \text{for } \forall x \in H, y \in D_f.$$

- (4).  $\|\Phi(f)\|^2 = \int_{\Omega} |f|^2 dE_{x,x}$  for  $\forall x \in D_f$ .
- (5). Calculus properties.

## Riesz projector

Definition (Separable Spectral subsets and Riesz projector)

$\sigma_1(A) \subseteq \sigma(A)$  is called separable spectral subset if it can be surrounded by a contour.  $P$  is a Riesz projector if

$$P = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda - A} d\lambda.$$

Theorem

*If  $A$  has separable spectrum, then  $P$  is a projection operator and  $\text{Ran}(P) \subseteq D(A^n)$  for each  $n \geq 1$ .*

Idea: Exchange the order of integrals and use Riemann sum to simulate the calculus.

# Applications

## Theorem (Riesz projector for isolated points)

Let  $\lambda_0$  be an isolated spectral point of  $A$  and  $P_{\lambda_0} = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} R(\lambda, A) d\lambda$ . Then

- (1).  $\ker(A - \lambda_0) \subseteq \text{Ran}(P_{\lambda_0})$ .
- (2). If  $X$  is a Hilbert space and  $A$  is self-adjoint, then  $\ker(A - \lambda_0) = \text{Ran}(P_{\lambda_0})$ .

Idea: Some tricky calculations and arguments of removable singularity.

## Theorem

If  $A$  is a self-adjoint operator and  $\lambda_0$  is an isolated spectral point, then it lies in point spectrum.

Idea:  $\lambda_0$  being not eigenvalue implies  $P_{\lambda_0} = 0$ . By calculus,  $R(\xi, A)$  at most has singularity of order 1 nearby. But  $P_{\lambda_0} = 0$  means residue is zero so that the singularities are removable by uniform boundness principle. Then we can prove  $\lambda_0 \in \rho(A)$ .

# Spectral sequence

## Proposition

*If  $A$  is self-adjoint, then  $\lambda \in \sigma(A)$  if and only if  $E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$  for each  $\varepsilon > 0$ .*

Idea:  $\text{supp}E = \sigma(A)$ .

## Theorem

*Let  $A$  be a self-adjoint operator, then  $\lambda \in \sigma(A)$  if and only if there is a sequence  $\{\psi_n\} \subseteq D(A)$  such that  $\|\psi_n\| = 1$  and  $(A - \lambda)\psi_n \xrightarrow{s} 0$ . This sequence is called spectral sequence.*

Idea: The necessity can be proved by considering  $\psi_n \in \text{Ran}(E(\lambda - \varepsilon_n, \lambda + \varepsilon_n))$  and using spectral integral.

## Discrete spectrum and essential spectrum

$$\sigma_d(A) = \{\lambda \in \sigma(A) : \exists \varepsilon_0 > 0, \dim \operatorname{Ran} E(\lambda - \varepsilon_0, \lambda + \varepsilon_0) < \infty\}.$$

$$\sigma_{\text{ess}}(A) = \{\lambda \in \sigma(A) : \exists \varepsilon_0 > 0, \dim \operatorname{Ran} E(\lambda - \varepsilon_0, \lambda + \varepsilon_0) = \infty\}.$$

### Proposition

*If  $A$  is self-adjoint and  $\lambda$  is an isolated spectral point, then  $\operatorname{Ran}(E(\{\lambda\})) = \ker(A - \lambda_0)$ .*

Idea: Calculus and approximation by  $f_n = \frac{1}{\lambda_0 - z} \mathbf{1}_{B^c(\lambda_0, \frac{1}{n})}$  from outside.

### Proposition

*Let  $A$  be self-adjoint, then  $\lambda \in \sigma_d(A)$  if and only if  $\lambda$  is an isolated spectral point and its space of eigenvectors is of finite dimensions.*

Idea: If  $\lambda$  is not isolated, then the dimensions will accumulate and blow up.

# Weyl criterion

## Theorem (Weyl criterion)

*Let  $A$  be self-adjoint.  $\lambda \in \sigma_{\text{ess}}(A)$  if and only if there is a sequence of  $\{\psi_n\} \subseteq D(A)$  that  $\|\psi_n\| = 1$ ,  $\psi_n \rightharpoonup 0$  and  $(A - \lambda)\psi_n \xrightarrow{s} 0$ . Here the sequence  $\{\psi_n\}$  is called Weyl sequence.*

Idea: For sufficiency,  $E(\{\lambda\})$  is compact and we can prove by contradiction. For necessity, we can choose  $\psi_n \in \text{Ran}E(\lambda - \varepsilon_n, \lambda + \varepsilon_n)$  which is orthonormal and then use spectral integral.

## Relative bounded

### Definition (Relative bounded)

$B$  is  $A$ -bounded if  $D(B) \subseteq D(A)$  and  $\|Bx\| \leq a\|Ax\| + b\|x\|$ . The infimum of  $a$  is called the relative bound of  $B$  with respect to  $A$ .

### Proposition

$A$  is self-adjoint,  $B$  is  $A$ -bounded. Then the relative bound  $a = \lim_{\lambda \rightarrow \infty} \|BR(A, i\lambda)\|_{op}$ .

Idea:  $Bx = BR(A, i\lambda)(A - i\lambda)x$ .

### Theorem (Kato-Rellich)

Let  $A$  be self-adjoint and  $B$  be closed symmetric and  $A$ -bounded with relative bound  $a < 1$ . Then we have  $A + B$  is self-adjoint on  $D(A)$ .

Idea: When the perturbation is relatively small, we can control it by resolvent identity  $A + B - i\lambda = (BR(A, i\lambda) + I)(A - i\lambda)$ .

## Relative compact

### Definition (Relative compact)

$A$  is closed.  $B$  is  $A$ -compact if  $D(A) \subseteq D(B)$  and there is  $z \in \rho(A)$  such that  $BR(A, z)$  is compact.

### Proposition

$A$  is self-adjoint,  $B$  is  $A$ -compact. Then the relative bound is 0.

Idea:  $BR(A, i\lambda) = BR(A, i)(A - i)R(A, i\lambda)$ .

### Theorem (Weyl compact perturbation)

$A$  is self-adjoint,  $B$  is symmetric and  $A$ -compact. Then  $\sigma_{\text{ess}}(A + B) = \sigma_{\text{ess}}(A)$ .

Idea:  $x_n$  is Weyl sequence of  $A$  and then  $Bx_n = BR(A, z_0)(A - z_0)x_n \xrightarrow{s} 0$ .

## An example

### Example

In  $\mathbb{R}^3$ ,  $V \in L^2(\mathbb{R}^3)$  is  $-\Delta$ -bounded and  $-\Delta$ -compact.

Idea: For boundness,

$$\|Vf\|_{L^2} \lesssim \|V\|_{L^2} \|f\|_{L^\infty} \lesssim \|V\|_{L^2} \|f\|_{H^2} \lesssim \|V\|_2 (\|f\|_{L^2} + \|-\Delta f\|_{L^2}).$$

For compactness,

$$R(A, i)f = \mathcal{F}^{-1} \mathcal{F}(R(A, i)f) = \mathcal{F}^{-1} \left( \frac{1}{|\xi|^2 - i} \hat{f} \right) = G * f = \int G(x - y) f(y) dy,$$

$$VR(A, i)f = \int V(x) G(x - y) f(y) dy.$$

Hilbert-Schmidt operator is compact. Combining with perturbation, we can see a family of potentials do not impact the essential spectrum of  $-\Delta$ .

# Strongly continuous semigroup

## Definition (Infinitesimal generator)

The infinitesimal generator  $A$  of a strongly continuous semigroup  $S$  is defined by

$$Ax = \lim_{t \searrow 0} \frac{S(t)x - x}{t} \quad D(A) = \{x \in X : \text{the limit above exists}\}.$$

Idea: We can regard the semigroup as  $e^{tA}$ .

## Proposition

- (1).  $\frac{d}{dt}S(t)x = S(t)Ax = AS(t)x$ .
- (2).  $\int_0^t S(s)x ds \in D(A)$ ,  $A \int_0^t S(s)x ds = S(t)x - x = \int_0^t S(s)Ax ds$ .
- (3).  $A$  is densely defined and closed.

# Contraction semigroup

If  $\|S(t)\| \leq 1$ ,  $S$  is called a contraction semigroup.

## Proposition

*If  $A$  is the generator of a contraction semigroup, then  $\{\lambda \in \mathbb{C} : \Re \lambda > 0\} \subseteq \rho(A)$  and  $R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t) dt$ .*

## Theorem (Hille-Yosida)

*Let  $A$  be a densely defined closed operator. Then  $A$  is a generator of some contraction semigroups if and only if  $(0, \infty) \subseteq \rho(A)$  and  $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ .*

Idea: Technical approximation  $A_h = \frac{S(h) - I}{h}$ ,  $M_t = \frac{1}{t} \int_0^t S(s) ds$  and  $A^\lambda = \lambda^2 R(\lambda, A) - \lambda$  and direct calculations.

The end

Thank you for listening.