Lecture Notes of Topics on Partial Differential Equation

Fu Chow PB23010309

Department of Mathematics, USTC

December 29, 2025

Abstract

These notes are written for the course "Topics on Partial Differential Equation". We start from functional analysis which is useful in the study of PDE. Based on Hilbert spaces and self-adjoint operators, we cover spectral theory, perturbation theory of operators, semigroup theory, and several related topics.

Keywords: functional analysis; spectral theory; operator perturbation; semigroup; partial differential equations.

1 Introduction

In a standard undergraduate course on functional analysis one learns the basic theory of bounded operators, but this framework is often too restrictive. When we study problems arising from partial differential equations, most of the operators we encounter are in fact unbounded. For instance, the most common example, the Laplace operator $-\Delta$, is not a bounded operator. Moreover, its domain imposes strong regularity requirements, which means that we can no longer demand that an operator be defined on the whole space. Instead, it is natural to merely require that the operator be densely defined. This also agrees with the spirit of analysis.

In what follows we begin with the basic theory of unbounded operators. We take for granted the standard material in undergraduate functional analysis and advanced real analysis; see [1][5], and then move directly to our main topic.

2 Unbounded Operators

2.1 Closed and closable operators

First of all, we recall the definition of bounded operators. Suppose A is a bounded operator, then its domain $D(A) = H$ and $\sup\limits_{u} \frac{||Au||}{||u||} < \infty$. Here we always let H denote Hilbert spaces. Then we can give the precise definition of unbounded operators.

Definition 2.1 (Unbounded operators)**.** A densely defined linear operator A is called a unbounded operator if there is no constant $C > 0$ such that $||Ax|| \leq C||x||$ holds for every $x \in D(A)$.

Now we have defined unbounded operators, but how can we similar continuity as the bounded cases? Then it comes to the conception of closed operators.

Definition 2.2 (Graphs and closed operators)**.** Suppose operator $A : D(A) \to H$, then graph is defined as the set $\{(x, y) \in H \times H : x \in D(A), y = Ax\}$, and the operator A is called a closed operator if $G(A)$ is closed in $H \times H$. Equivalently, A is closed if and only if we have $x \in D(A)$ and $y = Ax$ when $x_n \in D(A)$ and $(x_n, Ax_n) \to (x, y)$.

We emphasize that, in the unbounded setting, one cannot simply apply the usual closed graph theorem in the same way as for bounded operators.

Example 2.3. It's important to note that whether an operator is closed depends on its domain. Suppose $A = -\Delta$, if its domain is $C_c^\infty(\mathbb{R}^n)$, then is not a closed operator, but for the case $H^2(\mathbb{R}^n)$, it is. Here the range is $L^2(\mathbb{R}^n)$.

Proof. Consider $\varphi_k = e^{-|x|^2}\zeta_k$ where ζ_k is smooth localization, then $\varphi_k \to e^{-|x|^2}$ in L^2 norm which implies the $-\Delta$ is not closed. But for the H^2 case, we have already known that $||u||_{L^2(\mathbb{R}^n)} + ||-\Delta u||_{L^2(\mathbb{R}^n)}$ is an equivalent norm with respect to H^2 norm, so we can verify the limit of the sequence we considered has limit in H^2. Then it implies $-\Delta$ is a closed operator. $\square$

We can also weaken the condition of the definition of closed operators, and then it comes to the closable operators.

Definition 2.4 (Closable operators)**.** Let A_1 and A be defined on H with $G(A) \subseteq G(A_1)$, then we call A_1 is an extension of A. Equivalently, $D(A) \subseteq D(A_1)$ and $Ax = A_1x$ for $\forall x \in D(A)$.

If A has a closed extension, then we say A is closable.

The smallest closed extension of A is called the closure of A, written as $\bar{A}$. Equivalently we have the following relation

$$G(A) \subseteq G(\bar{A}) \subseteq G(A_1), \text{ where } A_1 \text{ is an arbitrary closed extension of } A.$$

However, the description of closable operators is not always convenient, and then we have the following lemma to better describe the closable property of an operator.

Lemma 2.5 (Closable property)**.** The followings are equivalent:
(1). A is closable;
(2). if $\{x_n\} \subseteq D(A)$, then $(x_n, Ax_n) \to (0, v)$ implies $v = 0$;
(3). if $\{x_n\} \subseteq D(A)$, then $(x_n, Ax_n) \to (u, v)$ implies $v = Au$.

Proof. (1)⇒(2): Suppose A_1 is a closed extension of A, then $(x_n, Ax_n) = (x_n, A_1x_n) \to (x = 0, Ax) = (0, 0)$.
(2)⇒(3): Consider $\tilde{x}_n = x_n - x$.
(3)⇒(1): It's a constructive proof. Let R be a new operator, whose domain is $D(R) = \{x : \exists\, y \in H \ s.t.\ (x, y) \in \overline{G(A)}\}$, and then we can verify it's the closure of A. First, let $y = Ax$, and then $D(A) \subseteq D(R)$. Secondly, y is unique thanks to (2). Finally, we can define $Rx = y$ and it's easy to see R is closed from the construction. Then A is closable. Here we recall the definition of $D(R)$, it's easy to verify it's the smallest closed graph, so $R = \bar{A}$. $\square$

Remark. From the proof above, we know that $\overline{G(A)} = G(\bar{A})$ if A is closable.

2.2 Conjugate Operators

In the bounded case, we can define the conjugate operators directly or by Riesz representation theorem, but how can extend this definition at the unbounded settings? Here we implement it in the form of Riesz representation. Suppose A is densely defined on H, let $D(A^*) = \{y \in H : \langle Ax, y\rangle \leq C_y||x||,\ \forall x \in D(A)\}$, and then we actually define a bounded operator $f_y(x) = \langle Ax, y\rangle$. By Riesz representation theorem, we have $\langle Ax, y\rangle = \langle x, A^*y\rangle$.

Definition 2.6 (Conjugate operators)**.** Suppose A is a densely defined operator on H, we say the operator A^* is the conjugate operator of A if $D(A^*) = \{y \in H : \langle Ax, y\rangle \leq C_y||x||,\ \forall x \in D(A)\}$ and

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ holds for } \forall x \in D(A) \text{ and } y \in D(A^*).$$

Conjugate operators are great objects to study since they have quantities of good properties.

Proposition 2.7. Suppose A, B are densely defined, and $A \subseteq B$, then $B^* \subseteq A^*$.

Proof.

$$\begin{aligned}
y \in D(B^*) &\Leftrightarrow \langle Bx, y\rangle = \langle x, B^*y\rangle && \text{for } \forall x \in D(B);\\
&\Rightarrow \langle Ax, y\rangle = \langle x, B^*y\rangle && \text{for } \forall x \in D(B);\\
&\Rightarrow B^*y = A^*y && \text{for } \forall y \in D(B^*)\\
&\Rightarrow D(B^*) \subseteq D(A^*).
\end{aligned}$$

□

Proposition 2.8. A^* is a closed operator.

Proof. Suppose $\{y_n\} \subseteq D(A^*)$ and $(y_n, A^*y_n) \to (y, z)$. Then we have

$$\langle Ax, y\rangle = \lim_{n\to\infty}\langle Ax, y_n\rangle = \lim_{n\to\infty}\langle x, A^*y_n\rangle = \langle x, z\rangle \text{ for } \forall x \in D(A).$$

It implies $y \in D(A^*)$. □

Example 2.9. Suppose $A = -\Delta$, $D(A) = C_c^\infty(\mathbb{R}^n)$, compute $D(A^*)$.

Proof. It's easy to verify that $H^2(\mathbb{R}^n) \subseteq D(A^*)$. Then we aim to show that $H^2 = D(A^*)$. Since Fourier transform is a unitary transform on L^2, we have

$$\langle -\Delta f, g\rangle = \langle |\xi|^2\hat{f}, \hat{g}\rangle = \langle \hat{f}, |\xi|^2\hat{g}\rangle.$$

Since C_c^∞ is dense in L^2, then it force that $|\xi|^2\hat{g} \in L^2$, and it implies that $D(-\Delta^*) = H^2$. □

In order to describe further properties of A^*, we give the following lemma.

Lemma 2.10. $G(A^*) = \{(z, A^*z) : z \in D(A^*)\} = \left[V\overline{G(A)}\right]^\perp$, where V is a unitary operator on $H \times H$ defined as $V(x, y) = (y, -x)$.

Proof. First observe that $\left[V\overline{G(A)}\right]^\perp = [VG(A)]^\perp$. Then $(y, z) \in [VG(A)]^\perp \Leftrightarrow (y, z)\perp V(x, Ax) \Leftrightarrow \langle Ax, y\rangle = \langle x, z\rangle$ for $\forall x \in D(A)$. □

Then we can prove an important theorem for conjugate operators.

Theorem 2.11. Suppose A is densely defined, then we have:
(1). A is closable $\Leftrightarrow D(A^*)$ is dense in H;
(2). if A is closable, then $A^{**} = \bar{A}$.

Proof. For the necessity part, we prove it by contradiction. If $\overline{D(A^*)} \neq H$, then there exists a $y \neq 0$, s.t. $y \in (D(A^*))^{\perp}$. But we note that

$$\langle (0,y), V(x, A^*x)\rangle = -\langle y, x\rangle = 0, \text{ for } \forall x \in D(A^*.)$$

However, it implies that

$$(0,y) \in [V(G(A^*))]^{\perp} = \left[V\left[V(G(A))^{\perp}\right]\right]^{\perp} = \overline{G(A)}.$$

Then it leads contradiction!

Conversely, we can consider A^{**}, and we aim to show that $G(A^{**}) = \overline{G(A)}$. Using the lemma above, we have

$$G(A^{**}) = \left[V\overline{G(A^*)}\right]^{\perp} = \left[V\overline{(V[G(A)]^{\perp})}\right]^{\perp} = \overline{G(A)}.$$

So A is closable.

The proof of (2) is contained in the proof of (1). □

In the undergraduate functional analysis, we have orthogonal decomposition theorem for the bounded operators, and here, the statement holds still.

Theorem 2.12 (Orthogonal decomposition)**.** Let A be a densely defined operator. $H = \overline{\text{Ran}(A)} \oplus \ker(A^*)$.

Proof. It suffices to show $\ker(A^*) = \text{Ran}(A)^{\perp}$. We compute the process as following:

$$\begin{aligned}
\forall y \in \ker(A^*) &\Leftrightarrow A * y = 0 \\
&\Leftrightarrow \langle x, A^*y\rangle = 0 \text{ for } \forall x \in D(A) \text{ since } D(A) \text{ is dense,} \\
&\Leftrightarrow \langle Ax, y\rangle = 0 \text{ for } \forall x \in D(A), \\
&\Leftrightarrow y \perp \text{Ran}(A) \\
&\Leftrightarrow ker(A^*) = \text{Ran}(A)^{\perp}
\end{aligned}$$

□

2.3 Self-adjoint Operators and Symmetric Operators

In this subsection, we generalize the definitions of self-adjoint operators and symmetric operators.

Definition 2.13 (Symmetric operators)**.** Let $A : D(A) \to H$ be a densely defined operator. A is called a symmetric operator if the formula

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

holds for any $x,\ y \in D(A)$.

We should notice the domain here since we usually don't know the relation between $D(A)$ and $D(A^*)$.

An important property of symmetric operators is the following lemma.

Lemma 2.14. Symmetric operators are closable operators.

Proof. By the criteria for the closable operators, let $\{x_n\} \subseteq D(A)$ converge to 0, and then it suffices to show $Ax_n \to 0$. Assume that $Ax_n \to y$. Then we have

$$\langle y, w\rangle = \lim_{n\to\infty} \langle Ax_n, w\rangle = \lim_{n\to\infty} \langle x_n, Aw\rangle = 0$$

hold for $\forall w \in D(A)$. Then it implies $y = 0$ since $D(A)$ is dense. $\square$

The following definition is slightly different from the definition of symmetric operators.

Definition 2.15 (Self-adjoint operators)**.** Let A be a densely defined operator. A is called a self-adjoint operator if $D(A) = D(A^*)$ and $Ax = A^*x$ holds for $\forall x \in D(A^*)$.

Using the properties above, we can have a useful observation that if A is symmetric, then A is closable, i.e. A^* is densely defined, which implies that $A^{**} = \bar{A}$.

Now we give a summary for the properties above.

Proposition 2.16. (1). If A is symmetric, then $A \subseteq A^{**} = \bar{A} \subseteq A^*$.

(2). If A is symmetric closed, then $A = A^{**} \subseteq A^*$.

(3). If A is self-adjoint, then $A = A^{**} = A^*$.

There are more conceptions for different kinds of operators, like essential self-adjoint operators and so on; see [3] for details.

Now it comes to a pratical problem, how can we find a self-adjoint operator? The following theorem helps a lot since we can always verify if an operator is symmetric.

Theorem 2.17. Let A be a symmetric operator, then the followings are equivalent:
(1). A is self-adjoint;
(2). A is a closed operator and $\ker(A^* \pm \mathrm{i}) = \{0\}$;
(3). A is a closed operator and $\mathrm{Ran}(A \mp \mathrm{i}) = H$.

Proof. We can prove that (2) and (3) are equivalent first. By orthogonal decomposition, (2) implies (3). Conversely, we can see $\mathrm{Ran}(A\mp)$ is dense. Since A is closed and $||(A \mp \mathrm{i})|| \geq ||x||$, we can see the range is actually closed.

Then, if A is self-adjoint, assuming $\phi \in \ker(A + \pm\mathrm{i})$, then we have

$$0 = \langle (A^* + \mathrm{i})\phi, \phi\rangle = \langle A^*\phi, \phi\rangle \pm \langle \phi, \phi\rangle.$$

And it implies $\phi = 0$, so we have (1)⇒(2).

Conversely, it suffices to show $D(A^*) \subseteq D(A)$ since $A \subseteq A^*$. Let $\phi \in D(A^*)$. Since $\mathrm{Ran}(A - i) = H$, there is an $\eta \in D(T)$ so that $(A - \mathrm{i})\eta = (A^* - \mathrm{i})\phi$. $D(A) \subseteq D(A^*)$. so $\phi - \eta \in D(A^*)$ and

$$(A^* - \mathrm{i})(\phi - \eta) = 0.$$

Since $\text{Ran}(A + \mathrm{i}) = H$, $\ker(A^* - \mathrm{i}) = \{0\}$, so $\phi = \eta \in D(A)$. This ends the proof. $\square$

2.4 Spectrum of Linear Operators

In the undergraduate functional analysis, we already define the conception of spectrum of bounded operators, here we just generalize slightly.

Definition 2.18 (Bounded inverse)**.** Let A be a densely defined operator. We say A has bounded inverse if there exists $A^{-1} : H \to D(A)$ satisfying the followings:
(1). $AA^{-1} = \boldsymbol{I}_H$ and $A^{-1}A = \boldsymbol{I}_{D(A)}$;
(2). A^{-1} is bounded.

Definition 2.19 (Resolvent sets and spectrum)**.** We say $\rho(A)$ is resolvent set of A if $\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ has bounded inverse}\}$ and $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the spectrum of A. More precisely, the point spectrum is $\sigma_p(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda\boldsymbol{I}) \neq \{0\}\}$, the continuous spectrum is $\sigma_c(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda\boldsymbol{I}) = \{0\} \text{ and } \text{Ran}(A - \lambda\boldsymbol{I}) \neq H \text{ but } \overline{\text{Ran}(A - \lambda\boldsymbol{I})} = H\}$ and residual spectrum is $\sigma_r(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda\boldsymbol{I}) = \{0\} \text{ and } \overline{\text{Ran}(A - \lambda\boldsymbol{I})} \neq H\}$.

Here we do not define discrete spectrum and essential spectrum for this time being, since we do not define spectral projections.

Now we begin to discuss the properties of the spectrum of self-adjoint operators.

Theorem 2.20. Let A be a self-adjoint operator. Then:
(1). $\sigma(A) \subseteq \mathbb{R}$;
(2). $\sigma_r(A) = \varnothing$;
(3). eigenfunctions of different eigenvalues are orthogonal.

Proof. (1). By the criteria of self-adjoint operators, we have $\ker(A + a + b\mathrm{i}) = \{0\}$ and $\text{Ran}(A + a + b\mathrm{i}) = H$ if $b \neq 0$. And we have coercive estimate

$$\Im\langle (A + a + b\mathrm{i})x, x\rangle = b\langle x, x\rangle.$$

It implies that

$$b||x|| \leq ||(A + a + b\mathrm{i})x||.$$

So we can see $a + b\mathrm{i} \in \rho(A)$.

(2). By orthogonal decomposition, we have

$$0 = \ker(A - \lambda) = \ker(A^* - \lambda) = (\text{Ran}(A - \lambda))^{\perp}$$

holding for $\forall \lambda \in \sigma(A) \setminus \sigma_r(A)$. And then we have $\text{Ran}(A - \lambda) = H$. So $\sigma_r(A) = \varnothing$.

(3). Note that we have

$$\lambda\langle \varphi, \psi\rangle = \langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle = \mu\langle \varphi, \psi\rangle.$$

$\square$

Example 2.21. Since the last example, we know that $-\Delta$ is a self-adjoint operator on $H^2(\mathbb{R}^d)$, and here we can compute its spectrum $\sigma(-\Delta) = [0, \infty)$.

Proof. Consider the resolvent equation $(A-\lambda)u=f$, then we have $(|\xi|^2-\lambda)\hat{u}=\hat{f}$. It implies that $\lambda\in\sigma(A)$ if and only if $\lambda\geq 0$. □

3 Spectral Theory

After a brief introduction to unbounded operators, we enter the most important part of our notes, the spectral theory. We will begin with the spectral theory of bounded operators, and then generalize it. This section is mainly based on [3][4].

3.1 Spectral Theory of Bounded Operators

In the undergraduate functional analysis, we have shown the following theorem.

Theorem 3.1. Let A be a bounded operator, then:
(1). $\rho(A)$ is open;
(2). $\sigma(A)$ is closed;
(3). $R_A(\lambda)=(A-\lambda\boldsymbol{I})^{-1}$ is an analytic function with respect to $\lambda\in\rho(A)$;
(4). if $\lambda,\ \mu\in\rho(A)$, then $R_A(\lambda)R_A(\mu)=R_A(\mu)R_A(\lambda)$;
(5). $R_A(\lambda)-R_A(\mu)=(\lambda-\mu)R_A(\lambda)R_A(\mu)$;
(6). if A and B are densely defined, then for $\forall z\in\rho(A)\cap\rho(B)$, we have $R_A(z)-R_B(z)=R_A(z)(B-A)R_B(z)$;
(7). (Spectral Radius Formula) $r(A)=\lim\limits_{n\to\infty}||A^n||^{\frac{1}{n}}$;
(8). if A is self-adjoint operator, then $||A^*A||_{op}=||A||_{op}^2$.

Our idea for proving continuous functional calculus is Stone-Weierstrass Approximation Theorem and polynomial functional calculus. To begin with, we have the following lemma.

Lemma 3.2. Let A be a bounded operator, and $p(x)$ be a polynomial $p(x)=\sum\limits_{i=0}^{n}a_ix^i$. Then we have $\sigma(p(A))=p(\sigma(A))=\{p(\lambda):\lambda\in\sigma(A)\}$.

Proof. First, we show that $p(\sigma(A))\subseteq\sigma(p(A))$. Suppose $\lambda\in\sigma(A)$, then we have $p(x)-p(\lambda)=(x-\lambda)q(x)$, which implies that $p(A)-p(\lambda)=(A-\lambda)q(A)$. If we have $p(\lambda)\in\rho(p(A))$, then we have $\boldsymbol{I}=(A-\lambda)q(A)(p(A)-p(\lambda))^{-1}$. But it implies that $A-\lambda$ is invertible, and then $\lambda\in\rho(A)$. It leads contradiction.

Conversely, it suffices to show $\sigma(p(A))\subseteq p(\sigma(A))$. Let $z\in\sigma(p(A))$. By fundamental theorem of algebra, we have $p(x)-z=\prod\limits_{i=1}^{n}(x-\lambda_i(z))$, and then it implies

$$p(A)-z\boldsymbol{I}=\prod_{i=1}^{n}(A-\lambda_i(z)\boldsymbol{I}).$$

Since $z\in\sigma(p(A))$, there is j s.t. $A-\lambda_j(z)$ is not invertible, so we have $\lambda_j(z)\in\sigma(A)$ such that $z=p(\lambda_j(z))$. Then we have $\sigma(p(A))\subseteq p(\sigma(A))$. □

Then we have a further lemma.

Lemma 3.3. Let A be a bounded self-adjoint operator, then we have

$$||p(A)||_{op} = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = ||p||_{L^\infty(\sigma(A))}.$$

Proof. First we have a simple observation that if A is self-adjoint, then $p(A)$ is also a self-adjoint operator.

Then we casn compute as following:

$$\begin{aligned} ||p(A)||_{op}^2 &= ||p(A)^* p(A)||_{op} = ||(\bar{p}p)(A)||_{op} \\ &= r((\bar{p}p)(A)) = \sup_{\lambda \in \sigma((\bar{p}p)(A))} |\lambda| \\ &= \sup_{\lambda \in (\bar{p}p)(\sigma(A))} |\lambda| = \sup_{\lambda \in \sigma(A)} |(\bar{p}p)(\lambda)| \\ &= \sup_{\lambda \in \sigma(A)} |p(\lambda)|^2. \end{aligned}$$

Then we show that $||p(A)||_{op} = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = ||p||_{L^\infty(\sigma(A))}$. □

After a long preparation, we can state the most important conclusion for this subsection.

Theorem 3.4 (Continuous functional calculus)**.** Let A be a bounded self-adjoint operator. Then there is a unique mapping $\phi_A : \mathcal{C}(\sigma(A)) \to L(H)$ satisfying the following properties.
(1). ϕ_A is an algebraic *-homomorphism, that is to say, for $\forall f,\ g \in \mathcal{C}(\sigma(A)), \lambda \in \mathbb{C}$, the following formulae hold

$$\begin{cases} \phi_A(\boldsymbol{I}) = \boldsymbol{I}_H \\ \phi_A(\bar{f}) = \phi_A(f)^* \\ \phi_A(fg) = \phi_A(f)\phi_A(g) \\ \phi_A(f+g) = \phi_A(f) + \phi_A(g) \\ \phi_A(\lambda f) = \lambda \phi_A(f). \end{cases}$$

(2). ϕ_A is isometric, that is to say, $||\phi_A(f)||_{op} = ||f||_{L^\infty(\sigma(A))}$.
(3). If $f(x) = x$, then $\phi_A(f) = A$.
(4). (Spectral mapping theorem) $\sigma(f(A)) = f(\sigma(A))$.
(5). If $f(x) \geq 0$ for $\forall x \in \sigma(A)$, then $f(A) \geq 0$ and $\langle f(A)x, x\rangle \geq 0$.
(6). If $A\psi = \lambda\psi$, then $f(A)\psi = f(\lambda)\psi$.

Proof. (1), (2) and (3) are easy to verify by the previous lemmas and the density of polynomials in $\mathcal{C}(\sigma(A))$(Stone Weierstrass theorem).

(4). We first show that $f(\sigma(A)) \subseteq \sigma(f(A))$. Suppose $\lambda \in \sigma(A)$, it suffices to show $f(\lambda) \in \sigma(f(A))$. Let $\chi_\varepsilon = \chi(\frac{x-\lambda}{\varepsilon})$ be a smooth and compact supported function with $||\chi||_{L^\infty} = 1$. Then there is a function ψ_ε with $||\psi_\varepsilon|| = 1$ such that

$$||\chi_\varepsilon(A)\psi_\varepsilon|| = ||\chi_\varepsilon||_{op} = ||\chi_\varepsilon||_{L^\infty(\sigma(A))} = 1.$$

So, we have

$$\lim_{\varepsilon\to 0}||(f(A)-f(\lambda))\chi_\varepsilon(A)\psi_\varepsilon|| \le \lim_{\varepsilon\to 0}(\sup_{\mu\in\sigma(A)}|f(\mu)-f(\lambda)|)||\chi_\varepsilon||_{L^\infty(\sigma(A))} = 0.$$

It implies that $f(\lambda)\in\sigma(f(A))$. Conversely, we show that $\sigma(f(A))\subseteq f(\sigma(A))$. Let $z\in\sigma(f(A))$. If $z\notin f(\sigma(A))$, then we can define a function $\frac{1}{f(\lambda)-z}$ which is a continuous function with respect to $\lambda\in\sigma(A)$. Then it implies that $z\in\rho(f(A))$, but it leads to contradiction.

(5) and (6) can be verified directly by (4). □

Corollary 3.5 (Square-root lemma). Let A be a positive defined and bounded self-adjoint operator, which is to say, $\langle Ax, x\rangle \ge 0$. Then $\sigma(A)\subseteq[0,\infty)$ and there is a self-adjoint operator $B\ge 0$ such that $A=B^2$.

Proof. Since the spectrum of self-adjoint operator is always real, then for any $\lambda\le 0$, we have

$$\langle (A-(-\lambda\boldsymbol{I}))x, x\rangle \ge \lambda||x||^2.$$

It implies that $\sigma(A)\subseteq[0,\infty)$. Then we can consider $f(x)=\sqrt{x}$, and Let $B=f(A)$. □

Corollary 3.6 (Spectral distance). Let A be a bounded self-adjoint operator. Then we have

$$||R_\lambda(A)||_{op} = \frac{1}{\mathrm{dist}(\lambda,\sigma(A))}.$$

Proof. It's the direct application of Spectral mapping theorem. Since

$$||R_\lambda(A)||_{op} = \left|\left|\frac{1}{A-\lambda}\right|\right|_{op} = \left|\left|\frac{1}{x-\lambda}\right|\right|_{L^\infty(\sigma(A))} = \frac{1}{\mathrm{dist}(\lambda,\sigma(A))}.$$

□

3.2 Borel Calculus and Spectral Theorem

Since we usually deal with Borel measurable functions, we need to generalize our theory further.

First, we introduce an acknowledgement theorem.

Theorem 3.7 (Riesz representation theorem). Let X be a compact Hausdorff space. Then $\mathcal{C}(X)^*$ is isometrically isomorphic to $M(X)$, where $M(X)$ denotes the space of the complex Radon measures on X.

Definition 3.8 ((Scalar)Spectral measure). Let A be a bounded self-adjoint operator. For $\forall x, y\in H$, we define a linear functional $l_{x,y}:\mathcal{C}(\sigma(A))\to\mathbb{C}$, $f\mapsto\langle x, f(A)y\rangle$. By Riesz representation theorem, there is a unique finite regular Borel measure $\mathrm{d}\mu_{x,y}(z)$ such that

$$l_{x,y}(f) = \int_{\sigma(A)} f(z)\mathrm{d}\mu_{x,y}(z)$$

holding for $\forall f\in\mathcal{C}(\sigma(A))$. Then we say $\mathrm{d}\mu_{x,y}$ is the spectral measure corresponding to A.

Remark. Now we get a family of measures indeed, so we cannot treat Null sets casually as usual.

We first generalize our theory to bounded Borel measurable functions. Let $\mathcal{B}_b(\sigma(A))$ denote the family of functions we consider now. And we endow the supremum norm on it, that is to say, $||f||_{\mathcal{B}_b} = ||f||_{L^\infty(\sigma(A))}$.

Theorem 3.9 (Borel functional calculus)**.** Let A be a bounded self-adjoint operator. Then there is a unique $\phi_A : \mathcal{B}_b(\sigma(A)) \to L(H)$, $f \mapsto f(A)$, satisfying the following properties:
(1). ϕ_A is an algebraic *-homomorphism.
(2). ϕ_A is norm continuous: $||\phi_A(f)||_{op} \leq ||f||_{\mathcal{B}_b}$.
(3). Let $f(x) = x$, and then $\phi_A(f) = A$.
(4). Suppose $f_n(x) \to f(x)$ for each x and $\sup_n ||f_n||_{\mathcal{B}_b}$ is bounded. Then $\phi_A(f_n) \to \phi_A(f)$ strongly.
Moreover ϕ_A has the properties:
(5). If $A\psi = \lambda\psi$, then $\phi_A(f)\psi = f(\lambda)\psi$.
(6). If $f \geq 0$, then $\phi_A(f) \geq 0$.
(7). If $BA = AB$, then $\phi_A(f)B = B\phi_A(f)$.

Before our proof, it is worth noting that we do not have the same Spectral Mapping Theorem as before and the conclusion here uses the conception of essential range whose definition is

$$\mathrm{essRan}_E(f) = \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, E(\{z \in \sigma(A) : |f(z) - \lambda| < \varepsilon\}) \neq 0\}$$

where E denotes the spectral projection measure. We will introduce this result later.

Proof. For $\forall f \in \mathcal{B}_b(\sigma(A))$, we can define a bilinear functional

$$a_f(x, y) = \int_{\sigma(A)} f(z) \mathrm{d}\mu_{x,y}(z)$$

for $\forall x, y \in H$. Then it is easy to see that

$$|a_f(x, y)| = ||f||_{\mathcal{B}_b} ||\mu_{x,y}||_{TV} \leq ||f||_{\mathcal{B}_b} ||x||_H ||y||_H.$$

By Riesz representation theorem, there is unique bounded linear operator $f(A)$ such that

$$\langle x, f(A)y \rangle = a_f(x, y) = \int_{\sigma(A)} f(z) \mathrm{d}\mu_{x,y}(z).$$

Now we successfully generalize our domain of *-homomorphism from continuous functions to bounded Borel functions by the integral of spectral measure. (1), (2) and (3) are easy to verify since we can always Approximate a measurable function almost everywhere by continuous function

(4). Now let f_n converge to f for each x. To show strong convergence, it suffices to show

$$||\phi_A(f_n)x - \phi_A(f)x||^2 = \langle \phi_A(f_n - f)x, \phi_A(f_n - f)x \rangle = \langle x, \phi_A(|f_n - f|^2)x \rangle \to 0.$$

Then it is equivalent to show

$$\int_{\sigma(A)} |f_n - f|^2 \mathrm{d}\mu_{x,x} \to 0.$$

However, we already know that f_n is uniformly bounded and converges pointwise, and then we can use dominated convergence theorem with respect to the measure $\mu_{x,x}$. Thus (4) is finished.

(5), (6) and (7) are quite direct. $\square$

In the last theorem, we constructed a functional calculus, $f \mapsto f(A)$ for any bounded Borel functions and any bounded self-adjoint operator A. The most important functions gained in passing from the continuous functional calculus to the Borel functional calculus are the characteristic functions of sets.

We first give the abstract definition of projection-valued measure(p.v.m.).

Definition 3.10 (Projection-valued measure(p.v.m.))**.** Let $E : \mathcal{B}(\mathbb{R}) \to P(H)$. The family $\{E(\cdot)\}$ of spectral projections of a bounded self-adjoint operator, which has the following properties:
(1). Each $E(\Omega)$ is an orthogonal projection.
(2). $E(\varnothing) = 0$, $E((-a, a)) = \boldsymbol{I}$ for some a.
(3). $E(B_1 \cap B_2) = \mathbf{1}_{B_1}\mathbf{1}_{B_2}$.
(4). Suppose $\{B_i\}_{i=1}^{\infty}$ is disjoint pairwise, then $E(\cup B_i) = s - \sum\limits_{i=1}^{\infty} E(B_i)$.
is called projection-valued measure(p.v.m.).

Remark. Here (1) and (4) imply (3). Condition (2) guarantees that the a p.v.m. must be associated with certain self-adjoint self-adjoint operator.

Now, we can introduce a natural example from what we have investigated above.

Definition 3.11 (spectral projections of A)**.** Let A be a bounded self-adjoint operator and ϕ_A the Borel functional calculus constructed above. For each Borel set $B \subset \sigma(A)$ define

$$E(B) := \chi_B(A),$$

where χ_B is the characteristic function of B.

Proposition 3.12. The map $E : \mathcal{B}(\sigma(A)) \to L(H)$ defined by $E(B) = \chi_B(A)$ is projection-valued and countably additive in the strong operator topology. In particular, $\{E(B)\}$ is a projection-valued measure, called the spectral measure of A.

Proof. Since $\chi_B^2 = \chi_B = \overline{\chi_B}$, by the *-homomorphism property we have

$$E(B)^2 = \chi_B(A)^2 = \chi_B^2(A) = \chi_B(A) = E(B), \quad E(B)^* = E(B),$$

so $E(B)$ is an orthogonal projection.

Clearly $E(\emptyset) = 0$ and $E(\sigma(A)) = \chi_{\sigma(A)}(A) = I$.

If B_1, B_2 are Borel sets, then $\chi_{B_1 \cap B_2} = \chi_{B_1}\chi_{B_2}$, hence

$$E(B_1 \cap B_2) = \chi_{B_1 \cap B_2}(A) = \chi_{B_1}(A)\chi_{B_2}(A) = E(B_1)E(B_2).$$

Now let (B_n) be pairwise disjoint Borel sets and put $B = \bigcup_{n=1}^{\infty} B_n$. Then for each λ,

$$\chi_B(\lambda) = \sum_{n=1}^{\infty} \chi_{B_n}(\lambda),$$

and defining $f_N = \sum_{n=1}^{N} \chi_{B_n}$ we have $f_N \to \chi_B$ pointwise and $\|f_N\|_\infty \leq 1$ for all N. By property (4) of the Borel functional calculus,

$$E(B) = \chi_B(A) = \lim_{N\to\infty} f_N(A) = \lim_{N\to\infty} \sum_{n=1}^{N} \chi_{B_n}(A) = s\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} E(B_n),$$

where the limit is in the strong operator topology. This shows that E is countably additive in the strong sense. $\square$

Although the p.v.m. is a definition which seems to be slightly abstract, we can see that it's just a generalization from the original scalar spectral measures.

Proposition 3.13. For each $x, y \in H$, the measure $\mu_{x,y}$ constructed from the Borel functional calculus satisfies

$$\mu_{x,y}(B) = \langle x, E(B)y \rangle, \qquad B \in \mathcal{B}(\sigma(A)).$$

Proof. For any Borel set B we have

$$\langle x, E(B)y \rangle = \langle x, \chi_B(A)y \rangle = \int_{\sigma(A)} \chi_B \mathrm{d}\mu_{x,y} = \mu_{x,y}(B),$$

by the very definition of $\mu_{x,y}$ as the representing measure for the functional $f \mapsto \langle x, f(A)y \rangle$. $\square$

Now, if we have a self-adjoint operator, by the characteristic functions, we can define $E_\Omega = \chi_\Omega(A)$ which forms a family of p.v.m.. Conversely, if we have a family of p.v.m. $\{E_\Omega\}$, then we can get a bounded self-adjoint operator A by the spectral integral $\int \lambda \mathrm{d}E_\lambda$, where $E_\lambda = E((-\infty, \lambda])$.

From the argument above, we can summarize as following.

Theorem 3.14. There is a one-one correspondence between bounded sel-adjoint operators A and bounded projection-valued measures $\{E_\Omega\}$ given by:

$$A \mapsto \{E_\Omega\} = \{\chi_\Omega(A)\}$$
$$\{E_\Omega\} \mapsto A = \int \lambda \mathrm{d}E_\Omega.$$

One may ask a subtle question where the projection-valued measures are supported. And the answer actually agrees with our intuition:

Proposition 3.15. Let the support of E be defined as $\operatorname{supp}(E) = \mathbb{R}\backslash\bigcup\{U : U \text{ is open and } E(U) = 0\}$. Then $\operatorname{supp}(E) = \sigma(A)$ where $A := \int \lambda \mathrm{d}E_\lambda$.

Proof. First, we show $\sigma(A) \subseteq \operatorname{supp}(E)$. If $\lambda_0 \notin \operatorname{supp}(E)$, then there is $\delta > 0$ such that $E((\lambda_0 - \delta, \lambda + \delta_0)) = 0$. Thus we can define a bounded Borel function

$$g(\lambda) = \begin{cases} \frac{1}{\lambda - \lambda_0} & \lambda \notin (\lambda_0 - \delta, \lambda + \delta_0) \\ 0 & \lambda \in (\lambda_0 - \delta, \lambda + \delta_0). \end{cases}$$

Then we can define a bounded self-adjoint operator

$$B = \int g(\lambda) \mathrm{d}E_\lambda.$$

Then we have

$$(A-\lambda_0\boldsymbol{I})B=(A-\lambda_0\boldsymbol{I})\int g(\lambda)\mathrm{d}E_\lambda=\int(\lambda-\lambda_0)g(\lambda)\mathrm{d}E_\lambda.$$

Since E_λ is a zero mapping on the inteval $(\lambda_0-\delta,\lambda+\delta_0)$, it has no contribution to the integral. So we have

$$(A-\lambda_0\boldsymbol{I})B=\int 1\mathrm{d}E_\lambda=\boldsymbol{I}.$$

Since B is bounded, then it implies $\lambda_0\notin\sigma(A)$. Thus we have shown that $\sigma(A)\subseteq\operatorname{supp}(E)$.

Conversely, if $\lambda_0\notin\sigma(A)$, then we can find a open set U containing λ_0 and not intersecting with $\sigma(A)$. Now, for any function h supported on U, by Spectral theorem, we have

$$h(A)=\int_{\sigma(A)}h(\lambda)\mathrm{d}E_\lambda=0.$$

Then for any $\xi\in H$, we have

$$0=\langle\xi,h(A)\xi\rangle=\int_{\sigma(A)}h(\lambda)\mathrm{d}\mu_\xi(\lambda).$$

If we assume $h\geq 0$, then μ_ξ is zero on U for any $\xi\in H$. Equivalently, it tells E is zero on U. And this implies that $\operatorname{supp}(E)\subseteq\sigma(A)$. □

3.3 Spectral Theory of Unbounded Operators

Now, we begin to extend our theory to the most general case, the Spectral theorem of unbounded self-adjoint operators. To generalize, we need to change the definition of p.v.m. slightly.

Definition 3.16 (p.v.m.)**.** Let E_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the measurable set $\Omega\subseteq\mathbb{R}$. The family of operators has the following properties:
(1). Each $E(\Omega)$ is an orthogonal projection.
(2). $E(\varnothing)=0$, $E((-\infty,\infty))=\boldsymbol{I}$.
(3). $E(B_1\cap B_2)=\mathbf{1}_{B_1}\mathbf{1}_{B_2}$.
(4). Suppose $\{B_i\}_{i=1}^\infty$ is disjoint pairwise, then $E(\cup B_i)=s-\lim_{i=1}^{\infty}E(B_i)$.

Then we can state the key theorem in this section. We first list some notations as following:

$$\Phi(f)=\int_\Omega f\mathrm{d}E,\quad E_{x,y}=\langle x,E(B)y\rangle,$$

$$D(\Phi(f))=D_f=\{y\in H:\int_\Omega|f|^2\mathrm{d}E_{y,y}<\infty\}.$$

Theorem 3.17 (Spectral theorem, spectral integral form)**.** Suppose $(E,\Omega,\mathcal{B}(\Omega))$ being a spectral measure space. f is a measurable function in it. Then we have
(1). D_f is a dense linear subspace.

(2). $\forall x \in H, y \in D_f$, the estimate holds

$$\int_\Omega |f| \mathrm{d}|E_{x,y}| \leq ||x|| \left(\int_\Omega |f|^2 \mathrm{d}E_{y,y} \right)^{\frac{1}{2}}.$$

(3). There is a unique densely defined closed operator denoted as $\Phi(f)$ whose domain is D_f and it satisfies

$$\langle x, \Phi(f)y \rangle = \int_\Omega f(\lambda) \mathrm{d}E_{x,y}(\lambda) \quad \text{for } \forall x \in H,\ y \in D_f.$$

(4). $||\Phi(f)||^2 = \int_\Omega |f|^2 \mathrm{d}E_{x,x}$ for $\forall x \in D_f$.

(5). calculus properties as following where f and g are both measurable functions:

(i). $\phi(f) + \Phi(g) \subseteq \Phi(f+g)$, $D(\Phi(f) + \Phi(g)) \subseteq D_f \cap D_g$.

(ii). $\Phi(\lambda f) = \lambda \Phi(f)$.

(iii). $\Phi(f)\Phi(g) \subseteq \Phi(fg)$, $D(\Phi(f)\Phi(g)) \subseteq D_g \cap D_{fg} \subseteq D_{fg}$.

(6). $\Phi(\bar{f}) = \Phi(f)^*$. In particular, $\Phi(f)$ is self-adjoint if and only if $f|_\Omega$ is a real valued function.

(7). $D_f = H$ if and only if $f \in \mathcal{B}_b(\Omega)$.

Proof. (1). We first show that D_f is a linear subspace. For scalar multiplication, we have

$$\int_\Omega |f|^2 \mathrm{d}E_{\alpha x, \alpha x} = \alpha^2 \int_\Omega |f|^2 \mathrm{d}E_{x,x}.$$

For addition, we have polarization identity, so

$$E_{x+y,x+y} \leq 2E_{x,x} + 2E_{y,y}.$$

Then it suffices to show the density. For $n = 1, 2, 3 \cdots$, let A_n be the subset of Ω in which $|f_n| < n$. If $x \in H$, let $x_n = E(A_n)x$. Since

$$E_{x_n,x_n}(A) = \langle x_n, A x_n \rangle = \langle x, E(A \cap A_n)x \rangle,$$

we have

$$\int_\Omega |f|^2 \mathrm{d}E_{x_n,x_n} = \int_\Omega |f|^2 \mathbf{1}_{A_n} \mathrm{d}E_{x,x} < n^2 ||x||^2 < \infty.$$

Thus $x_n \in D_f$, and then it implies that D_f is dense.

(2). If $x \in H$, $y \in D_f$, the Radon Nikodym theorem, there is a function h_n with modulus 1 such that

$$h_n f \mathbf{1}_{A_n} \mathrm{d}E_{x,y} = \mathbf{1}_{A_n} |f| \mathrm{d}|E_{x,y}|.$$

Then we have

$$\begin{aligned}
\int_\Omega \mathbf{1}_{A_n} |f| \mathrm{d}|E_{x,y}| &= \langle x, \Phi(\mathbf{1}_{A_n} h_n f) y \rangle \\
&\leq ||x|| ||\Phi(\mathbf{1}_{A_n} h_n f) y|| \\
&= ||x|| \left(\int_\Omega |f \mathbf{1}_{A_n} h_n|^2 \mathrm{d}E_{y,y} \right)^{\frac{1}{2}} \\
&\leq ||x|| \int_\Omega |f|^2 \mathrm{d}E_{y,y}.
\end{aligned}$$

Using dominated convergence theorem leads the desired estimate.

(3). For a fixed $y \in D_f$, consider

$$l_y(x) = \int_\Omega f \mathrm{d}E_{x,y}.$$

Since y is fixed, by (2), we have

$$|l_y(x)| \lesssim_y ||x||,$$

which means l_y is a bounded linear functional. By Riesz representation theorem, there is a unique operator $T_f : D_f \to H$ such that

$$\langle x, T_f y\rangle = \int_\Omega f \mathrm{d}E_{x,y}.$$

It remains to show $\Phi(f) = T_f$ is a closed operator. After showing (6), we can use $\Phi(f) = \Phi(\bar{f})^*$ to prove it.

(4). Considering approximation sequence $f_n = f\mathbf{1}_{A_n}$, since f_n is bounded, we can see $D_{f-f_n} = D_f$. Then we have

$$\langle x, \Phi(f - f_n)y\rangle = \int_\Omega f - f_n \mathrm{d}E_{x,y} = \langle x, \Phi(f)y\rangle - \langle x, \Phi(f_n)y\rangle = \langle x, (\Phi(f) - \Phi(f_n))y\rangle.$$

Thus we have

$$\Phi(f - f_n) = \Phi(f) - \Phi(f_n).$$

Then using norm inequality and dominated convergence theorem, we have

$$\lim_n ||\Phi(f)y - \Phi(f_n)y||^2 = \lim_n ||\Phi(f - f_n)y||^2 \leq \lim_n \int_\Omega |f - f_n|^2 \mathrm{d}E_{y,y} = 0.$$

Then using the conclusions of bounded case, we have

$$||\Phi(f)y||^2 = \lim_n ||\Phi(f_n)y||^2 = \lim_n \int_\Omega |f_n|^2 \mathrm{d}E_{y,y} = \int_\Omega |f|^2 \mathrm{d}E_{y,y}.$$

(5). (i) and (ii) are quite direct, and then it remains to show (iii). In the bounded case, we have shown that $\Phi(f_n)\Phi(g_m) = \Phi(f_n g_m)$. Now, for any $x \in D_g \cap D_{fg}$, by the strong limit(see (4)), we have

$$\Phi(f_n)\Phi(g)x = \lim_m \Phi(f_n)\Phi(g_m)x = \lim_m \Phi(f_n g_m)x = \Phi(f_n g)x.$$

Let $y = \Phi(g)x$. Then we have

$$\int_\Omega |f_n|^2 \mathrm{d}E_{y,y} = \int_\Omega |f_n g|^2 \mathrm{d}E_{x,x}.$$

By monotone convergence theorem, we see that $y \in D_f$ if and only if $x \in D_{fg}$. So we see that $D(\Phi(f)\Phi(g)) = D_g \cap D_{fg} \subseteq D_{fg}$.

(6). We first show that $D(\Phi(\bar{f})) \subseteq D(\Phi(f)^*)$. Suppose $x \in D(f)$ and $y \in D_{\bar{f}} = D_f$. It follows from the strong limit and bounded calculus that

$$\langle \Phi(f)x, y\rangle = \lim_n \langle \Phi(f_n)x, y\rangle = \lim_n \langle x, \Phi(f_n)^* y\rangle = \lim_n \langle x, \Phi(\bar{f}_n)y\rangle = \langle x, \Phi(\bar{f})y\rangle.$$

Thus $y \in D(\Phi(f)^*)$, and

$$\Phi(\bar{f}) \subseteq \Phi(f)^*.$$

To get another side, we have to show that every $y \in D(\Phi(f)^*)$ lies in D_f. For any $x \in H$, since $f_n = f\mathbf{1}_{A_n}$, then the multiplication theorem gives

$$\begin{aligned}\langle x, \Phi(\bar{f}_n)y\rangle &= \langle \Phi(f_n)x, y\rangle = \langle \Phi(f)\Phi(\mathbf{1}_{A_n})x, y\rangle \\ &= \langle \Phi(\mathbf{1}_{A_n})x, \Phi(f)^* y\rangle = \langle x, \Phi(\mathbf{1}_{A_n})\Phi(f)^* y\rangle\end{aligned}$$

Thus $\Phi(\bar{f}_n)y = \Phi(\mathbf{1}_{A_n})\Phi(f)^* y$ for any $y \in D(\Phi(f)^*)$. Let $v = \Phi(f)^* y$, then

$$\int_\Omega |f_n|^2 \mathrm{d}E_{y,y} = ||\Phi(f)_n y||^2 = \langle \Phi(f_n)y, \Phi(f_n)y\rangle = \int_\Omega |\mathbf{1}_{A_n}|^2 \mathrm{d}E_{v,v} \leq ||v||^2.$$

Thus $y \in D_f = D_{\bar{f}}$.

(7). It suffices to the necessity thanks to the bounded Borel calculus. Assume $D_f = H$. Since $\Phi(f)$ is a closed operator, then $\Phi(f)$ is a bounded operator. Here we need an equality stronger then norm continuity which need to agree $\mathcal{B}_b$ with L^∞ whose elements are actually equivalent classes. That is to say, we need

$$||f_n||_{L^\infty} = ||\Phi(f_n)||_{op},$$

where f_n is a truncate function. Then we have

$$||f_n||_{L^\infty} = ||\Phi(f_n)||_{op} = ||\Phi(\mathbf{1}_{A_n})\Phi(f)||_{op} \leq ||\Phi(\mathbf{1}_{A_n})||_{op}||\Phi(f)||_{op} \leq ||\Phi(f)||_{op}.$$

Thus $||f||_{L^\infty} < \infty$. □

Remark. In the previous section, we seem not to treat $\mathcal{B}_b$ the same as L^∞. So Φ cannot become an injection. But if Φ is an injection, then we can prove that $\Phi : L^\infty \to L(H)$ is actually isometric by an argument of measure. If $f \in L^\infty$, then there is x with $||x|| = 1$ such that

$$||\Phi(f)||_{op} = \int_\Omega |f|^2 \mathrm{d}E_{x,x} \leq M\mu_{x,x}(B),$$

where $B = \{|f| \leq M\}$. Let $M = ||f||_{L^\infty}$, and it implies that $||\Phi(f)||_{op} \leq ||f||_{L^\infty}$.

3.4 Cayley Transform and Decomposition of Spectral Measure

For this part, we just introduce some results and do not give the concrete proofs of them.

We first consider Cayley transform, which establish a connection between self-adjoint operators and unitary operators.

Definition 3.18 (Cayley transform)**.** For a self-adjoint operator A, let $U = (A - \mathrm{i}\boldsymbol{I})(A + \mathrm{i}\boldsymbol{I})^{-1}$ be the Cayley transform of A.

Proposition 3.19. If A is self-adjoint, then U is unitary.

In fact, every unitary operator whose spectrum does not contain 1 can be obtained by the inverse of Cayley transform.

If we have a self-adjoint operator A, then we can investigate the structure of its spectral measure. When considering the scalar spectral measures, we can classify them as what we do in Lebesgue-Radon-Nikodym theorem. And we can pull the conclusion back to the general spectral measure. Then we obtain the following theorem.

Theorem 3.20. If A is a self-adjoint operator, then

$$H = H_{pp} \oplus H_c = H_{pp} \oplus H_{ac} \oplus H_{sc}.$$

Furthermore, we can restrict the operator A to these spectrums by p.v.m. to get the decomposition of operator itself.

3.5 Riesz Projector

In this subsection, we introduce Riesz projectors to treat the operator which is not self-adjoint. We first introduce some basic conceptions.

Definition 3.21 (Vector-valued analytic function)**.** Let X be a Banach space and f is Banach-valued. If for each $z \in D$, there is $g(z) \in X$ such that

$$\lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} = g(z),$$

then we say that f is a strongly analytic function on D.

If for each $x^* \in X^*$, $x^* f$ is analytic, then we say it is a weak analytic function on D.

Definition 3.22 (Line integral)**.** Suppose that Γ being a rectifiable curve and f is a continuous vector-valued function on D, then we can define the line integral

$$\int_\Gamma f(\xi) d\xi = \int_{[a,b]} f(\gamma(t)) \mathrm{d}m(t),$$

where $m(t)$ is a finite complex line measure.

Theorem 3.23. Weak analyticity implies strong analyticity.

Proof. We first show that weak analyticity implies continuity. Let

$$x_n = \frac{f(z + \xi_n) - f(z)}{\xi_n},$$

where $\xi_n \to 0$. By weak continuity, we see $\lim_n x^*(x_n)$ exist for each $x^* \in X^*$. Then by principle of uniform boundness, we see $||x_n|| < M$.

Using Riemann sum and the analyticity, we have

$$x^* \int_\Gamma f(\xi) \mathrm{d}\xi = \int_\Gamma x^* f(\xi) \mathrm{d}\xi = 0.$$

Let $z \in \text{Int}\,(\Gamma)$, and then we see

$$x^* f(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma x^* \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = x^* \left(\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \right).$$

Then

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi.$$

As the standard argument in complex analysis, we know f is strongly analytic. □

Then we can begin to consider Riesz projectors.

Lemma 3.24. *A is injective and closed if and only if A^{-1} is closed.*

Proof. For the sufficiency, let $y_n \in D(A^{-1}) = \text{Ran}(A)$ and $Ax_n = y_n$. Then

$$\begin{cases} y_n \to y \\ x_n \to x \end{cases} \Rightarrow \begin{cases} Ax_n \to y \\ x_n \to x \end{cases}$$

Since A is closed, then we have $x \in D(A)$ and $y \in \text{Ran}(A) = D(A^{-1})$.

For the necessity, we can prove at the same way. □

Lemma 3.25. *If A is not closed, then $\sigma(A) = \mathbb{C}$.*

Proof. Suppose $\lambda \in \rho(A)$. Then we have $R_\lambda(A) = (A - \lambda \boldsymbol{I})^{-1}$ is bounded. Since $(A - \lambda \boldsymbol{I})^{-1}$ is defined on H, then it must be closed. By the previous lemma, we know that $A - \lambda \boldsymbol{I}$ is a closed operator. Then A is closed. contradiction! □

Definition 3.26 (Separable spectral subset)**.** $\sigma_1(A) \subseteq \sigma(A)$ is called a separable spectral subset if there is a contour Γ such that $\sigma_1(A)$ is contained in Γ and $\sigma_1(A)$ and $\sigma(A) \setminus \sigma_1(A)$ are both closed in $\mathbb{C}$. We call such a contour as an admissable contour.

Definition 3.27 (Riesz projectors)**.** Let $R(\lambda, A) = \frac{1}{\lambda - A}$. We say P is a Riesz projector if

$$P = \frac{1}{2\pi\mathrm{i}} \int_\Gamma R(\lambda, A) \mathrm{d}\lambda.$$

Theorem 3.28. *If A is a densely defined closed operator and has separable spectrum, then P defined above is a projection operator and $\text{Ran}(P) \subseteq D(A^n)$ for each $n \geq 1$.*

Proof. We first show that P is a projection operator. Choose two admissable contours γ and γ' with $\gamma \subseteq \gamma'$. Then we have

$$\int_{\gamma'} (\xi - \lambda)^{-1} \mathrm{d}\xi = 2\pi\mathrm{i} \quad \forall \lambda \in \text{Int}\,\gamma$$

and

$$\int_\gamma (\xi - \lambda)^{-1} \mathrm{d}\lambda = 0 \quad \forall \xi \in \gamma'.$$

Also, we know the resolvent identity

$$\frac{1}{\lambda - A} - \frac{1}{\xi - A} = \frac{\xi - \lambda}{(\lambda - A)(\xi - A)}.$$

Then we compute as follows

$$\begin{aligned}
P^2 &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\gamma \int_{\gamma'} \frac{1}{\lambda - A} \frac{1}{\xi - A} \mathrm{d}\xi \mathrm{d}\lambda \\
&= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\gamma \int_{\gamma'} \frac{1}{\xi - \lambda} \left(\frac{1}{\lambda - A} - \frac{1}{\xi - A}\right) \mathrm{d}\xi \mathrm{d}\lambda \\
&= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\gamma \int_{\gamma'} \frac{1}{\xi - \lambda} \frac{1}{\lambda - A} \mathrm{d}\xi \mathrm{d}\lambda - \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\gamma \int_{\gamma'} \frac{1}{\xi - \lambda} \frac{1}{\xi - A} \mathrm{d}\xi \mathrm{d}\lambda \\
&= P - 0 = P.
\end{aligned}$$

Now we aim to show $\mathrm{Ran}(P) \subseteq D(A)$. Using Riemann sum for an admissable contour Γ, we can define

$$x_n = \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{n} R(\xi_i, A)(z_{i+1} - z_i)x.$$

Here $x_n \in D(A)$ since $R(\lambda, A)$ is a bounded operator for $\lambda \in \Gamma$. And we see that $x_n \to Px$. Since

$$AR(\xi_i, A) = A(\xi_i - A)^{-1} = -\boldsymbol{I} + \xi_i R(\xi_i, A),$$

we have

$$Ax_n = \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{n} \xi_i R(\xi_i, A)(z_{i+1} - z_i)x \to \frac{1}{2\pi \mathrm{i}} \int_\Gamma \xi R(\xi, A) x \mathrm{d}\xi.$$

That is to say, we have

$$\begin{cases} x_n \to Px \\ Ax_n \to \frac{1}{2\pi \mathrm{i}} \int_\Gamma \xi R(\xi, A) \mathrm{d}\xi. \end{cases}$$

Since A is closed, then we can see

$$APx = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \xi R(\xi, A) x \mathrm{d}\xi.$$

Likewise, we have

$$A^n Px = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \xi^n R(\xi, A) x \mathrm{d}\xi.$$

Our proof finishes. □

Now we consider the isolated spectral point and investigate its properties.

Theorem 3.29 (Riesz projector for isolated points)**.** Let λ_0 be an isolated spectral point of A and $P_{\lambda_0} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} R(\lambda, A) \mathrm{d}\lambda$. Then we have the properties as follows:
(1). P_{λ_0} is a projection.
(2). $\ker(A - \lambda_0) \subseteq \mathrm{Ran}(P_{\lambda_0})$.
(3). If X is a Hilbert space and A is self-adjoint, then $\ker(A - \lambda_0) = \mathrm{Ran}(P_{\lambda_0})$ and P_{λ_0} is orthogonal projection($P_{\lambda_0} = E(\{\lambda_0\})$).

Remark. The property (3) reveales that Riesz projectors are a generalization of the previous spectral projections.

Proof. (1). It is direct.

(2). Suppose that $x \in \ker(A - \lambda_0)$. Then we have

$$(A - \lambda)x = (\lambda_0 - \lambda)x.$$

So we can compute as follows

$$\begin{aligned} P_{\lambda_0} x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} R(\lambda, A) x \mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} R(\lambda, A) \frac{\lambda - A}{\lambda - \lambda_0} x \mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} \frac{1}{\lambda - \lambda_0} x \mathrm{d}\lambda = x. \end{aligned}$$

So we have $\ker(A - \lambda_0) \subseteq \mathrm{Ran}(P_{\lambda_0})$.

(3). It suffices to show $(A - \lambda_0)R_{\lambda_0} = 0$. First we have

$$(A - \lambda_0) \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} R(\lambda, A) \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} (\lambda - \lambda_0) R(\lambda, A) \mathrm{d}\lambda.$$

Since A is self-adjoint now, we can use calculus which leads to

$$||R(\lambda, A)||_{op} \leq \frac{1}{\mathrm{dist}(\lambda, \sigma(A))}.$$

Then on the deleted ball $B(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, we have

$$||(\lambda - \lambda_0) R(\lambda, A)||_{op} \leq \frac{|\lambda - \lambda_0|}{\mathrm{dist}(\lambda, \sigma(A))} \leq 1.$$

Using generalized complex analysis theory(here we omit the corresponding argument), we can see λ_0 is a removable singularity. And then $(\lambda - \lambda_0)R(\lambda, A)$ can be seen as a holomorphic function so that $(A - \lambda_0)P_{\lambda_0} = 0$. ☐

Since there are some examples showing that an isolated spectral point may not lie in point spectrum, such as Volterra integral operator. We can show the isolated spectral points of a self-adjoint operator always lies in point spectrum.

Theorem 3.30. If A is a self-adjoint operator and λ_0 is an isolated spectral point, then it lies in point spectrum.

Proof. If λ_0 is not an eigenvalue, then $\ker(\lambda_0 - A) = 0$. Because of the last theorem, we see $P_{\lambda_0} = 0$. Since we know $||(\xi - \lambda_0)R(\xi, A)||_{op} \leq 1$ for $\xi \in B(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, then $\langle x, R(\xi, A)y \rangle$ can only have singularity of one order. However, from $P_{\lambda_0} = 0$, we see

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} R(\xi, A) \mathrm{d}\xi = P_{\lambda_0} = 0.$$

So we have $\langle x, R(\xi, A)y\rangle < \infty$ on $B(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. By the principle of uniform boundness, we see $R(\xi, A)$ has removable singularity, and it implies that $||R(\xi, A)||_{op} = M < \infty$. By resolvent identity, we see
$$||R(\lambda_1, A) - R(\lambda_2, A)||_{op} \leq M^2|\lambda_1 - \lambda_2|.$$
Then we can construct a convergent sequence $\{\lambda_n\}$ such that
$$R(\lambda_0, A) = N - \lim_n R(\lambda_n, A).$$
It remains to show $\lambda_0 \in \rho(A)$ to get contradiction. Consider that
$$\begin{cases} R(\lambda, A)x \to R(\lambda_0, A)x \\ (\lambda_0 - A)R(\lambda, A)x = x + (\lambda_0 - \lambda)R(\lambda, A)x \to x \end{cases}$$
and
$$R(\lambda_0, A)(\lambda_0 - A)x = \lim_\lambda R(\lambda, A)(\lambda_0 - A)x = x + \lim_\lambda (\lambda_0 - \lambda)R(\lambda, A)x = x.$$
Here we need to use that $A - \lambda_0$ is a closed operator since A is self-adjoint. □

3.6 Discrete Spectrum And Essential Spectrum

There are several different definitions of essential spectrums, and since here we aim to introduce Weyl criterion, we choose the definition which is compatible with it.

Proposition 3.31. If A is self-adjoint, then $\lambda \in \sigma(A)$ if and only if $E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$ for each $\varepsilon > 0$.

Proof. In the spectral theorem we have already seen that $\mathrm{supp}E = \sigma(A)$. □

Now we would like to use sequence to describe the spectrum of the self-adjoint operator A.

Theorem 3.32. Let A be a self-adjoint operator, then $\lambda \in \sigma(A)$ if and only if there is a sequence $\{\psi_n\} \subseteq D(A)$ such that $||\psi_n|| = 1$ and $(A-\lambda)\psi_n \xrightarrow{s} 0$. This sequence is called spectral sequence.

Proof. The sufficiency is easy to verify since it gives that $A - \lambda$ does not have a non-zero lower bound.

For the necessity, there are two ways to prove it.

(Method I). we can construct the sequence directly. If $\lambda \in \sigma_p(A)$, we can take its eigenvectors. If not, since A is self-adjoint, then $\lambda \in \sigma_c(A)$. So $(A - \lambda)^{-1}$ is densely defined and $(A - \lambda)^{-1}$ is unbounded. Then we can take a sequence $\{y_n\}$ such that $||y_n|| = 1$ and $||(A - \lambda)^{-1}y_n|| \to \infty$. Let
$$x_n = \frac{(A - \lambda)^{-1}y_n}{||(A - \lambda)^{-1}y_n}.$$
This completes the proof.

(Method II). Take $||\psi_n|| = 1$ with $\psi_n \in \text{Ran}(E(\lambda - \varepsilon_n, \lambda + \varepsilon_n))$. Using spectral integral theory, we have

$$\begin{aligned} ||(A-\lambda)\psi_n||^2 &= \int_{\sigma(A)} |z-\lambda|^2 \mathrm{d}E_{\psi_n,\psi_n} \\ &= \int_{\sigma(A)} |z-\lambda|^2 \mathbf{1}_{\{|z-\lambda|<\varepsilon_n\}} \mathrm{d}E_{\psi_n,\psi_n} \\ &\leq \varepsilon_n^2 ||\psi_n|| = \varepsilon^2 \to 0. \end{aligned}$$

This also completes the proof. □

We introduce the definitions of discrete spectrum and essential spectrum in the language of spectral projections.

Definition 3.33 (Discrete spectrum and essential spectrum)**.**

$$\sigma_d(A) = \{\lambda \in \sigma(A) : \exists\, \varepsilon_0 > 0,\ \dim \text{Ran} E(\lambda - \varepsilon_0, \lambda + \varepsilon_0) < \infty\}.$$

$$\sigma_{\text{ess}}(A) = \{\lambda \in \sigma(A) : \exists\, \varepsilon_0 > 0,\ \dim \text{Ran} E(\lambda - \varepsilon_0, \lambda + \varepsilon_0) = \infty\}.$$

Proposition 3.34. If A is self-adjoint and λ is an isolated spectral point, then $\text{Ran}(E(\{\lambda\})) = \ker(A - \lambda_0)$.

Proof. By the unbounded functional calculus, we can see

$$(A - \lambda_0)E(\{\lambda_0\}) = \int_{\sigma(A)} (z - \lambda_0)\mathbf{1}_{\lambda_0} \mathrm{d}E(z) = 0.$$

Then it means that $\text{Ran}(E(\{\lambda_0\})) \subseteq \ker(A - \lambda_0)$.

Conversely, let

$$f_n = \frac{1}{\lambda_0 - z}\mathbf{1}_{B^c(\lambda_0, \frac{1}{n})}.$$

Then f_n is bounded. Using bounded calculus, we can see

$$f_n(A)(\lambda_0 - A) = E(B^c(\lambda_0, \frac{1}{n})).$$

For each $x \in \ker(A - \lambda_0)$, we have

$$0 = f_n(A)(\lambda_0 - A)x = E(B^c(\lambda_0, \frac{1}{n}))x \to E(\mathbb{C} \setminus \{\lambda_0\})x.$$

It implies that $E(\{\lambda_0\})x = x$.

In particular, we can see any isolated spectral point is a eigenvalue since $\text{supp}E = \sigma(A)$. □

Proposition 3.35. Let A be self-adjoint, then $\lambda \in \sigma_d(A)$ if and only if λ is an isolated spectral point and its space of eigenvectors is of finite dimensions.

Proof. For the sufficiency, there is $\varepsilon > 0$ s.t. $E(B(\lambda, \varepsilon)) = E(\{\lambda\})$. Since $\dim \text{Ran} E(\{\lambda\}) < \infty$, we have $\lambda \in \sigma_d(A)$.

Conversely, we can assume λ is not isolated and there is a sequence $\lambda_n \to \lambda$. We can choose $B(\lambda_n, \varepsilon_n)$ are disjoint to each other. Since different eigensubspaces are orthogonal to each other, we have

$$\dim \mathrm{Ran} E\left(\sum_{i=1}^{\infty} B(\lambda_n, \varepsilon_n)\right) = \infty.$$

However, there must be a neighborhood of λ being of finite dimensions with respect to the spectral projection, and then it shows contradiction. $\square$

Now we turn to the description of essential spectrum.

Proposition 3.36. Let A be self-adjoint. Then $\sigma_{\text{ess}}(A)$ is closed. $\lambda \in \sigma_{\text{ess}}(A)$ if and only if at least one of the followings holds.
(1). $\lambda \in \sigma_c(A)$.
(2). λ is a limit point of $\sigma_p(A)$.
(3). $\lambda \in \sigma_p(A)$ and $\dim \ker(A - \lambda) = \infty$.

Proof. $\sigma_{\text{ess}}(A)$ is closed because of the same dimensional argument in the last proposition. And the equivalence is clear. $\square$

The following theorem is one of the most important conclusions in the whole section which gives a clear description of essential spectrum.

Theorem 3.37 (Weyl criterion)**.** Let A be self-adjoint. $\lambda \in \sigma_{\text{ess}}(A)$ if and only if there is a sequence of $\{\psi_n\} \subseteq D(A)$ that $||\psi_n|| = 1$, $\psi_n \rightharpoonup 0$ and $(A - \lambda)\psi_n \xrightarrow{s} 0$. Here the sequence $\{\psi_n\}$ is called Weyl sequence.

Proof. We first introduce the conception of Weyl spectrum $\omega(A)$ whose elements have corresponding Weyl sequences. Then it suffices to show $\omega(A) = \sigma_{\text{ess}}(A)$.

(1). We show $\omega(A) \subseteq \sigma_{\text{ess}}(A)$ first. Assume there is a $\lambda \in \omega(A)$ but $\lambda \notin \sigma_{\text{ess}}(A)$. Since we have

$$(A - \lambda)\psi_n \xrightarrow{s} 0,$$

we can see $\lambda \in \sigma(A)$. If $\lambda \in \sigma_d(A)$, then $E(\{\lambda\})$ is a finite dimensional projection operator. Then it's a compact operator. By the total continuity, we have

$$E(\{\lambda\})\psi_n \xrightarrow{s} 0.$$

Then let $\omega_n = \psi_n - E(\{\lambda\})\psi_n$. we see $||\omega_n|| \to 1$. Using the fact $\ker(A - \lambda) = \mathrm{Ran}(E(\{\lambda\}))$, we can consider that $T_\lambda = (A - \lambda)|_{\ker(A-\lambda)^\perp}$.

Acknowledgement. If A is self-adjoint, then (1). $\ker(A)^\perp$ is an invariant subspace; (2). $A_1 = A|_{\ker A^\perp}$ is self-adjoint, $\ker A_1 = \{0\}$ and $\sigma(A_1) \subseteq \sigma(A)$.

Using the acknowledgement, we see T_λ is self-adjoint. And since λ is an isolated spectral point, we have spectral gap estimate which implies that

$$||T_\lambda^{-1}|| < \infty.$$

Then we have

$$\omega_n = T_\lambda^{-1}(T_\lambda \omega) \xrightarrow{s} 0.$$

But $\omega_n \xrightarrow{s} 1$. It leads to contradiction.

(2). We can construct the sequence directly. Since $\lambda \in \sigma_{\text{ess}}(A)$, we have $\dim \operatorname{Ran}(\lambda - \varepsilon, \lambda + \varepsilon) = \infty$. Then we choose $\varepsilon_n \searrow 0$ and $\psi_n \in \operatorname{Ran} E(\lambda - \varepsilon_n, \lambda + \varepsilon_n)$ with $||\psi_n|| = 1$. It's clear that ψ_n is orthonormal. Then we can see $\psi_n \rightharpoonup 0$. It ramains to show the strong convergence. Using spectral integral, we have

$$\begin{aligned} ||(A-\lambda)\psi_n||^2 &= \int |z-\lambda|^2 \mathrm{d}E_{\psi_n,\psi_n} \\ &\leq \int |z-\lambda|^2 \mathbf{1}_{(\lambda-\varepsilon_n, \lambda+\varepsilon_n)} \mathrm{d}E_{\psi_n,\psi_n} \\ &\leq \int \varepsilon_n^2 \mathrm{d}E_{\psi_n,\psi_n} \leq \varepsilon_n^2. \end{aligned}$$

Then we finish the proof. □

4 Perturbation Theory of Self-Adjoint Operators

We cannot always wish that we are dealing with some self-adjoint operators, but we still want to know the behaviors of their spectrums. Then it comes to the perturbation theory. In this section, we introduce some conceptions to describe their difference from self-adjoint operators and finally give a nice conclusion to show that we can regard them as self-adjoint operators in some cases. This section is mainly based on [2].

Definition 4.1 (Relative bounded operators)**.** A densely defined operator B is called A-bounded if the followings hold:
(1). $D(A) \subseteq D(B)$.
(2). There is $z_0 \in \rho(A)$ such that $BR(A, z_0)$ is bounded.

Remark. Using resolvent identity, we can see that the existence can be improved to be arbitrariness. For (2), we have an equivalent statement (2'): there are a, $b > 0$ such that

$$||Bx|| \leq a||Ax|| + b||x||.$$

(2) implies (2') is quite direct by considering $||BR(A, z_0)(A - z_0)x|| \leq C||(A - z_0)x||$. Another side can be proved by contradiction that $BR(A, z_0)$ cannot be unbounded with the upper bound of A . We call the infimum of a as the relative bound of B with respect to A.

Example 4.2. Let $V \in L^2(\mathbb{R}^3)$. Then V is $-\Delta$-bounded.

Proof. Using embedding theorem, we have

$$||Vf||_{L^2} \lesssim ||V||_{L^2}||f||_{L^\infty} \lesssim ||V||_{L^2}||f||_{H^2} \lesssim ||V||_2(||f||_{L^2} + ||-\Delta f||_{L^2}).$$

It implies that V is $-\Delta$-bounded. □

We have a description for the relative bound with limit.

Lemma 4.3. Let A be a self-adjoint operator, B be A-bounded. Then the relative bound

$$a = \lim_{\lambda \to \infty} ||BR(A, \mathrm{i}\lambda)||_{op}.$$

Proof. Since A is self-adjoint, for $\lambda \in \mathbb{R}$, we have $\mathrm{i}\lambda \in \rho(A)$. Then we can compute as follows:

$$\begin{aligned} ||Bx|| &\leq ||BR(A, \mathrm{i}\lambda)||||(A - \mathrm{i}\lambda)x|| \\ &\leq ||BR(A, \mathrm{i}\lambda)||(||Ax|| + ||\lambda x||). \end{aligned}$$

It implies that $a \leq ||BR(A, \mathrm{i}\lambda)||$.

On the other hand, we can use spectral theorem, and then we have

$$\begin{aligned} ||BR(A, \mathrm{i}\lambda)x|| &\leq (a + \varepsilon)||AR(A, \mathrm{i}\lambda)x|| + b_\varepsilon ||R(A, \mathrm{i}\lambda)x|| \\ &\leq (a + \varepsilon)||x|| + \frac{b_\varepsilon}{\lambda}||x||. \end{aligned}$$

Then letting $\lambda \to \infty$, we have

$$\liminf_{\lambda \to \infty} ||BR(A, \mathrm{i}\lambda)|| \leq a + \varepsilon \to a.$$

So that we see the relative bound $a = \lim\limits_{\lambda \to \infty} ||BR(A, \mathrm{i}\lambda)||$. □

Example 4.4. The relative bound of V with respect to $-\Delta$ is 0.

Proof. It suffices to show for $\varepsilon > 0$, we have

$$||f||_{L^\infty} \leq \varepsilon|| - \Delta f||_{L^2} + C_\varepsilon ||f||_{L^2}.$$

We have two methods to deal with this question. One is to use scaling. we can consider $f_\lambda(x) = f(\lambda x)$, then we have

$$||f||_{L^\infty} = ||f_\lambda||_{L^\infty} \leq C_1 \lambda^\alpha || - \Delta f||_{L^2} + C_2 \lambda^{-\beta} ||f||_L^2,$$

where α, $\beta > 0$. Then let $\lambda \to \infty$.

Another method is to use compactness for the case whose domain is bounded. If the inequality does not hold. Then we have a sequence f_n which satisfies that

$$||f_n||_{L^\infty} > \varepsilon|| - \Delta f_n||_{L^2} + n||f_n||_{L^2}.$$

We can assume $||f_n||_{L^\infty} = 1$, and then we have $||f_n||_{L^2} \to 0$ and $|| - \Delta f||_{L^2} \leq \frac{1}{\varepsilon}$. This means f_n is uniformly bounded in H^2. Then there is a subsequence g_n converging to 1 in L^∞. Since the domain is bounded, then we also have L^2 strong convergence. But $||g_n|| \to 0$, and it leads to contradiction. □

Then we can prove a fundamental theorem in the perturbation theory.

Theorem 4.5 (Kato-Rellich)**.** Let A be self-adjoint and B be closed symmetric and A-bounded with relative bound $a < 1$. Then we have $A + B$ is self-adjoint on $D(A)$.

Proof. Suppose $|\lambda| \gg 1$. Then we have $||BR(A, \mathrm{i}\lambda)|| \leq a + \varepsilon < 1$. Consider that

$$A + B - \mathrm{i}\lambda = (BR(A, \mathrm{i}\lambda) + \boldsymbol{I})(A - \mathrm{i}\lambda)$$

has bounded inverse. Then $\mathrm{Ran}(A + B - \mathrm{i}\lambda) = H$. Using the criterion, it suffices to show $A + B$ is closed symmetric. Since $D(A) \subseteq D(B)$, then we see $A + B$ is symmetric. To show closeness, let $(A + B)x_n \to y$ and $x_n \to x$. Since the relative bound is lower than 1, we have

$$||Bx|| \leq (a + \varepsilon)||Ax|| + b||x||.$$

Then we have

$$\begin{aligned} ||A(x_m - x_n)|| &\leq ||(A + B)(x_n - x_m)|| + ||B(x_n - x_m)|| \\ &\leq ||(A + B)(x_n - x_m)|| + \underbrace{(a + \varepsilon)||A(x_n - x_m)}_{\text{absorbed to the left}}|| + b||x_n - x_m|| \end{aligned}$$

Then it means that $Ax_n \to y_1 = Ax$ and $Bx_n \to y - y_1$. □

A further conception is relative compact.

Definition 4.6 (Relative compact)**.** Let A be closed. We say B is A-compact if
(1). $D(A) \subseteq D(B)$.
(2). There is $z_0 \in \rho(A)$ such that $BR(A, z_0)$ is compact.

Remark. The existence can be improved to arbitrariness for the same reason.

Lemma 4.7 (Lifting convergence)**.** Let K be a compact operator, B_n be a bounded sequence with $B_n \xrightarrow{s} B$. Then

$$\lim_n ||KB_n - KB||_{op} = 0 \quad \lim_n ||B_n K - BK||_{op} = 0.$$

Proof. We only prove the convergence of $B_n K$. If K is of finite rank, then we have $Fx = \sum_{i==1}^N \langle x, e_i \rangle Fe_i$. We have

$$||(B_n - B)F||_{op} = \sup_{||x||=1} ||(B_n - B)Fx|| \leq \sum_{i=1}^N |\langle x, e_i \rangle| ||(B_n - B)Fe_i|| \to 0.$$

For the general case, we know the fact that compact operators in Hilbert space are the limit of finite rank operators with respect to the operator norm. □

Lemma 4.8 (Relative bound of relative compact operator)**.** Let A be self-adjoint, B be A-compact. Then the relative bound of B with respect to A is 0. Equivalently, we have $\lim_{\lambda \to \infty} ||BR(A, \mathrm{i}\lambda)||_{op} = 0$.

Proof. We first observe that

$$||(A - \mathrm{i})R(A, \mathrm{i}\lambda)x|| = ||R(A, \mathrm{i}\lambda)(A - \mathrm{i})x|| \leq \frac{1}{|\lambda|}||(A - \mathrm{i})x|| \to 0.$$

Then we see

$$BR(A, \mathrm{i}\lambda) = \underbrace{BR(A, \mathrm{i})}_{\text{compact}} \underbrace{(A - \mathrm{i})R(A, \mathrm{i}\lambda)}_{\text{strongly converge to } 0}$$

converges to 0 in the operator norm. $\square$

Now we give the most important conclusion in the whole section.

Theorem 4.9 (Weyl compact perturbation)**.** Let A be self-adjoint and B be symmetric and A-compact. Then $\sigma_{\text{ess}}(A + B) = \sigma_{\text{ess}}(A)$.

Proof. We first show that $\sigma_{\text{ess}}(A) \subseteq \sigma_{\text{ess}}(A + B)$. If $\lambda \in \sigma_{\text{ess}}(A)$, by Weyl criterion, we have a sequence x_n such that $||x_n|| = 1$, $x_n \rightharpoonup 0$, $(A - \lambda)x_n \xrightarrow{s} 0$. Then we can see $Ax_n = (A - \lambda)x_n + \lambda x_n \rightharpoonup 0$. Thus we have

$$Bx_n = \underbrace{BR(A, z_0)}_{\text{compact}} \underbrace{(A - z_0)x_n}_{\text{weak to } 0} \xrightarrow{s} 0.$$

Hence x_n is also the Weyl sequence of $A + B$ which implies that $\lambda \in \sigma_{\text{ess}}(A + B)$.

Now we show that $\sigma_{\text{ess}}(A + B) \subseteq \sigma_{\text{ess}}(A)$. We claim that B is $A + B$-compact. Since

$$B + A - \mathrm{i}\lambda = (BR(A, \mathrm{i}\lambda) + 1)(A - \mathrm{i}\lambda),$$

we have

$$BR(A + B, \mathrm{i}\lambda) = \underbrace{BR(A, \mathrm{i}\lambda)}_{\text{compact}} \underbrace{(BR(A, \mathrm{i}\lambda) + 1)^{-1}}_{\lambda \gg 1}$$

be compact. So does $-B$. Then using Kato-Rellich theorem, we see $A + B$ is self-adjoint, and then we can repeat the process above. $\square$

Example 4.10. $V \in L^2$ is $-\Delta$-compact.

Proof. It suffices to show that $VR(A, \mathrm{i})$ is compact where we denote $-\Delta$ as A. Our idea is to get a concrete form of $R(A, \mathrm{i})f$ first. Using Fourier transform, we have

$$R(A, \mathrm{i})f = \mathcal{F}^{-1}\mathcal{F}(R(A, \mathrm{i})f) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - \mathrm{i}}\hat{f}\right) = G * f = \int G(x - y)f(y)\mathrm{d}y,$$

Where $G = \mathcal{F}^{-1}(\frac{1}{|\xi|^2 - \mathrm{i}})$. Then we have

$$VR(A, \mathrm{i})f = \int V(x)G(x - y)f(y)\mathrm{d}y.$$

Let $K(x, y) = V(x)G(x - y)$, and then it remains to show $K(x, y) \in L^2_{x,y}$ because Hilbert-Schmidt theory. In fact, we see

$$\int_x \int_y V(x)^2 G(x - y)^2 \mathrm{d}y\mathrm{d}x = \int_x V(x)^2 \underbrace{\int_y G(x - y)^2 \mathrm{d}y}_{\text{uniformly bounded to } x} \mathrm{d}x < \infty.$$

Then we finish our proof. $\square$

A big reason for introducing such perturbation theory is to deal with a class of potential operators.

Definition 4.11 (Kato potential)**.** A multiplication operator V is called as Kato potential if $V \in L^2 + \varepsilon L^\infty$ for each $\varepsilon > 0$.

Proposition 4.12. Kato potentials are $-\Delta$-compact.

Proof. Since $V \in L^2 + \varepsilon L^\infty$, it tells that V have a nice decay near the infinity point, so we can regard it as a localization. Then we can see the following relation

$$L^2 \overset{(-\Delta+1)^{-1}}{\longrightarrow} H^2 \overset{V \text{ compact embedding}}{\hookrightarrow\hookrightarrow} L^2.$$

The graph says it all. $\square$

5 Semigroup Theory

This theory may originate from ODE theory and Duhamel principle that we can describe the solution of the equation $u_t = Au$ in the form $u = e^{tA}u_0$, where A can be not only a number or a matrix, but also a general operator. This section is mainly based on [2].

5.1 Semigroups

Definition 5.1 (C_0 semigroup)**.** Let X be a Banach space and $S = \{S(t) : t \geq 0\}$ be a family of bounded operator. S is called a C_0 semigroup(also known as strongly continuous one-parameter semigroup) if
(1). $S(0) = \boldsymbol{I}$.
(2). S is a semigroup.
(3). $S(t)$ is strongly continuous that $||S(t)x - x|| \to 0$ as $t \searrow 0$ for each $x \in X$.

Remark. Using (2) we can prove that $S(t)$ has strong continuity everywhere.

We naturally want to know something about its generator.

Definition 5.2 (Infinitesimal generator)**.** The infinitesimal generator A of of a strongly continuous semigroup S is defined by

$$Ax = \lim_{t\searrow 0} \frac{S(t)x - x}{t}.$$

And $D(A) = \{x \in X : \text{ the limit above exists}\}$.

Then we can describe the infinitesimal generator.

Lemma 5.3. Let S be a C_0 semigroup on X. There is $M \geq 1$ and $\omega \in \mathbb{R}$ such that $||S(t)|| \leq M\mathrm{e}^{\omega t}$

Proof. Since $S(t)x$ is continuous with respect to t, we have $\sup_{t\in[0,1]} ||S(t)x|| \leq C_x$. By uniform boundness principle, we have $\sup_{[0,1]} ||S(t)|| < \infty$. Then we have

$$||S(t)|| = ||S([t])S(\{t\})|| \leq ||S(\{t\})|| ||S(1)||^{[t]} = ||S(\{t\})||\mathrm{e}^{[t]\log||S(1)||}.$$

Then let $M = \sup_{t\in[0,1]} ||S(t)||$ and $\omega = \log ||S(1)||$. $\square$

Proposition 5.4. Let $S(t)$ be a C_0 semigroup and A be the generator of $S(t)$ defined above. We have the following properties:
(1). $t \mapsto S(t)x$ is continuous on $t \geq 0$ for every $x \in X$.
(2). For each $x \in D(A)$, the mapping $t \mapsto S(t)x$ is continuously differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)x = S(t)Ax = AS(t)x.$$

(3). For each $x \in X$, we have

$$\int_0^t S(s)x\mathrm{d}x \in D(A), \quad A\int_0^t S(s)x\mathrm{d}s = S(t)x - x = \int_0^t S(s)Ax\mathrm{d}s.$$

(4). A is a densely defined closed operator.

Proof. (1). For the left continuity, we see

$$||S(t)x - S(t-h)x|| \leq ||S(t-h)(S(h)x - x)|| \leq M\mathrm{e}^{\omega(t-h)}||S(h)x - x|| \to 0.$$

The right side can be proven similarly.

(2). We can prove it similar as what we do in (1). And note that we can see if $x \in D(A)$, then $S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$ from the proof.

(3). We compute directly first

$$\begin{aligned}
&\lim_{h\searrow 0} \frac{S(h) - \boldsymbol{I}}{h} \int_0^t S(s)x\mathrm{d}s \\
=&\frac{1}{h}\left[S(h)\int_0^t S(s)x\mathrm{d}s - \int_0^t S(s)x\mathrm{d}s\right] \\
=&\frac{1}{h}\left[\int_h^{t+h} S(s)x\mathrm{d}s - \int_0^t S(s)x\mathrm{d}s\right] \\
=&\frac{1}{h}\left[\int_t^{t+h} S(x)x\mathrm{d}s - \int_0^h S(s)x\mathrm{d}s\right] \\
=&S(t)x - x.
\end{aligned}$$

From the computation, we see $A\int_0^t S(s)x\mathrm{d}s = S(t)x - x$.

(4). For any $x \in D(A)$, we have $\frac{1}{t}\int_0^t S(s)x\mathrm{d}s \to x$ since (3) and strong continuity. It implies that A is densely defined. To show A is closed, we define two approximation operators $A_\varepsilon x = \frac{S(\varepsilon)-\boldsymbol{I}}{\varepsilon}x$ and $M_t = \frac{1}{t}\int_0^t S(s)x\mathrm{d}s$. Then

$$A_\varepsilon M_t x = A_t M_\varepsilon x.$$

Suppose $x_n \to x$ and $Ax_n \to y$. We have $A_\varepsilon x_n = A_\varepsilon M x_n = AM_\varepsilon x_n = M_\varepsilon A x_n \to y$ and then $A_\varepsilon x = M_\varepsilon y$. Let $\varepsilon \to 0$. So that $x \in D(A)$ and $Ax = y$. □

Now we give a specific kind of semigroups.

Definition 5.5 (Contraction semigroup)**.** A C_0 semigroup is called a contraction semigroup if $||S(t)|| \leq 1$ for every $t \geq 0$.

Lemma 5.6. Suppose A is the generator of the contraction semigroup $S(t)$, then $\{\lambda \in \mathbb{C} : \Re\lambda > 0\} \subseteq \rho(A)$ and

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)\mathrm{d}t$$

where the integral is convergent in the sense of operator norm.

In particular, we see

$$||R(\lambda, A)|| \leq \int_0^\infty \mathrm{e}^{-\Re\lambda t}\mathrm{d}t = \frac{1}{\Re\lambda}.$$

Proof. Let $\tilde{R}(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)\mathrm{d}t$. It suffices to show that
(1).$\tilde{R}(\lambda, A)x \in D(A)$, $(\lambda - A)\tilde{R}(\lambda, A) = \boldsymbol{I}_X$.
(2). $\tilde{R}(\lambda, A)(\lambda - A) = \boldsymbol{I}_{D(A)}$.

(1). Compute in definition

$$\begin{aligned}
A_h\tilde{R}(\lambda, A)x &= \int_0^\infty \mathrm{e}^{-\lambda t}\frac{S(t+h) - S(t)}{h}x\mathrm{d}t \\
&= \frac{\mathrm{e}^{\lambda h} - 1}{h}\int_h^\infty \mathrm{e}^{-\lambda t}S(t)x\mathrm{d}t - \frac{1}{h}\int_0^h \mathrm{e}^{-\lambda t}S(t)x\mathrm{d}t \\
&\xlongequal{h \to 0} \lambda\int_0^\infty e^{-\lambda t}S(t)\mathrm{d}t - S(0)x \\
&= \lambda\tilde{R}(\lambda, A)x - x.
\end{aligned}$$

Thus we see $\tilde{R}(\lambda, A)x \in D(A)$ and $(\lambda - A)x = x$ for every $x \in X$.

(2). We can still compute directly

$$\begin{aligned}
\tilde{R}(\lambda, A)Ax &= \int_0^\infty \mathrm{e}^{-\lambda t}S(t)Ax\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}S(t)x\mathrm{d}t \\
&= [\mathrm{e}^{-\lambda t}S(t)x]_0^\infty + \lambda\int_0^\infty \mathrm{e}^{-\lambda t}S(t)x\mathrm{d}t \\
&= -x + \lambda\tilde{R}(\lambda, A)x,
\end{aligned}$$

where the x can be any elements in $D(A)$. □

Theorem 5.7 (Hille-Yosida)**.** Let A be a densely defined closed operator. Then A is a generator of some contraction semigroups if and only if $(0, \infty) \subseteq \rho(A)$ and $||R(\lambda, A)|| \leq \frac{1}{\lambda}$.

Proof. Last lemma tells the necessity, so here we only show the sufficiency. Our idea is to use approximation sequence to construct the semigroup. Let $A^\lambda = \lambda^2 R(\lambda, A) - \lambda$. Then we can see

$$A^\lambda x - Ax = \lambda R(\lambda, A)Ax - Ax = A(\lambda - A)^{-1}Ax \to 0$$

for $x \in D(A)$. Then since A is closed, we get the strong convergence.

Since A^λ is bounded, the semigroup e^{-tA^λ} can be defined by power series. Since

$$\begin{aligned}||\mathrm{e}^{tA^\lambda}|| &\leq \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n\lambda^{2n}}{n!}||(\lambda-A)^{-1}||^n \\ &\leq \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda t}=1,\end{aligned}$$

they are contraction semigroups. For all μ, λ, $t>0$, and all $x\in D(A)$, we have

$$\mathrm{e}^{tA^\lambda}x-\mathrm{e}^{tA^\mu}=\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{sA^\lambda}\mathrm{e}^{(t-s)A^\mu}\mathrm{d}s.$$

So,

$$\begin{aligned}||\mathrm{e}^{tA^\lambda}x-\mathrm{e}^{tA^\mu}x|| &= \int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{sA^\lambda}\mathrm{e}^{(t-s)A^\mu}x\mathrm{d}s \\ &\leq \int_0^t||\mathrm{e}^{sA^\lambda}\mathrm{e}^{(t-s)A^\mu}||||A^\lambda x-A^\mu x||\mathrm{d}s \\ &\leq t||A^\lambda x-A^\mu x||.\end{aligned}$$

Using strong convergence, we can define $S(t)x=\lim_{\lambda\to\infty}\mathrm{e}^{tA^\lambda}x$ for every $x\in D(A)$. Since $D(A)$ is dense and $||\mathrm{e}^{tA^\lambda}||\leq 1$, we can extend its domain to the X. And since $\mathrm{e}^{t\lambda}\xrightarrow{s}A$, we can easily verify $S(t)$ is a C_0 semigroup.

It remains to show the generator of $S(t)$, called $\tilde{A}$, is A. For all $t>0$ and $x\in D(A)$, we have

$$\mathrm{e}^{tA^\lambda}x-x=\int_0^t\mathrm{e}^{sA^\lambda}Ax\mathrm{d}s.$$

Because of the strong convergence, we have

$$S(t)x-x=\int_0^t\mathrm{e}^{sA}Ax\mathrm{d}s.$$

Then we can see $\tilde{A}_tx\to Ax$, which implies that $D(A)\subseteq D(\tilde{A})$ and $\tilde{A}|_{D(A)}=A$. Since $(\lambda-A)^{-1}$ and $(\lambda-\tilde{A})^{-1}$ both exist, then

$$(\lambda-A)D(A)=X=(\lambda-\tilde{A})D(\tilde{A}).$$

It implies that $D(A)=D(\tilde{A})$. □

Theorem 5.8 (Hille-Yosida)**.** We say $P(M,\omega)$ holds if $||S(t)||\leq M\mathrm{e}^{\omega t}$. Then we have
(1). $P(1,0)\Leftrightarrow\Re\{\lambda>0\}\subseteq\rho(A)$, $||R(\lambda,A)||\leq\frac{1}{\Re\lambda}$.
(2). $P(1,\omega)\Leftrightarrow\Re\{\lambda>\omega\}\subseteq\rho(A)$, $||R(\lambda,A)||\leq\frac{1}{\Re\lambda-\omega}$.
(3). $P(M,\omega)\Leftrightarrow\Re\{\lambda>\omega\}\subseteq\rho(A)$, $||R(\lambda,A)^n||\leq\frac{M}{(\Re\lambda-\omega)^n}$.

Proof. We omit the proof; see [4]. □

We shall note that we do not always have semigroup solutions for a certain PDE since the semigroup solutions have nice regularity and we cannot expect such good properties. What's more, such solutions have dependence on the initial data, and we do not know the well-posedness of most equations.

5.2 Accretive Operators And Dissipative Operators

Now we introduce a conception generalizing the definition of self-adjoint operators from the perspective of quadratic forms.

Definition 5.9 (Accretive operators and dissipative operators)**.** A unbounded operator A is call accretive if $\Re\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$. Likewise, B is called dissipative if $\Re\langle Bx, x\rangle \leq 0$ for every $x \in D(B)$.

Since the definitions of accretive operators and dissipative operators are totally corresponded, for simplicity, we only consider accretive operators below.

Proposition 5.10. Let A be densely defined and accretive, then A is closable and its closure is also accretive.

Proof. Suppose $x_n \in D(A)$ and $x_n \to 0$, $Ax_n \to l$. It suffices to show $l = 0$. If not, let $l \neq 0$. The we have

$$\lim_{n\to\infty} \Re\langle A(y + cx_n), y + cx_n\rangle = \Re\langle Ay, y\rangle + c\langle l, y\rangle.$$

But since c and y can be arbitrary, the formula above can be negative when $l \neq 0$, and then it leads to contradiction. ☐

Definition 5.11 (Maximal accretive)**.** Let A be accretive. A is called maximal accretive if A do not have a proper accretive extension.

Theorem 5.12. The followings are equivalent:
(1). $\bar{A}$ is maximal accretive.
(2). There is $\lambda_0 > 0$, such that $A^* + \lambda_0$ is injective.
(3). There is $\lambda_1 > 0$, such that $\overline{\text{Ran}(A + \lambda_1)} = H$.

There is a equivalent statement for the theorem above.

Theorem 5.13. A is closed and accretive, and then the followings are equivalent:
(1). A is maximal accretive.
(2). There is $\lambda_0 > 0$, such that $A^* + \lambda_0$ is injective.
(3). There is $\lambda_1 > 0$, such that $\text{Ran}(A + \lambda_1) = H$.

Proof. (2)⇔(3). $\ker(A^* + \lambda) = \text{Ran}(A + \lambda)^{\perp}$. And we have the coercity estimate

$$\Re\langle Ax, x\rangle \geq \lambda\langle x, x\rangle$$

for every $\lambda > 0$. Since A is closed, we can see (2) and (3) are equivalent.

(1)⇒(3). Since $A + \lambda$ has closed range, if $\text{Ran}(A + \lambda) \neq H$, there is $0 \neq y \in \text{Ran}(A + \lambda)^{\perp}$ such that

$$\langle (A + \lambda)x, y\rangle = 0,$$

which implies that $y \notin D(A)$. Then we can define $\tilde{A}$ on $D(A) \oplus \{y\}$ and $\tilde{A}$ is actually an accretive operator.

(3)⇒(1). If $A \subseteq \tilde{A}$, then for each $y \in D(\tilde{A})$, let $x = (\tilde{A} + \lambda)y$. Since $\mathrm{Ran}(A + \lambda) = H$, there is $z \in D(A)$ such that $(A + \lambda)z = (\tilde{A} + \lambda)y$ which implies that $(\tilde{A} + \lambda)(y - z) =$. And the coercity estimate gives that $y = z$. □

The nost important conclusion for accretive operators is the following theorem.

Theorem 5.14. Let A be densely defined and closed. A is the generator of a contraction semigroup if and only if A is maximal dissipative.

Proof. For the necessity, using Hille-Yosida theorem, it remains to show A is dissipative. Since

$$\langle x, (S(t) - \boldsymbol{I})x \rangle \leq 0.$$

Then it implies that $\langle x, Ax \rangle \leq 0$ so that A is dissipative.

For the sufficiency, we can see the coercity estimate

$$\lambda ||x|| \leq ||(\lambda - A)x||$$

holding for every $\lambda > 0$ which implies $(\lambda - A)$ is injective. $\lambda_0 \in \rho(A)$. Then we can use iteration by resolvent identity to show $\lambda - A$ is surjective. Then we can use Hille-Yosida theorem. □

References

[1] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, 2 edition, 1999.

[2] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics Vol.2: Fourier Analysis, Self-Adjointness*. Academic Press, 1 edition, 1975.

[3] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics Vol.1: Functional Analysis*. Academic Press, 1 edition, 1980.

[4] Walter Rudin. *Functional Analysis*. McGraw-Hill, 2 edition, 1991.

[5] Gongqing Zhang and Yuanqu Lin. *Lecture Notes on Functional Analysis*. Peking University Press, 2 edition, 2021. in Chinese.