概率论外篇

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$(\Omega, \mathcal{F}, P)$ $\quad X: \Omega \to \mathbb{R}\ (\mathbb{C})$

$$\mathbb{R}\ (\mathbb{C}) \subseteq \mathrm{Mat}_{n\times n}(\mathbb{R})\ (\mathbb{C}) \subseteq \mathrm{Mat}_{m\times n}(\mathbb{R})\ (\mathbb{C}).$$

样本协方差矩阵 $\quad X_i = \begin{pmatrix} x_{1i} \\ \vdots \\ x_{pi} \end{pmatrix}$

$$W = \frac{1}{n}\sum X_i X_i^T = \frac{1}{n}\sum_i (x_{ji}x_{ki})_{jk}.$$

$$= \frac{1}{n} XX^T \quad (X = (X_1 \cdots X_p))$$

设 $X_{ij} \sim N(0,1)$ iid X 联合密度为 $\prod_j \prod_i \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_{ij}^2} = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\mathrm{tr}(XX^T)}$

谱的渐近行为.

$\mathrm{Tr}(H^k) \longrightarrow$ 对称多项式 $\longrightarrow$ 矩方法

定义 1.1 $A_n = (a_{ij})_n$ 对称

(1) a_{ij} 相互独立

~~其~~ $\{a_{ii}\} \sim Y$. $\{a_{ij}\} \sim Z$.

(2) 规范化 $\mathbb{E}[Y] = \mathbb{E}[Z] = 0$ $\quad \mathrm{Var}(Y) < \infty$. $\mathrm{Var}(Z) = 1$

(3) $\forall k \geq 3$. $\mathbb{E}[Y^k]$. $\mathbb{E}[Z^k] < \infty$

定理 1.2 $\forall k \in \mathbb{N}$. $\frac{1}{n}\mathbb{E}\left[\mathrm{tr}\left(\frac{A_n}{\sqrt{n}}\right)^k\right] \longrightarrow \gamma_k = \begin{cases} \frac{1}{m+1}\binom{2m}{m} & k = 2m \\ 0 & k = 2m+1 \end{cases}$ as $n \to \infty$

$$\gamma_k = \int_{-2}^{2} \frac{x^k}{2\pi}\sqrt{4-x^2}\,dx$$

Rmk 算 $k=2$ 知 $n\mathbb{E}[a_{ij}^2] \sim n^2$

Proof: 证明仍是组合配对

$$\mathbb{E}[\mathrm{tr}A_n^k] = \sum_{(i_1\cdots i_k)} \mathbb{E}[a_{i_1i_2}\cdots a_{i_ki_1}]$$

~~先假定 $i_3 = i_1$. $i_4 \neq i_1 \neq i_2$ $\Rightarrow a_{i_1i_2}a_{i_2i_1}a_{i_1i_4}\cdots a_{i_ki_1}$~~

一些组合的想法. ~~要求~~ 自由度大的贡献大 $\Rightarrow i_1 \cdots i_k$ 成环, $\frac{k}{2}$

这个对应到合法 () 表示. $\longrightarrow$ Catalan 数. $\frac{1}{n+1}\binom{2n}{n}$.

非可交配对

2.

GOE : $X_n(\omega) = (X_{ij}(\omega))_{ij}$ $A_n = \frac{1}{2}(X_n + X_n^T)$

$\Rightarrow a_{ij} = a_{ji} \sim N(0, \frac{1}{2}\sigma^2)$ $a_{ii} \sim N(0, \sigma^2)$

$dA_n = \prod da_{ii} \prod_{i<j} da_{ij}$ $f = 2^{-\frac{1}{2}}(\pi\sigma^2)^{-\frac{1}{4}n(n+1)} \exp(-\frac{1}{2\sigma^2}\mathrm{tr}A_n^2) dA_n$

GOE在正交变换下不变

GUE $X_{ij} = N(0,\sigma^2) + i N(0,\sigma^2)$

$A_n = \frac{1}{2}(X_n + X_n^H)$ $f = 2^{-\frac{n}{2}}(\pi\sigma^2)^{-\frac{n^2}{2}} \exp(-\frac{1}{2\sigma^2}\mathrm{tr}A_n^2) dA_n$

转化为特征值联合密度

$\boxed{\text{Thm}}$ $A_n \sim GOE_n(\sigma)$ 联合密度 $\rho(x_1, \cdots x_n) = \frac{1}{C_n}\exp(-\frac{1}{2\sigma^2}\sum x_i^2)\prod_{j<k}|x_k - x_j|$

其中 $x_1, \cdots x_n$ 为特征值

这里 $C_n = (2\pi)^{n/2}\sigma^{\frac{1}{2}n(n+1)}\prod_{k=1}\frac{\Gamma(1+\frac{k}{2})}{\Gamma(1+\frac{1}{2})}$

proof. $X = R\Lambda R^T$ $R^TR = I$ $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_n)$

R 有 $p_1 \cdots p_l$ ($l = \frac{n(n-1)}{2}$ 个) 不同的变量

$\frac{\partial X}{\partial \lambda_\gamma} = R\frac{\partial \Lambda}{\partial \lambda_\gamma}R^T \Rightarrow R^T\frac{\partial X}{\partial \lambda_\gamma}R = \frac{\partial \Lambda}{\partial \lambda_\gamma}$

$$\Rightarrow \boxed{\sum_{j,k} r_{j\alpha}\frac{\partial x_{jk}}{\partial \lambda_\gamma}r_{k\beta} = \delta_{\alpha\beta}\delta_{\alpha\gamma}}$$

$$\boxed{\frac{\partial X}{\partial p_\mu} = \frac{\partial R}{\partial p_\mu}\Lambda R^T + R\Lambda\frac{\partial R^T}{\partial p_\mu}}$$

$$\Rightarrow R^T\frac{\partial X}{\partial p_\mu}R = S^{(\mu)}\Lambda - \Lambda S^{(\mu)}$$

其中 $S^{(\mu)} = R^T\frac{\partial R}{\partial p_\mu} = -\frac{\partial R^T}{\partial p_\mu}R$

$$\longrightarrow \sum_{j,k} r_{j\alpha}\frac{\partial x_{jk}}{\partial \lambda_\gamma}r_{k\beta} = S_{\alpha\beta}^{(\mu)}(\lambda_\beta - \lambda_\alpha)$$

接下来一步我不会. 反正 $|J| = \prod_{\alpha<\beta}|\lambda_\beta - \lambda_\alpha|\, f(p_1 \cdots p_l)$

$\boxed{\text{Thm}}$ 设 $A_n \sim GUE_n(1)$ 记 $x_1, \cdots x_n$ 为 A_n 的特征值. 则其联合密度为

$$\rho_n(x_1, \cdots x_n) = \frac{1}{C_n} \exp\left(-\frac{1}{2}\sum x_i^2\right) \prod |x_k - x_j|^2$$

$$C_n = (2\pi)^{\frac{1}{2}n} \prod_{j=1}^{n} \frac{\Gamma(1+j)}{\Gamma(2)}$$

整合版 $\rho_n = \frac{1}{Z_{n\beta}} \exp\left(-\frac{\beta}{4}\sum \lambda_i^2\right) |\Delta_n \lambda|^\beta$

GOE $\Rightarrow \beta = 1$

GUE $\Rightarrow \beta = 2$

$$Z_{n\beta} = (2\pi)^{n/2} \left(\frac{2}{\beta}\right)^{\frac{1}{2}n + \frac{\beta}{4}n(n-1)} \prod \frac{\Gamma(1+\frac{\beta}{2}j)}{\Gamma(1+\frac{\beta}{2})}$$

GUE的关联核

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{1}{k!} h_k(x) h_k(y) \sqrt{\phi(x)\phi(y)}$$

$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$ $\quad h_k(x)$ 是Hermite多项式

$h_0(x) = 1.\quad h_k(x)\phi(x) = (-1)^k \frac{d^k}{dx^k}\phi$

Hermite多项式有正交关系 $\int_{-\infty}^{\infty} h_i(x) h_j(x) \phi(x)\, dx = j!\,\delta_{ij}$

此时 $\int K_n(x,x)\,dx = n$, $\int K_n(x,y) K_n(y,z) = K_n(x,z)$

k点关联函数 $\rho_n^{(k)} = \frac{n!}{(n-k)!} \int \cdots \int \rho_n(\lambda)\, d\lambda_{k+1} \cdots d\lambda_n$

$\boxed{\text{Thm}}$ (1) $\rho_n(\lambda) = \frac{1}{n!} \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{n}$

(2) $\rho^{(k)}(\lambda_1 \cdots \lambda_k) = \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{k}$

(3) 区间 $I \subseteq \mathbb{R}$ 内不含特征值的概率为

$$P(\lambda_j \in I^c, \forall j) = 1 + \sum \frac{(-1)^k}{k!} \int_I \cdots \int_I \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{k} d\lambda_1 \cdots d\lambda_k$$

Dyson lem

$$\boxed{\int_{\mathbb{R}} \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j}^{k+1} d\lambda_{k+1} = (n-k) \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{k}}.$$

$$\det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{k+1} \stackrel{\text{Laplace}}{=} \det\left[K_n(\lambda_i, \lambda_j)\right]_{i,j=1}^{k} K_n(\lambda_{k+1}, \lambda_{k+1}) + T$$

T 积分完了?

$\square$

$$\begin{array}{cc} \begin{bmatrix} K_{11} & \cdots & \\ \vdots & & \\ \vdots & & \end{bmatrix} & \begin{bmatrix} K_{1,n+1} \\ \\ \\ \end{bmatrix} \\ K_{n+1,1} \cdots 0 & K_{n+1,n+1} \\ \ell & \end{array}$$

看不懂. 想看完证明

道理和Dyson lem类似

4. 想考虑 λ_{max} 的极限分布　　以下使用拿来主义，看看得了

Hermite 多项式：

$$h_k = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (it)^k \exp\left[-\frac{1}{2}(t-ix)^2\right] dt$$

$$x h_k = h_{k+1} + k h_{k-1} \qquad x h_k = h_{k+1} + h_k'$$

$$h_k'' - x h_k' + k h_k = 0$$

C-D: $x \neq y$. $$\sum_{k=0}^{n-1} \frac{1}{k!} h_k(x) h_k(y) = \frac{h_n(x)h_{n-1}(y) - h_{n-1}(x)h_n(y)}{(n-1)!\,(x-y)}$$

$$\cancel{\Rightarrow \sum_{k=0}^{n-1} \frac{1}{k!} h_k(x)h_k(x) = h_n(x)}$$

$x=y$ 的特殊情况可以取极限，略去

$$h_k(x+t) = \sum_{j=0}^{k} \binom{k}{j} h_j(x)\, t^{k-j}.$$

简谐振子波函数：

$$x\psi_k = \sqrt{k+1}\,\psi_{k+1} + \sqrt{k}\,\psi_{k-1}$$

$$\psi_k' = -\frac{x}{2}\psi_k + \sqrt{k}\,\psi_{k-1}$$

$$-\psi_k'' + \frac{x^2}{4}\psi_k = \left(k+\frac{1}{2}\right)\psi_k.$$

$x \neq y$ C-D: $$K_n(x,y) = \frac{\psi_n(x)\psi_n'(y) - \psi_n(y)\psi_n'(x)}{x-y} - \frac{1}{2}\psi_n(x)\psi_n(y).$$

定义 $$K_{Ai}(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y} \qquad Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{izw + \frac{i}{3}w^3}\,dw$$

Airy统计：$$\lim_{n\to\infty} \frac{1}{n^{k/6}}\, \rho_n^{(k)}\left(\prod_{i=1}^{k} \sqrt{n}\left(2+\frac{x_i}{n^{2/3}}\right)\right) = \det\left(K_{Ai}(x_i,x_j)\right)_{ij}$$

Idea 将 Ai 联系到有波函数的形式

$$\hat{K}_n(\hat{x},\hat{y}) = \frac{1}{n^{1/6}} K_n\left(\sqrt{n}\left(2+\frac{\hat{x}}{n^{2/3}}\right), \sqrt{n}\left(2+\frac{\hat{y}}{n^{2/3}}\right)\right)$$

$$\hat{\psi}_n(\hat{x}) = n^{1/12}\, \psi_n\left(\sqrt{n}\left(2+\frac{\hat{x}}{n^{2/3}}\right)\right)$$

只要 $\lim \hat{\psi}_n(\hat{x}) = Ai(\hat{x})$　✓

用 laplace 方法（鞍点）$\Rightarrow$ 只需算 ψ

Tracy-Widom F_2 分布：$\forall t \in \mathbb{R}$ $\lim_{n\to\infty} P\left(\lambda_{max} \le \sqrt{n}\left(2+\frac{t}{n^{2/3}}\right)\right) = F_2(t)$

$$F_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\int_t^{\infty}\cdots\int_t^{\infty} \det\left[K_{Ai}(x_i,x_j)\right]_{ij}\, dx_1\cdots dx_k.$$

推论 $\frac{\lambda_{max}}{\sqrt{n}} \xrightarrow{P} 2$.（前情提要 $P(\lambda_{max} \le a_n y + b_n) = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!}\int_y^{\infty}\cdots\int_y^{\infty}\cdots$）

Fredholm 行列式

Ising 模型. 1维周期情

Gibbs测度 $\mu_{\Lambda;\beta;h}(\sigma) = \frac{1}{Z_{\Lambda;\beta;h}} e^{-\beta H(\sigma)}$

$H(\sigma) = -J \sum_{i,j\text{相邻}} \sigma_i\sigma_j - \underset{\text{外场}}{h} \sum_{i\in\Lambda} \sigma_i$

Λ 是有限格点集 $= [1, L]^d \subseteq \mathbb{Z}^d$

σ_i 在构形空间 $\{-1, 1\}$ 中

磁化强度 $M_\Lambda(\sigma) = \sum_{i\in\Lambda} \sigma_i$

$Z_{\Lambda;\beta;h}$ 是配分函数 $= \sum_{\sigma_1\cdots\sigma_N=\pm1} \exp\left(\beta J \sum \sigma_i\sigma_{i+1} + h\sum_i \sigma_i\right)$.

将 $\mu_{\Lambda;\beta;h}$ 归一化

$$P^{\text{转移矩阵}} = \begin{array}{c|cc} & + & - \\ \hline + & e^{\beta J+\beta h} & e^{-\beta J} \\ - & e^{-\beta J} & e^{\beta J-\beta h} \end{array}$$

用 $\langle \sigma | P | \sigma' \rangle$ 表示 σ, σ' 处的元素

$\Rightarrow \langle \sigma | P | \sigma' \rangle = \exp\left(\frac{1}{2}(\sigma+\sigma')\beta h + \beta J \sigma\sigma'\right)$

$\Rightarrow Z_{\Lambda;\beta;h} = \sum_{\sigma_1\cdots\sigma_N=\pm1} \prod_{i=1}^{N} \exp\left(\beta J \sigma_i\sigma_{i+1} + \beta h \frac{\sigma_i+\sigma_{i+1}}{2}\right)$

神秘观察 $= \sum_{\sigma_1\cdots\sigma_N=\pm1} \prod_{i=1}^{N} \langle \sigma_i | P | \sigma_{i+1} \rangle$

$= \mathrm{tr}(P^N)$

$\lambda_+ := e^{\beta J} A_+(h)$

$\lambda_- := e^{\beta J} A_-(h)$

$\Rightarrow Z_{\Lambda;\beta;h} = e^{\beta J N}(A_+^N + A_-^N)$

$\mathbb{E}\left[\frac{M_n(\sigma)}{n}\right]$ 是什么. $= \sum_{\sigma} \frac{M_n(\sigma)}{n Z_{\Lambda;\beta;h}} \exp(-\beta H(\sigma))$

$\frac{\partial}{\partial h} \log Z_{\Lambda;\beta;h} = \frac{1}{Z_{\Lambda;\beta;h}} \sum_{\sigma} \cancel{\exp(\beta J \sigma_i\sigma_{i+1})}\ \beta M(\sigma) \exp -\beta H(\sigma)$

$\Rightarrow \mathbb{E}\left[\frac{M_N(\sigma)}{N}\right] = \frac{1}{\beta N} \frac{\partial}{\partial h} \log Z_{\Lambda;\beta;h}$

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$$\mathbb{E}\left[\frac{M_N(\sigma)}{N}\right] = \frac{1}{\beta N}\frac{\partial}{\partial h}\log Z_{N;\beta;h} \qquad Z_{\Lambda;\beta;h} = e^{\beta J N}(A_+^N + A_-^N)$$

$$= \frac{1}{\beta N}\frac{\partial}{\partial h}\left[\beta J N + \log(A_+^N + A_-^N)\right]$$

对 N 求。

$$= \frac{1}{\beta(A_+^N + A_-^N)}\left(A_+^{N-1}\frac{\partial}{\partial h}A_+ + A_-\cdots\right)$$

$$\approx \frac{1}{\beta\,\left(1+\left(\frac{A_-}{A_+}\right)^N\right)}\frac{1}{A_+}\frac{\partial}{\partial h}A_+ \qquad \beta\left(h + \frac{t}{\beta N^\delta}\right)$$

$$= \frac{1}{\beta A_+}\frac{\partial}{\partial h}A_+$$

用矩母函数 $\mathbb{E}\left[\exp\left(\frac{t}{N^\delta}M_N\right)\right]$ $\delta = 1, \frac{1}{2}$

可以导出 WLLN & CLT.

Cuire-Weiss 模型

$$\Lambda = [1, L]^d$$

平均场近似：$\sum_{j:i\sim j} \sigma_j \sigma_i = 2d\sigma_i \frac{1}{2d} \sum_{j:j\sim i} \sigma_j \approx \boxed{2d} \sigma_i \frac{1}{|\Lambda|} \sum \sigma_j$

相邻数

原 $H(\sigma) = -J \sum_{i\sim j} \sigma_i \sigma_j - h \sum_{i\in\Lambda} \sigma_i$

$$\approx -\frac{Jd}{N} \left(\sum_{i=1}^{N} \sigma_i\right)^2 - h \sum_{i=1}^{N} \sigma_i$$

C-W模型的 Gibbs 测度

$$\mu_{\Lambda;\beta;h}(\sigma) = \frac{1}{Z_{\Lambda;\beta;h}} \exp(-\beta H(\sigma))$$

$$Z_{\Lambda;\beta;h} = \sum_{\sigma} \exp(-\beta H(\sigma))$$

临界逆温度

定理 LLN意义下相变. $\beta_c = \frac{1}{2d}$ $h=0$. $M_N = \sum_{i=1}^{N} \sigma_i$

(1) 当 $\beta \in [0, \beta_c]$, $\forall \varepsilon > 0$ $\exists$ $C_1 = C_1(\beta, N)$ s.t. 高温

$$\mu_{N;\beta;0}\left(\left|\frac{M_N}{N}\right| \geq \varepsilon\right) \leq 2e^{-C_1 N}$$

(2) 当 $\beta \in (\beta_c, \infty)$, 存在自发磁化强度 $m^* = m^*(\beta) > 0$, s.t. $\forall \varepsilon > 0$, $\exists C_2 = C_2(\beta, \varepsilon) > 0$

对充分大的 N 有

$$\mu_{N;\beta;0}\left(\left|\left|\frac{M_N}{N}\right| - m^*\right| \geq \varepsilon\right) \leq 2e^{C_2 N}.$$

Rmk: $\frac{M_N}{N} \longrightarrow \begin{cases} \delta_0 & \beta \leq \beta_c \\ \frac{1}{2}(\delta_{m^*} + \delta_{-m^*}) & \beta > \beta_c \end{cases}$

引理 1. 自由能函数 $f_\beta(m) = -\beta d m^2 - S(m)$ $m \in [1, 1]$

$$S(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}$$

β是逆温

$dm^2 - \frac{1}{\beta} S$ 物理中的自由能

$E(m) - TS(m)$

m^*就是自由能极小

$\Rightarrow$最可能的宏观态

则有 $\lim_{N\to\infty} \frac{1}{N} \log Z_{N;\beta;0} = -\min_{m\in[1,1]} f_\beta(m)$

更一般的 $K \subseteq [-1, 1]$ $\lim \frac{1}{N} \log \mu_{N;\beta;0}\left(\frac{M_N}{N} \in K\right) = -\min_{m\in K} f_\beta(m) + \min_{m\in[-1,1]} f_\beta(m)$

证. $A_N = \left\{-1 + \frac{2k}{N} \,\middle|\, k = 0, \cdots, N\right\}$ $I = mN$

$$\mu_{N;\beta;0}\left(\frac{M_N}{N} \in K\right) = \frac{1}{Z_{N;\beta;0}} \sum_{m\in K\cap A_N} \binom{N}{\frac{(1+m)N}{2}} e^{\beta d m^2 + \beta h m N}$$

反解 k

证明 $\lim_{N\to\infty} \frac{1}{N}\log Z_{N;\beta;0} = -\min_{m\in[-1,1]} f_\beta(m)$

有 $\Delta \le Z_{N;\beta;0} \le (N+1)\Delta$ ①, $\Delta := \max_{m\in A_N} \binom{N}{(1+m)N/2} \exp(\beta d m^2 N)$

$$\frac{C_1}{\sqrt{N}}\exp(N\,S(m)) \le \binom{N}{(1+m)N/2} \le C_2\sqrt{N}\exp(N\,S(m)) \quad ②$$

代入 $\Rightarrow$ 上界：

$$Z_{N;\beta;0} \le (N+1)\sqrt{N}\,C_2 \exp N \max\left(S(m)+\beta d m^2\right)$$

$$\le C_2 (N+1)\sqrt{N} \exp\left(-N\min f_\beta\right)$$

$$\Rightarrow \limsup \frac{1}{N}\log Z_{N;\beta;0} \le -\min f_\beta$$

$\exists\, m^* = \min f_\beta$

$N\to\infty$ $\forall\delta.$ $|f_\beta(m) - f_\beta(m^*)| < \delta$ $\exists\, m\in A_N$

$$Z_{\beta;N;0} \ge \frac{C_1}{\sqrt{N}} \exp N\left(-\min f_\beta + \delta\right)$$

$$\frac{1}{N}\log Z_{\beta;N;0} > -\min f_\beta(m^*) - \delta \qquad \delta\to 0.$$

□

β: $f'(m) = -2\beta d m + \frac{1}{2}\log\frac{1+m}{1-m}$ $\qquad I_\beta = f_\beta(m) - \min_{m'\in[-1,1]} f_\beta(m')$

$f''(m) = -2\beta d + \frac{1}{1-m^2}$

I_β

$\boxed{2\beta d \le 1}$ $\quad f''\ge 0$ $\quad f'(m)=0$ 解唯一 $m=0$

$m=0$

$K=[-1,-\varepsilon]\cup[\varepsilon,1]$. I_β 最小值为 $C_1(\beta,\varepsilon)>0$

$$\frac{1}{N}\log \mu_{N,\beta;0}\left(\frac{M_N}{N}\in K\right) \to -C_1$$

$$\Rightarrow \mu_{N;\beta;0}\left(\left|\frac{M_N}{N}\right|\ge\varepsilon\right) \le 2e^{-C_1 N}$$

$2\beta d > 1$ $\quad m = \pm\sqrt{1-\frac{1}{2\beta d}} = \pm m^*$ $\quad K=[-1,1]\setminus\left(B(m^*,\varepsilon)\cup B(-m^*,\varepsilon)\right)$

□

CLT: of C-W model

$\beta_c = \frac{1}{2d}$ $h=0$ $M_N = \sum_{i=1}^N \sigma_i$

(1) $\beta < \beta_c$ $\sqrt{\frac{1-2\beta d}{N}} M_N \xrightarrow{D} N(0,1)$

(2) $\beta = \beta_c$ $\frac{M_N}{N^{3/4}} \xrightarrow{D} Y$ $f_Y(y) = \frac{1}{C} e^{-\frac{1}{12}y^4}$

证:
$M(t) = \mathbb{E}\left[\exp\left(t\frac{M_N}{N^\delta}\right)\right] = \frac{Z_{N;\beta;h}}{Z_{N;\beta,0}}$ $h = \frac{t}{\beta N^\delta}$ (上一节技术)

$Z_{N;\beta;h} = \sum_{\sigma_i} \exp\left(\frac{\beta d}{N}\left(\sum_{i=1}^N \sigma_i\right)^2 + \beta h \sum_{i=1}^N \sigma_i\right)$

处理二次项! H-S 变换 $e^{\alpha x^2} = \frac{1}{\sqrt{\pi\alpha}}\int dy \exp\left(-\frac{y^2}{\alpha} + 2xy\right)$

$\Rightarrow Z_{N;\beta;h} = \sqrt{\frac{N}{\pi\beta d}} \int \sum_{\sigma_i} \exp\left(-\frac{N}{\beta d}y^2 + (2y+\beta h)\sum_{i=1}^N \sigma_i\right) dy$

$= \sqrt{\frac{N}{\pi\beta d}} \int dy \left(\exp(2y+\beta h) + \exp(-2y-\beta h)\right)^N \exp\left(\frac{N}{\beta d} y\right)$

$= \sqrt{\frac{N}{4\pi\beta d}} \int dy \exp\left(-\frac{\beta h^2 N}{4d} + \frac{hN}{2d} y - N g(y)\right)$

$y \mapsto \frac{y-\beta h}{2}$ $g(y) = \frac{1}{4\beta d} y^2 - \log(e^y + e^{-y})$

总之, 令 $I_N(t) = \int_{\mathbb{R}} \exp\left(\frac{N^{1-\delta} t y}{2\beta d} - N g(y)\right) dy$

$M(t) = \exp\left(-\frac{N^{1-2\delta} t^2}{4\beta d}\right) \frac{I_N(t)}{I_N(0)}$

$g'(y) = \frac{y}{2\beta d} - \frac{e^y - e^{-y}}{e^y + e^y}$

$g''(y) = \frac{1}{2\beta d} - \frac{4e^{2y}}{(e^{2y}+1)^2}$

$\Rightarrow$ $2\beta d \le 1$ $y_0 = 0$ 唯一 zero

$\beta < \beta_c$ $\delta = \frac{1}{2}$

$I_N(t) \underset{\text{Laplace}}{\sim} e^{-Ng(0)} \int_{-\delta_0}^{\delta_0} \exp\left(\frac{1}{2\beta d} N^{\frac{1}{2}} t y - \frac{N}{2}\left(\frac{1}{2\beta d} - 1\right) y^2\right) dy$

$\sim e^{-Ng(0)} \frac{1}{\sqrt{N}} \cdot 2\sqrt{\frac{\pi\beta d}{1-2\beta d}} \exp\left(\frac{t^2}{4\beta d(1-2\beta d)}\right)$ $\delta_0 \ll 1$

$\Rightarrow$ $M(t) \xrightarrow{\Phi} e^{\frac{1}{2}(bt)^2}$ $b = \frac{1}{\sqrt{1-2\beta d}}$

$\beta = \beta_c$. $\delta = \frac{3}{\varphi}$

$$g(y) - g(y_0) = \frac{1}{12} y^4$$ 比较2阶

李杨单位圆

铁磁模型 $H(\sigma) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i$

逆温度 $\beta \geq 0$ 铁磁耦合 J_{ij} 令 $J = J_{ij} \Rightarrow$ Ising 模型

记号 $[N] = \{1, \cdots N\}$; $\sigma = (\sigma_1, \cdots \sigma_N)$

$$X(\sigma) = \{ i \in [N] : \sigma_i = -1 \}$$

$$Z_{N;\beta;h} = \sum_{\sigma} \exp\left(\beta \sum_{i<j} J_{ij}\sigma_i\sigma_j + \beta h \sum_{i=1}^{N} \sigma_i \right)$$

$$= \exp\left(\beta h N + \beta\sum_{i<j} J_{ij}\right) \cdot \sum_{\sigma} \exp\left(\beta h \sum_{i=1}^{N}(\sigma_i - 1) + \beta \sum_{i<j} J_{ij}(\sigma_i\sigma_j - 1)\right)$$

$$= \exp\left(\beta h N + \beta \sum_{i<j} J_{ij}\right) \cdot \sum_{X\subseteq[N]} z^{|X|} \prod_{i\in X}\prod_{j\notin X} a_{ij}$$

$z = e^{-2\beta h}$

$a_{ij} = a_{ji} = e^{-2\beta J_{ij}}$

杨李多项式 $P_A(z_1, \cdots z_N) = \sum_{S\subseteq[N]} z^{S} \prod_{i\in S}\prod_{j\notin S} a_{ij}$

$$\boxed{z^S := \prod_{i\in S} z_i}$$

李杨单位圆定理 $A = (a_{ij})_{i,j}^{N}$ Hermite $|a_{ij}| \leq 1 \Rightarrow$ 当 $|z_i| \leq 1$ $i = 1, \cdots N$

则 $P_A(z_1, \cdots z_N) \neq 0$

Proof 1: 对N归纳 . 没看懂.

$$P_A(z_1, \cdots z_N) = \sum_{S \not\ni N} z^S \prod_{i\in S}\prod_{j\notin S} a_{ij} + \sum_{S\ni N} z^S \prod_{i\in S}\prod_{j\notin S} a_{ij}$$

Proof 2: ~~term1: $a\in[-1,1]$~~ $\mathcal{A}$ 类: (1) $P_I(z_I)$ 为 z_J $(J\subseteq I)$ 的单项式之线性组合

(2) $|z_i| < 1$. $P_I(z_I) \neq 0$ (李杨性)

(1) $a \in [-1,1]$ 则 $P(z_1, z_2) = z_1 z_2 + a(z_1+z_2) + 1 \in \mathcal{A}$

(2) $P_I(z_I) \in \mathcal{A}$. $P_J(z_J) \in \mathcal{A} \Rightarrow P_{I\cup J}(z_{I\cup J}) := P(z_I)P(z_J) \in \mathcal{A}$ ($I \cap J = \emptyset$)

(3) Asano 缩并: $I \cap \{\alpha, \beta\} = \emptyset$

$P_{I\cup\{\alpha,\beta\}}(z_{I\cup\{\alpha,\beta\}}) = A z_\alpha z_\beta + B z_\alpha + C z_\beta + D$ ABCD是 z_I 多项式

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对 $\gamma \notin I$ 假设 $P_{I\cup\{\gamma\}}(z_{I\cup\{\gamma\}}) = Az_\gamma + D$

则 $P_{I\cup\{\alpha,\beta\}}(z_{I\cup\{\alpha,\beta\}}) \in \mathcal{A} \Rightarrow P_{I\cup\{\gamma\}}(z_{I\cup\{\gamma\}}) \in \mathcal{A}$

因为 $\left|-\frac{D}{A}\right| \geq 1$

不知道啥情况, 反正 $\prod_{i<j} A_{ij} \xrightarrow{\text{Asano}} P(z_1, \cdots z_N)$

计值结论得了.

清楠某本: $M_X(h) = \mathbb{E}[\exp hX] = z^{-\frac{N}{2}} \frac{P(z, \cdots z)}{P(1, \cdots 1)}$

$z = e^{-2\beta h}$

$\Rightarrow M_X(h)$ 只有纯虚零点.

复习补缺

1. Wigner 矩阵与半圆律

$$\begin{cases} A_n = (a_{ij})_{i,j=1}^{n} \quad a_{ij} = a_{ji} \\ a_{ii} \sim Y,\ a_{ij} \sim Z \quad \mathbb{E}[Y] = 0 = \mathbb{E}[Z] \quad Var(Y) < \infty \quad Var(Z) = 1 \\ Y, Z \text{ 高阶矩存在} \end{cases}$$

$$\lim \mathbb{E}\left[\frac{1}{n} tr\left(\frac{A_n}{\sqrt{n}}\right)^k\right] \longrightarrow \gamma_k = \int_{-2}^{2} \frac{x^k}{2\pi}\sqrt{4-x^2}\, dx$$

$$\gamma_k = \begin{cases} \frac{1}{m+1}\binom{2m}{m} & k = 2m \\ 0 & k = 2m+1 \end{cases}$$

$$LHS = \lim \frac{1}{n^{\frac{k}{2}+1}} \sum_{i_1 \cdots i_k} a_{i_1 i_2} \cdots a_{i_k i_1}$$

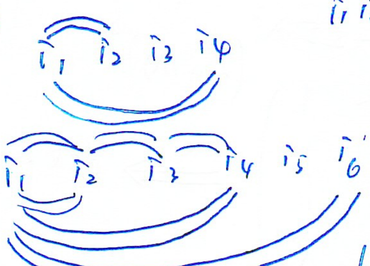
自由度大 ⟶ 贡献大 $\boxed{\text{主项}}$

顶点计算：计算可见 顶点有 $\frac{k}{2}+1$ 个 （k 偶 $= 2m$）

$(i_1\ i_2 \cdots i_{2m})$ 中 每条边出现2次，$m+1$个顶点

$k=4$时 $a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} a_{i_4 i_1}$，$i_3 = i_1$

$i_1\ i_2\ i_1\ i_4$



总之，对应到非交叉配对

$k=6$时 或

~~$i_1 i_2 i_1 i_3 i_1$~~

$(i_1 i_2 i_3 i_4 i_3 i_2)$

↓

$1\ 1\ 1\ {-1}\ {-1}\ {-1}$

对应 路径 $(i_1, \cdots i_{2m}) \longrightarrow (a_1 \cdots a_{2m})$

$$a_j = \begin{cases} 1 & i_j i_{j+1} \text{ 首次出现} \\ -1 & \text{otherwise} \end{cases} \Rightarrow \text{随机游走 } S_i \geq 0$$

$$\frac{1}{m+1}\binom{2m}{m}$$

↑

主项贡献总数

□

2. GOE & GUE 随机矩阵

$$X=(X_{ij})_{i,j=1}^{n} \qquad X_{ij}\sim N(0,1)$$

$$H=\frac{1}{\sqrt{2}}(X+X^T) \qquad H_{ij}\sim N(0,1) \qquad H_{ij}\ \ 1\le i<j\le n$$

$$H_{ii}\sim N(0,2)$$

复：$H=\frac{1}{2}\left((X+iY)+(X^T-iY^T)\right)$ $\quad X_{ij},\ Y_{ij}\sim N(0,1)$

Wigner —— 想研究对称矩阵的谱 —— GOE & GUE

结论 $\rho_{n\beta}(\lambda_1,\cdots\lambda_n)=\frac{1}{Z_{n\beta}}e^{-\frac{\beta}{4}\sum_{i=1}^{n}\lambda_i^2}\prod_{j<k}|\lambda_j-\lambda_k|^{\beta}$

$\beta=1$ GOE

$\beta=2$ GUE

$$Z_{n\beta}=(2\pi)^{\frac{n}{2}}\left(\frac{2}{\beta}\right)^{\frac{n}{2}+\frac{\beta}{4}n(n-1)}\prod_{j=1}^{n}\frac{\Gamma(1+\frac{\beta}{2}j)}{\Gamma(1+\frac{\beta}{2})}$$

$\beta=1$

~~由~~数几何而微分+矩阵计算后

$$\Rightarrow J=\boxed{g(p_1\cdots p_{\frac{n(n-1)}{2}})}\prod_{\alpha<\beta}|\lambda_\beta-\lambda_\alpha| \qquad \square$$

Selberg 积分.

定义 k点关联 $\rho_n^{(k)}(\lambda_1,\cdots\lambda_n)=\frac{n!}{(n-k)!}\int\cdots\int_{\mathbb{R}^{n-k}}\rho_n(\lambda)\,d_{k+1}\cdots d_n$

GUE: $\rho_n(\lambda)=\frac{1}{Z_{n2}}e^{-\frac{1}{2}\sum\lambda_i^2}(\Delta_n(\lambda))^2$

$$\Delta_n=\begin{pmatrix}1&\cdots&1\\ \vdots&&\vdots\\ \lambda_1^{n-1}&\cdots&\lambda_n^{n-1}\end{pmatrix}\Longrightarrow \det\Delta_n=\det[h_{i-1}(\lambda_j)]_1$$

首一

$$\stackrel{?}{=}\prod_{i=0}^{n-1}c_i\det\left(\frac{1}{c_{i-1}}h_{i-1}(\lambda_j)\right)$$

感觉性

~~感觉可证：$P(\lambda_{max}<ay+b)=1+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\int_y^\infty\cdots\int_y^\infty\det\ \ldots$~~

Laplace方法

例 $\cancel{I_n = \int e^{nf(t)} e^{\frac{it z}{e}} dt \quad f(t) = -\frac{1}{2}t^2 + \log(1+t) + iut}$

$\cancel{f'(t) =}$ $\cancel{形 \int e^{nf(x)} dx}$

这个例太困难. Concept: 利用 n 将小区间大权重积分拉平至R. 凑高斯积分.

作业题回顾:

随机矩阵: 主项贡献来自二次不重复项.

1. $X = (X_{ij})_{i,j}^{n}$ $\quad X_{ij} \overset{iid}{\sim} N(0,1) \quad \alpha = n - p$ fixed.

$1 \le i \le p, 1 \le j \le n$

$$\frac{1}{p} \mathbb{E}\left[tr\left(\left(\frac{XX^T}{p}\right)^m\right)\right] = \frac{1}{p^{1+m}} \mathbb{E} \sum Y_{i_1 i_2} Y_{i_2 i_3} \cdots Y_{i_m i_1}$$

$$Y = (XX^T) = \left(\sum_{k=1}^{n} X_{ik} X_{jk}\right)_{ij}^{p}$$

$$= \frac{1}{p^{1+m}} \mathbb{E}\left[\sum_{\substack{i_1 \cdots i_m \\ k_1 \cdots k_m}} X_{i_1 k_1} X_{i_2 k_1} X_{i_2 k_2} X_{i_3 k_2} \cdots X_{i_m k_m} X_{i_1 k_m}\right]$$

$i_1 \cdots i_m$ 取值于 $1, \cdots p$

$k_1 \cdots k_m$ 取值于 $1, \cdots n$.

$m=2$ 时 $\quad X_{i_1 k_1} X_{i_2 k_1} X_{i_2 k_2} X_{i_1 k_2}$

$k_1 = k_2 \quad$ 3个自由

3 $\quad X_{i_1 k_1} X_{i_2 k_1} X_{i_2 k_2} X_{i_3 k_2} X_{i_3 k_3} X_{i_1 k_3}$

$k_1 = k_3 \quad k_2 = k_1$

自由指标数 m+1个 有 2m个指标 每条路两次

$\Downarrow$

非交叉配对

主项 $\frac{1}{m+1}\binom{2m}{m} p^{m+1}$ 近同指数

$$\Rightarrow \frac{1}{p} \mathbb{E}\, tr\left(\frac{XX^T}{p}\right)^m \longrightarrow \frac{1}{m+1}\binom{2m}{m}. \quad \square$$

Idea: 找到非交叉配对 $\Longrightarrow \frac{1}{m+1}\binom{2m}{m}$

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2. 矩方法：$A_n = (a_{ij})_{i,j}^{n}$ Wigner $\{a_{ij} : i \le j\}$ 独立，$\mathbb{E}[a_{ij}] = 0$ $Var(a_{ij}) = 1$

高阶矩一致有界. 证 $\lim_{n\to\infty} P(\|A_n\|_2 \ge n^{\frac{1}{2}+\delta}) = 0$ $\forall \delta > 0$

Note：由于实对称，所以不用区别奇异值和特征值了.

$$\|A_n\|_2 = \max|\lambda_n| \le \left(\sum \lambda_i^{2m}\right)^{\frac{1}{2m}}$$

$$P\left(\|A_n\|_2 \ge n^{\frac{1}{2}+\delta}\right) \le \frac{\mathbb{E}\,|\lambda_i^{2m}|}{n^{m+2m\delta}} \le \frac{\mathbb{E}\left(\sum \lambda_i^{2m}\right)}{n^{m+2m\delta}}$$

$$\le \frac{\mathbb{E}\,[tr(A_n^{2m})]}{n^{m+2m\delta}} \sim \frac{\frac{1}{m+1}\binom{2m}{m} n^{m+1}}{n^{m+2m\delta}} \to 0$$

令 $2m\delta > 1$ ✓ □

3. 复版本. $A_n = A_n^{*}$ a_{ii}, $Re\, a_{ij}$ $Im\, a_{ij}$ 独立
$a_{ii} \sim Y$, $Re\, a_{ij} \sim Z$, $Im\, a_{ij} \sim Z$

$\mathbb{E}[Y] = 0 = \mathbb{E}[Z]$ $Var(Y) < \infty$ $Var(Z) = 1$ 高阶矩有界

证半圆律 $\frac{1}{n}\mathbb{E}\left[tr\left(\frac{A_n}{\sqrt{n}}\right)^{k}\right] \to \gamma_k$.

这里的主项是 ij 一次 ji 一次 $\rightsquigarrow \frac{1}{m+1}\binom{2m}{m}$.

$\mathbb{E}[|a_{ij}|^2] = 1$. □

4 GOE的谱分布 $n=2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}^2 = 0$$

$$0 = \left(\lambda - \frac{a_{11}+a_{22}}{2}\right)^2 + a_{11}a_{12} - a_{12}^2 - \frac{(a_{11}+a_{22})^2}{4}$$

$$= \left(\lambda - \frac{a_{11}+a_{22}}{2}\right)^2 - \frac{4a_{12}^2 + (a_{11}-a_{12})^2}{4}$$

$$\begin{cases} \lambda_+ = \dfrac{(a_{11}+a_{22}) + \sqrt{4a_{12}^2 + (a_{11}-a_{22})^2}}{2} \\ \lambda_- = \dfrac{(a_{11}+a_{22}) - \sqrt{4a_{12}^2 + (a_{11}-a_{22})^2}}{2} \end{cases}$$

$$f(\lambda_+,\lambda_-,a_{12}) = \frac{1}{2\sqrt{2\pi}} \exp -\frac{1}{4}(\lambda_+^2+\lambda_-^2) \cdot \left|\frac{\partial(\lambda_+,\lambda_-,a_{12})}{\partial(a_{11},a_{22},a_{12})}\right|^{-1}$$

$$|J| = \begin{vmatrix} \dfrac{1+\frac{a_{11}-a_{22}}{\sqrt{\Delta}}}{2} & \dfrac{1-\frac{a_{11}-a_{22}}{\sqrt{\Delta}}}{2} & \dfrac{a_{12}}{2\sqrt{\Delta}} \\ \dfrac{1-\frac{a_{11}-a_{22}}{\sqrt{\Delta}}}{2} & \dfrac{1+\frac{a_{11}-a_{22}}{\sqrt{\Delta}}}{2} & -\dfrac{a_{12}}{\sqrt{\Delta}} \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{\left(1+\frac{a_{11}-a_{22}}{\sqrt{\Delta}}\right)^2-\left(1-\frac{a_{11}-a_{22}}{\sqrt{\Delta}}\right)^2}{4} = \frac{a_{11}-a_{22}}{\sqrt{\Delta}}$$

$$\lambda_+ + \lambda_- = a_{11}+a_{22}$$

$$\lambda_+ - \lambda_- = \sqrt{\Delta}$$

$$\lambda_+\lambda_- = a_{11}a_{22} - a_{12}^2$$

$$(a_{11}-a_{22})^2 = \cancel{(\lambda_++\lambda_-)^2 - 4(\lambda_+\lambda_- + a_{12}^2)} = \sqrt{(\lambda_+-\lambda_-)^2 - 4a_{12}^2}$$

$$\Rightarrow |J| = \frac{\sqrt{(\lambda_++\lambda_-)^2 - 4\cancel{(\lambda_+\lambda_- + a_{12}^2)}-4a_{12}^2}}{\lambda_+-\lambda_-}$$

$$f(\lambda_+,\lambda_-,a_{12}) = \frac{1}{\cancel{2\sqrt{2\pi}}}(\cdot) \exp\left(-\frac{1}{4}(\lambda_+^2+\lambda_-^2)\right) \frac{\lambda_+-\lambda_-}{\sqrt{(\lambda_+-\lambda_-)^2-4a_{12}^2}}$$

$$\frac{1}{2}\int_{-\frac{\lambda_+-\lambda_-}{2}}^{\frac{\lambda_+-\lambda_-}{2}} \frac{da_{12}}{\sqrt{\left(\frac{\lambda_+-\lambda_-}{2}\right)^2-a_{12}^2}} = \frac{1}{2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

$$\Rightarrow f(\lambda_+,\lambda_-) = \frac{1}{4\sqrt{2\pi}} \exp\left(-\frac{1}{4}(\lambda_+^2+\lambda_-^2)\right)\cdot(\lambda_+-\lambda_-) \qquad \square$$

5. GUE. 密度推导

$$H = \frac{1}{2}\left((X+iY) + (X+iY)^H\right).$$

上三角. $f(z) = \frac{1}{\pi} e^{-|z|^2}$

对角 $f(x) = \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}x^2$

$$f(H) = \frac{1}{(2\pi)^{\frac{n}{2}}\cdot\pi^{\frac{n(n-1)}{2}}} \prod_{i=1}^{N} \exp^{-\frac{1}{2}x^2} \prod_{i<j} \exp -|z|^2 = 2^{-\frac{n}{2}}\pi^{-\frac{n^2}{2}} \cdot \exp -\frac{1}{2}\mathrm{tr}H^2.$$

酉不变性.

$$f(H) \propto \exp -\frac{1}{2}\operatorname{tr}H^2 = \exp -\frac{1}{2}\operatorname{tr} H H^H$$

$$Y = UH \qquad U\text{ 酉}$$

$$f(Y) \propto \exp -\frac{1}{2}\operatorname{tr} UHH^H U^H = \exp -\frac{1}{2}\operatorname{tr}Y^2.$$

密度相同（同分布）.

GUE 和 GOE 的矩. Wick 公式.

1. $a_p(N) = \mathbb{E}[\operatorname{tr}(H^{2p})]$

$$= \sum_{i_1\cdots i_{2p}} \mathbb{E}[H_{i_1 i_2}\cdots H_{i_{2p} i_1}]$$

$$= \sum_{i_1\cdots i_{2p}} \sum_{\text{pair}} \prod \mathbb{E}[H_{..} H_{..}]$$

GOE $H_{ii} \sim N(0,2)$, $H_{ij} \sim N(0,1)$

GUE $H_{ii} \sim N(0,1)$, $H_{ij} \sim N_{\mathbb{C}}(0,1)$

$p=1.$ $\cancel{a_p(1)}$ $a_1(N) = \cancel{N}\ 2N + 2\cdot\frac{N(N-1)}{2} = N^2.$

$p=2.$ $\sum_{i_1\cdots i_4} \mathbb{E}[H_{i_1i_2}H_{i_2i_3}H_{i_3i_4}H_{i_4i_1}]$ 无交叉

$= \sum_{i_1\cdots i_4}$ $\cancel{\langle i_1i_2\rangle\langle i_2i_3\rangle +}$ $\langle i_1i_2, i_2i_3\rangle\langle i_3i_4, i_4i_1\rangle + \langle i_1i_2, i_3i_4\rangle\langle i_2i_3, i_4i_1\rangle + \langle i_1i_2, i_4i_1\rangle\langle i_2i_3, i_3i_4\rangle.$

$= \sum_{\substack{i_1\cdots i_4\\ i_3=i_1}}$ $N^3\cancel{(N-1)(N-2)}$ 3个自由度. $\quad \sum N$ $\cancel{i_3=i_1, i_4=i_1\ i}$ 1个自由度. $\quad \sum N^3\cancel{(N-1)(N-2)}$ GUE 高度自由!

$\cancel{2(N^3-3N^2+2N)} + N$ $\quad 2N^3+N.$ $\quad$ 1 交叉

$i_1 \quad i_2 \quad i_3 \quad i_4$

$i_1 \quad i_2 \quad i_3 \quad i_4$

3

$p=3.$ $1\ 2\ 3\ 4\ \ 5\ 6$ 非交叉 配对 5

$\frac{6\ 5\ 4}{6\cdot 4}$

一交叉

非交 $\frac{1}{4}\binom{6}{3}$ 5 $\ 1\ 2\ 3\ 4\ \ 5\ 6.$ $3+2\cdot2+1$

$1\ 2\ 3\ 4\ 5\ 6$

$5N^4 + 10N^2$

$1\ 2\ 3\ 4\ 5\ 6$

$(123456)(13)(25)(46) = (14)(2653).$

利用代表 $(1234)(12)(34) = (13)(2)(4)$

$(1234)(14)(23) = (1)(24)(3)$ 3

$(1234)(13)(24) = (1432)$ 1

$\cancel{3\times}$ 5. 3 $(123456)(12)(34)(56) =$

$(135)(2)(4)(6)$ 4

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$(123456)(13)(24)(56) =$ 2

$(1432\ 5)(6)$

p=4. 非交 $\frac{1}{5}\binom{8}{4} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 6\cdot 5} = 14$ 5.

一交

$(12345678)(12)(34)(5)(68) = (135876)(2)(4)$ 3

二交 $(12345678)(13)(24)(57)(68) = (14325876)$ 1

三交 $(12345678)(15)(26)(37)(48) = (16385274)$ 1.

$7\ \cancel{6}\ 5\ 3 = 105$

~~排除 8×2+~~

段了. 没有去轮椅   搜 Zagier .

GOE 低维可以 轮椅 -T-

$(1234)(13)(24) = (1432)$ (14)

p=1. n^2+n

p=2.

$2(n^2+n) + 4n + (n^2 ? n)$

轮椅都以人砸了. T∧T .

...

p=3

熵 $H(X) = -\int \ln f(x)\cdot f(x)dx$

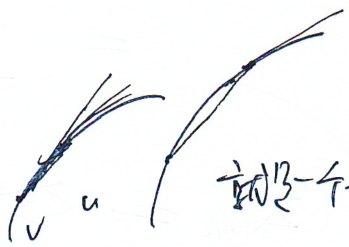
条件概率 ⇒ 相对熵 $H_Y(X) = -\sum P(X_i|y)\ln P(X_i|y)$

$$\begin{cases} H_Y(X) \le H(X) \\ H(X,Y) = H_Y(X) + H(Y) \end{cases}$$

Gibbs $u - u\log u \le v - u\log v$ $u,v>0$

$\Leftrightarrow u - v \le u(\log u - \log v)$ $u>v$

$\Leftrightarrow \frac{1}{u} \le \frac{\log u - \log v}{u-v}$



就是一个切线不等式.

$\Rightarrow -\int f\log f \le \int f\log g$

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如何用 Gibbs：1. $\mathbb{E}[X]=0$ $Var(X)=1$ 何时熵最大 ϕ.

f 在 $(-\infty,\infty)$ 上 $\exists$.

$$-\int f\log f \le -\int f\log \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$$

$$= -\int f\log\frac{1}{\sqrt{2\pi}} + \frac{1}{2}\int x^2 f$$

$$= -\log\frac{1}{\sqrt{2\pi}} + \frac{1}{2} = \log\sqrt{2\pi e}.$$

2. $D=[0,\infty)$ $\mathbb{E}[X]=\frac{1}{\lambda}$

$$-\int f\log f \le -\int f\log \lambda e^{-\lambda x}$$

$$= -\int f\log\lambda + \int \lambda x f$$

$$= -\log\lambda + 1 = \log\frac{e}{\lambda}.$$

3. $D(0,a)$

$$-\int f\log f \le -\int f\log\frac{1}{a} = \log a.$$

$\sum_{\omega}$

1. Jensen 离散.

Ising 模型

$$1.\ \langle e^{tM_n}\rangle = \frac{1}{Z_{N,\beta,h}}\sum_{\{\sigma\}} e^{t\sum\sigma_i}\cdot e^{-\beta H(\sigma)}$$

$$= \frac{1}{Z_{N,\beta,h}}\sum_{\{\sigma\}}\exp\left(\beta J\sum_{\langle ij\rangle}\sigma_i\sigma_j + \beta\left(h+\frac{t}{\beta}\right)\sum\sigma_i\right)$$

$$= \frac{Z_{N,\beta,h+\frac{t}{\beta}}}{Z_{N,\beta,h}}$$

$$Z_{N,\beta,h} = \mathrm{Tr}(P^N) \qquad P \stackrel{\text{转移}}{=} \begin{pmatrix} e^{\beta J+\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta h}\end{pmatrix}$$

$$\frac{\left(\lambda^+_{h+\frac{t}{\beta}}\right)^N + \left(\lambda^-_{h+\frac{t}{\beta}}\right)^N}{(\lambda^+_h)^N + (\lambda^-_h)^N} \approx \left(\frac{\lambda^+_{h+\frac{t}{\beta}}}{\lambda^+_h}\right)^N \xrightarrow{N\to\infty} = \exp N\log\left(1+\frac{\lambda^+_{h+\frac{t}{\beta}}-\lambda^+_h}{\lambda^+_h}\right.$$

尺度变换.

$$\cancel{\left\langle \exp t\frac{M_N}{N}\right\rangle} \overset{Asy}{=} \exp\left(N\frac{t}{\beta\lambda_h^+}\frac{d\lambda_h^+}{dh}\right)$$

Scale :
$$= \cancel{\exp\left(t\frac{d\log\lambda_h^+}{dh}\right)}$$

$$\Rightarrow \left\langle \exp t\frac{M_N}{N}\right\rangle \longrightarrow \exp\left(\frac{t\lambda_h^{+\prime}}{\beta\lambda_h^+}\right)$$

$$\frac{M_N}{N} \xrightarrow{P} \frac{\lambda_h^{+\prime}}{\beta\lambda_h^+}$$ 本来是D. 但 h fixed.

$$\langle \exp t M_N\rangle \overset{Asy}{=} \exp N\left(\frac{\lambda_{h+\frac{t}{\beta}}^+-\lambda_h^+}{\lambda_+^h} - \frac{1}{2}\left(\frac{\lambda_{h+\frac{t}{\beta}}-\lambda_h^+}{\lambda_+}\right)^2\right)$$

$$= \exp\left(N\left(\frac{t}{\beta}\frac{d\lambda_h^+}{dh} - \left(\frac{t}{\beta}\right)^2\frac{d^2\lambda_h^+}{dh^2}\right) - \frac{1}{2}\left(\frac{t}{\beta}\right)^2\frac{d\lambda_h^+}{dh}\right)$$

$$= \exp\left(N\left(\lambda_h^{+\prime}\frac{t}{\beta} - \cancel{\tfrac{1}{2}}\left(\frac{t}{\beta}\right)^2\underbrace{\left(\frac{d^2\lambda_h^+}{dh} - \frac{1}{2}\frac{d\lambda_h^+}{dh}\right)}_{\sigma^2}\right)\right)$$

$$\log'' = \left(\frac{1}{\lambda_+^h}\frac{d\lambda_+}{dh}\right)' = -\frac{1}{\lambda_+^2}\frac{d\lambda}{dh} + \frac{1}{\lambda_+}\frac{d^2\lambda_+}{dh^2}$$

也存在 √.

6.3.

$h=0$
$$P(|M_N| \geq N\varepsilon) = 2P(M_N \geq N\varepsilon)$$
$$= 2P\left(\frac{M_N}{\sqrt{N}} \geq \sqrt{N}\varepsilon\right)$$ 看谁更快 的合符量.
$$\leq 2\frac{\left\langle e^{\frac{M_N}{\sqrt{N}}}\right\rangle}{e^{\sqrt{N}\varepsilon}} \lesssim \exp^{-\sqrt{N}\varepsilon}$$

BC √.

22

C-W 模型

$$\mu_{\Lambda;\beta;h}(\sigma) = \frac{1}{Z_{\Lambda;\beta;h}} \exp(\beta H(\sigma))$$

$$H(\sigma) = -\frac{d}{N}\left(\sum \sigma_i\right)^2 - h\sum \sigma_i$$

$M_N = \sum_{i=1}^{N} \sigma_i$ 对 $\frac{M_N}{N}$ 进行极限估计和涨落估计

$m_N = \frac{M_N}{N}$ $d=1$

$$p(m_N = m) = \frac{1}{Z_{N;\beta;h}} \binom{N}{\frac{1+m}{2}N} \exp\left(\frac{\beta d}{N}(mN)^2 + \beta h m N\right)$$

~~$m_N N = N$~~

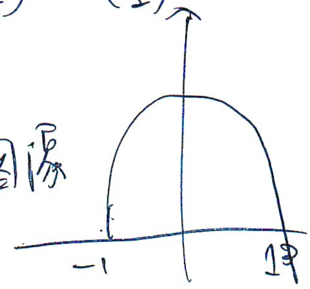
$$= \frac{1}{Z_{N;\beta;h}} \binom{N}{\frac{1+m}{2}N} \exp\left(\beta d m^2 N + \beta h m N\right)$$

6.5.1:

$$p(m_N = m) = \frac{1}{Z_{N;\beta;0}} \binom{N}{\frac{1+m}{2}N} \exp\left(\frac{\beta d m^2 N^2}{\varphi(N)}\right)$$

$$\binom{N}{\frac{1+m}{2}N} = \frac{N!}{\left(\frac{1+m}{2}N\right)!\left(\frac{1-m}{2}N\right)!} \approx \frac{\left(\frac{N}{e}\right)^N}{\left(\frac{\frac{1+m}{2}N}{e}\right)^{\frac{1+m}{2}N}\left(\frac{\frac{1-m}{2}N}{e}\right)^{\frac{1-m}{2}N}} \approx \frac{1}{\left(\frac{1+m}{2}\right)^{\frac{1+m}{2}N}\left(\frac{1-m}{2}\right)^{\frac{1-m}{2}N}}$$

$$= \exp\left(N\left(-\frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}\right)\right)$$ → 图像



$$= \exp N(S(m) + o(1))$$

$$\approx \frac{1}{Z_{N;\beta;0}} \exp\left(N S(m) + \beta \frac{d m^2 N^2}{\varphi(N)} + o(1)\right) \qquad S(m)$$

$\frac{N}{\varphi(N)} = 0$ $$\approx \frac{1}{Z_{N;\beta;0}} \exp N\left(S(m)\right)$$

$$p(m_N \notin (-\varepsilon,\varepsilon)) = \frac{\sum_{m\notin(-\varepsilon,\varepsilon)} \exp(N S(m))}{Z_{N;\beta;0}} \leq \frac{N \exp N S(\varepsilon)}{e^{N S(0)}}$$

$$\approx N \exp^{-C(\varepsilon)S(0)N} \to 0$$

$\frac{N}{\varphi(N)} = \infty$ $$p\left(m_N \in (-1+\varepsilon, 1-\varepsilon)\right) \leq \frac{N \exp C(\varepsilon)\frac{N^2}{\varphi(N)}(1-\varepsilon)^2}{\exp^{\beta d \frac{N^2}{\varphi(N)}}} \to 0$$

$(1-\varepsilon) - 1 < 0$

$\psi_\beta(h) = \lim_{N\to\infty} \frac{1}{N}\log Z_{N;\beta;h}$ 关于 h 凸

$$Z_{N;\beta;h} = \sum_i^N \binom{N}{i} \exp\left(\beta\left(\frac{N-2i}{N}\right)^2 N + h\frac{N-2i}{N}\cdot N\right)$$

$$\frac{1}{N}\log Z_{N;\beta;h} \xrightarrow{\;\le\;} \cancel{\binom{N}{\frac{1+m}{2}N}}\ \cancel{\exp}\left(S(m) + (hm + \beta m^2)\right)$$

$\exists$

$f = -\beta - S_m$ ✓

$$\max \frac{1}{N}\log Z_{N;\beta;h} \overset{N\to\infty}{=} \max\left(S(m) + hm + \beta m^2\right)$$

$$\max\left(hm - f(m)\right)$$

$$\psi(h+a) + \psi(h-a) = \max\left(S(m) + (h+a) + \beta m^2\right) + \max\left(S(m) + (h-a) + \beta m^2\right)$$

取一次 max 和两次 max.

知识再现：

1. Wigner 半圆　　半交叉配对　　$2m$ 点 $\Rightarrow \frac{1}{m+1}\binom{2m}{m}$　　$m+1$ 个自由度（指标）

非交叉配对 $\Leftrightarrow$ cycle 自由度

$$\frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left(\frac{A}{\sqrt{n}}\right)^k\right] \underset{k=2m}{\sim} \frac{1}{n^{1+m}}\cdot\frac{1}{m+1}\binom{2m}{m}\cdot n^{1+m}$$

GUE 每个元素独立

GOE 麻烦一些

2. GOE 密度 $f(H) =$　　对角 Var 2　非对角 Var 1

$$\frac{1}{(4\pi)^{\frac{n}{2}}}\cdot\frac{1}{(2\pi)^{\frac{n(n-1)}{2}}}\exp\left(\sum -\frac{1}{4}H_{ii}^2 + \sum_{i<j} -\frac{1}{2}H_{ij}^2\right)$$

$$= \frac{1}{2^{\frac{n(n+1)}{4}}}\cdot\frac{1}{\pi^{\frac{n(n+1)}{2}}}\exp\left(-\frac{1}{4}\operatorname{tr}H^2\right).$$

GUE　　$\frac{1}{\pi}\exp\left(-|z|^2\right)$　非对角

对角 $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$.

(4π) (2π)

π

24. Gibbs 不等式 $u - u\log u \leq v - u\log v$

$$\Rightarrow -\int u\log u \leq -\int u\log v$$

Boltzmann $P(X=E_i) = p_i$

$$q_i = f(p_i) = \frac{\exp}{Z}\left(-\frac{E_i}{\beta}\right)$$

由Gibbs $\Rightarrow$.

Ising $H(\sigma) = -J\sum_{i\sim j}\sigma_i\sigma_j - \beta\sum\sigma_i$

1维.

转移 $P = \begin{pmatrix} \exp(\beta J + \beta h) & \exp(-\beta J) \\ \exp(-\beta J) & \exp(\beta J - \beta h) \end{pmatrix}$ 半

$$Z_{N;\beta;h} = tr(P^N) = \exp(\beta J N)\cdot(\lambda_+^N + \lambda_-^N)$$

$$\boxed{\mathbb{E}\left[\exp(t M_N)\right] = \frac{Z_{N;\beta;h+\frac{t}{\beta}}}{Z_{N;\beta;h}}} \approx \left(\frac{\lambda^+_{h+\frac{t}{\beta}}}{\lambda^+_h}\right)^N$$

一阶展开

$$= \frac{\sum \exp t\sum\sigma_i \cdot \exp \beta J\sum\sigma_i\sigma_{i+1} + \beta h\sum\sigma_i}{Z_{N;\beta;h}}$$

中心极限 则二阶展开.

a.s. 收敛可以用 分布 + exp-Markov.

$$P(|M_N| > N\varepsilon) \leq 2P(M_N > N\varepsilon) = 2P\left(e^{\frac{M_N}{\sqrt{n}}} > e^{\varepsilon\sqrt{n}}\right) \leq \frac{2\,\mathbb{E}\left[e^{\frac{M_N}{\sqrt{n}}}\right]}{e^{\varepsilon\sqrt{n}}}$$

有界量

Ising v.s. Curie-Weiss

1 2.

1: $H(\sigma) = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i$ $\mu_{N,\beta,h}(\sigma) = \frac{\exp\left(\beta J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + \beta h\sum_{i=1}^{N}\sigma_i\right)}{Z_{N,\beta,h}}$

2: $H(\sigma) = -\frac{J}{N}\left(\sum_{i=1}^{N}\sigma_i\right)^2 - h\sum_{i=1}^{N}\sigma_i$ $\mu_{N,\beta,h}(\sigma) = \frac{\exp\left(\frac{J\beta}{N}\left(\sum_{i=1}^{N}\sigma_i\right)^2 + \beta h\sum_{i=1}^{N}\sigma_i\right)}{Z_{N,\beta,h}}$

1: 配分函数局部展开

期望 $\mathbb{E}\left[\frac{M_N}{N}\right] = \frac{\sum_{\sigma}\left(\sum_{i=1}^{N}\sigma_i\right)\cdot\exp\left(J\beta\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + \beta h\sum_{i=1}^{N}\sigma_i\right)}{Z_{N,\beta,h}}$

$$= \frac{1}{\beta N}\frac{\partial \log Z_{N,\beta,h}}{\partial h} = \frac{1}{\beta N}\frac{\partial}{\partial h}\log\left(e^{\beta J N}\left(\lambda_+^N + \lambda_-^N\right)\right) \Rightarrow \frac{1}{\beta}\frac{\partial \lambda_+}{\partial h}\cdot\frac{1}{\lambda_+}$$

$Z_{N,\beta,h} = tr(P^N)$

$$P = \begin{pmatrix} e^{\beta J+\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta h} \end{pmatrix}$$

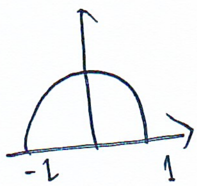
$$\lambda^2 - e^{\beta J}\left(e^{\beta h}+e^{-\beta h}\right)\lambda + \left(e^{2\beta J} - e^{-2\beta J}\right) = 0$$

$$\frac{e^{\beta J}\left(e^{\beta h}+e^{-\beta h}\right) \pm \sqrt{e^{2\beta J}\left(e^{2\beta h} + 2e^{-2\beta h} + 2\right) - 4\left(e^{2\beta J} - e^{-2\beta J}\right)}}{2}$$

$$= e^{\beta J}\left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + 4e^{-\beta J}}\right]$$

LLN 用矩母函数1阶展开计算

CLT 用矩母函数二阶展开计算



2: 组合数估计 $\binom{N}{\frac{1+m}{2}N} \approx \exp N\left(S(m) + o(1)\right)$ $\cancel{S(m)} = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}$

扰动项在 $\frac{1}{N}\log$ 作用下消失.

对 m_N 进行粗糙估计

$P(m_N \in \cdots) = \frac{\Sigma\cdots}{Z_N}$ N 边界项 → Vanish ⋯ 中心项

磁场

$$H(\sigma) = -\sum J_{ij}\sigma_i\sigma_j - h\sum \sigma_i$$

uNspin:

$$X = \beta \sum_{i=1}^{N} \sigma_i$$

$M_N(h)$ 只有纯虚零点