Topology

lec 1 : metric space

Def. X set A metric structure ~~is a~~ on X is a map

$d: X \times X \to \mathbb{R}_{(\geq 0)}$

s.t. (1) $\underbrace{d(x,y) \geq 0}$; $d(x,y) = 0 \Leftrightarrow x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) + d(y,z) \geq d(x,z)$

(X, d) metric space.

Rmk. (1) $\cancel{d(x,y) \geq 0}$ ~~is~~ ~~trut~~ $d(x,y) + d(y,x) \geq 0$

In (X, d)

① ~~diam~~ $A \subseteq X \to \operatorname{diam} A = \sup_{x,y \in A} d(x,y)$

if. $\operatorname{diam} A < +\infty$. A is a bounded set.

② open ball $B(x_0, r) = \{ y : d(x_0,y) < r\}$

closed $\overline{B(x_0,r)} = \{y : d(x_0, y) \leq r\}$ can be ϕ.

sphere $S(x_0, r) = \{y : d(x_0, y) = r\}$

③ $U \subseteq (X, d)$ is a open set if $\boxed{\forall x \in U.\ \exists \varepsilon > 0.\ s.t\ B(x, \varepsilon) \subseteq U.}$

F is closed set if $F^c = X \setminus F$ is open

open ball is open but it's not so trivial as intuition
(closed) (closed)

Example.

① discrete metric For any $X \neq \phi$. define $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

② $X = \mathbb{R}$.

a. $d(x,y) = |x-y|$

b. $\bar{d}(x,y) = \min\{|x-y|, 1\}$

c. $\bar{d}(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$

$X = \mathbb{R}^n$ $d_E(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$

$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{\frac{1}{p}} & 1 \leq p < +\infty \\ \sup_{1 \leq i \leq n} |x_i - y_i| \end{cases}$

③ $X = \mathbb{R}^{\mathbb{N}} = \{\{x_n\} : x_i \in \mathbb{R}\}$

~~unif~~ $d_{unif}((x_n), (y_n)) = \sup_{n \in \mathbb{N}} \bar{d}(x_n, y_n)$

$d_{prod}((x_n), (y_n)) = \sum_{n=1}^{\infty} 2^{-n} \bar{d}(x_n, y_n)$

$X_1 = \{(x_n) : \|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < +\infty\}$

$d((x_n), (y_n)) = |(x_n - y_n)|_p$

④ $C([a,b])$

L^p metric : $d_p(f,g) = \begin{cases} \left(\int_a^b |f-g|^p \, dx\right)^{\frac{1}{p}} & 1 \le p < +\infty \\ \sup\limits_{x\in[a,b]} |f-g| & p = +\infty. \end{cases}$

$C^k([a,b])$ $\quad d_{p,k} = \cancel{\sum_{i\le k} \| f^{(i)} - g^{(i)} \|_p} \ \sum_{i\le k} d_p(f^{(i)}, g^{(i)}).$

⑤ $A, B \subseteq \mathbb{R}^n$ bounded, closed

hausdorff metric. $d(A,B) = \inf\{\varepsilon > 0.\ A \subseteq B_\varepsilon,\ B \subseteq A_\varepsilon\}$

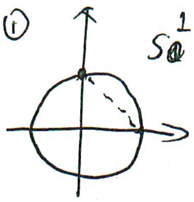
∞.

open ball & closed ball

(X,d) $\quad Y \subseteq X. \Rightarrow Y \times Y \subseteq X \times X.$ $\quad$ sub

Prop. $d|_{Y\times Y}$ is a metric on $X \times X$ $\quad$ proof is trivial

e.g. $\quad$ ① $S^1 \subseteq \mathbb{R}^2$ $\quad$ ② $[1,3] \cup (4,5) \subseteq \mathbb{R}$ $\quad$ subspace



$B_Y(3,1)$ is not a open ball in $\mathbb{R}$

open set

$(X, d_X),\ (Y, d_Y) \rightsquigarrow (X \times Y, d)$?

Prop $\quad d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$ is a metric on $X \times Y$

e.g. $(\mathbb{R}, d_E) \times (\mathbb{R}, d_E) \rightsquigarrow (\mathbb{R}^2, d^1)$

in fact

$(X_1, d_1) \times \cdots (X_n, d_n) \rightsquigarrow d_p$ on $X_1 \times \cdots \times X_n$ by

$$d_p((x_1, \cdots x_n), (y_1, \cdots y_n)) = \left(\sum_{i=1}^n d_i(x_i, y_i)^p\right)^{\frac{1}{p}}$$

Def. $\quad$ We say $f: (X, d_X) \to (Y, d_Y)$ is an <u>isometry</u> if $\quad$ isometric embedding

① $f: X \to Y$ is <u>bijective</u> $\quad \longrightarrow$ injective

② $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$

and in that case we say (X, d_X) and (Y, d_Y) isometry

Def. we say $f:(X,d_X)\longrightarrow(Y,d_Y)$ is a lipschitz map (with $\underbrace{\text{constant}}_{\text{lipschitz}}$ L) if $d_Y(f(x_1),f(x_2))\le L\, d(x_1,x_2)$. $\forall\, x_1,x_2\in X$.

e.g. $f:(\mathbb{R},\bar{d})\longrightarrow(\mathbb{R},d)$, $x\mapsto x$. is not a lipschitz map.

depend on the d, not a topological conception

Def. (X,d) We say (x_n) convergent to x_0 in X. if.

$$\boxed{\forall\varepsilon>0,\ \exists\, N>0.\ \text{s.t}\ \forall\, n>N.\ d(x_n,x_0)<\varepsilon}$$

Notation $x_n\xrightarrow{d}x_0$

eg. d_{dis}: $x_n\xrightarrow{d_{dis}}x_0 \Longleftrightarrow \exists N>0,\ \text{s.t}\ n>N,\ x_n=x_0$.

Def. $f:(X,d_X)\longrightarrow(Y,d_Y)$ $x_0\in X$.

We say f is $\underbrace{\text{continous}}_{\text{at } x_0}$ if any sequence $x_n\xrightarrow{d_X}x$ we have $f(x_n)\xrightarrow{d_Y}f(x_0)$

Def. f is continous on if f is continous at any $x_0\in X$.

Example (X,d) "d is continous."

① Fix $x_0\in X$, consider $f_{x_0}:X\to\mathbb{R}$. $x\mapsto d(x,x_0)$

$$\left|f_{x_0}(x)-f_{x_0}(y)\right| = \left|d(x,x_0)-d(y,x_0)\right| \le d(x,y)$$

② For any $A\subseteq X$, $d_A: X\to\mathbb{R}$. $x\mapsto \inf_{a\in A} d(x,a)$

d_A is continous

③ $d: X\times X\longrightarrow\mathbb{R}$ is continous if we ~~have~~ endow $X\times X$ any one of the product metric.

?

Example. $C[a,b]$. d_p $\int: f\mapsto\int_a^b f\,dx\in\mathbb{R}$ is continous

but derive is not e.g.

Example $f:(X,d_{dis})\longrightarrow(Y,d_Y)$ is continous.

what about the reverse $f:(Y,d_Y\to X,d_{dis})$

$$y_n\xrightarrow{d_Y}y_0 \overset{?}{\Longrightarrow} f(y_n)\xrightarrow{d_{dis}}f(y_0) \Longleftrightarrow f \text{ is locally constant}$$

$\Updownarrow$

$f(y_n) = f(y_0)$ $n>N$

Lec 2. topological space

prop. the following are ~~equi~~ equivalant $f: (X, d_X) \to (Y, d_Y)$

① f is continous at x_0

② $\forall \varepsilon > 0, \exists \delta > 0$. s.t $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

③ $\forall \varepsilon > 0, \exists \delta > 0$. s.t $B_{d_X}(x_0, \delta) \subseteq f^{-1}(B_Y(x_0, \varepsilon))$

what we def'ed before should be seen as (sequentially continous)

proof ①⟺② is almost the same as mathematical analysis "Heine & common continous"

Observation if $d_1(x,y) \le C d_2(x,y) \Rightarrow B^{d_2}(x_0, r) \subseteq B^{d_1}(x_0, cr)$

prop. If $d_1 \le C d_2$ are two metrics on X. and $f: (X, d_1) \to (Y, d_Y)$ is continous then $f: (X, d_2) \to (Y, d_Y)$ is also continous

Similarly study Y

Def. we say two metrics are ~~str~~ strong equivalant if. $\exists C_1, C_2 > 0$. $C_1 d_2 \le d_1 \le C_2 d_2$

~~$C_1 d_2 \le d_1 \le C_2 d_2$~~

eg. L^p metrics d_p $(1 \le p < +\infty)$

Thm. if d_X, d_X^2 are strong equivalant on X

d_Y d_Y^2 Y

$\Rightarrow$ "$f: (X, d_X) \to (Y, d_Y)$ is continous $\Leftrightarrow f: (X, d_X^2) \to (Y, d_Y^2)$ is continous"

Example. d_∞ & is bounded $\bar{d}_\infty := \min\{1, d_\infty\}$ on $\mathbb{R}^n$

~~But~~ d_∞ & $\bar{d}_\infty$ are NOT strong equivalant

But $f: (X, d_\infty) \to Y$ continous $\Leftrightarrow f: (X, \bar{d}_\infty) \to Y$ continous

Since continuity is about the ~~beha~~ behaviors of f locally

Def. (neighborhood) If $N \subseteq (X, d_X)$ contains an open ball that contains x_0

we call N is a Neighborhood of x_0.

prop. f is continous at $x_0 \Leftrightarrow f^{-1}(\text{neighborhood of } f(x_0)) = \text{Ne.. of } x_0$

rmk Neighborhood needs ~~not~~ not to be open

prop. $f: (X, d_X) \to (Y, d_Y)$ continous $\Leftrightarrow f^{-1}(\text{open}) = \text{open}$

proof $(\Rightarrow)$ f is continous. assume V is an open set of Y. $x_0 \in f^{-1}(V) \Rightarrow f(x_0) \in V$.

$\Rightarrow \exists B(f(x_0), \varepsilon) \subseteq V. \Rightarrow B(x_0, \delta) \subseteq f^{-1}(V)$

$\Rightarrow f^{-1}(B(f(x_0), \varepsilon)) \supseteq B(x_0, \delta)$. □

$(\Leftarrow)$ $f^{-1}(B(f(x_0), \varepsilon)) \ni x_0 \Rightarrow$

WARNING. f is continuous at x_0 $\not\Leftrightarrow$ f^{-1}(open nei... of $f(x_0)$) = open nei... of x_0

Def. We say d_1, d_2 are topological equivalent if "open in d_1 $\Leftrightarrow$ open in d_2"

Still in metric space. Let $\mathcal{N}(x) = \{$ all neighborhoods of $x_0 \} \subseteq \cancel{P(P(X))}$ $P(X)$

Thm (N1) if $N \in \mathcal{N}(x)$, then $x \in N$

(N2) if $N \in \mathcal{N}(x)$, $M \supseteq N$, then $M \in \mathcal{N}(x)$

(N3) if $N_1, N_2 \in \mathcal{N}(x)$, then $N_1 \cap N_2 \in \mathcal{N}(x)$

about one point

(N4) if $N \in \mathcal{N}(x)$, $\exists M \in \mathcal{N}(x)$ s.t. $M \subseteq N$ and $N \in \mathcal{N}(y)$, $\forall y \in M$

Def. A neighborhood structure on a set X is a map

$\mathcal{N}: X \to P(P(X)) \setminus \{\phi\}$ that satisfy (N1) ~ (N4)

e.g. $\mathcal{N}(x) = X$

$\mathcal{N}(x) = \{A : x \in A\}$

$\mathcal{N}(x) :=$ neighborhood for (X, d)

$(X, \mathcal{N})$, (neighborhood) topological space.

We can define use it

$f: (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ cts at x_0

$\Leftrightarrow f^{-1}(\mathcal{N}_{f(x_0)}) = \mathcal{N}_{x_0}$

in $(X, \mathcal{N})$

Def. We say U is open if $U \in \mathcal{N}(x)$ for $\forall x \in U$

$(X, \mathcal{N}) \rightsquigarrow (X, \mathcal{T})$

prop. (O1) ϕ, X is open

(O2) U_1, U_2 are open $\Rightarrow U_1 \cap U_2$ is open

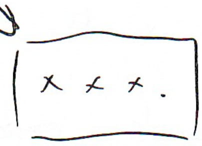
(O3) U_α are open $\Rightarrow \bigcup_\alpha U_\alpha$ is open

can be translated in open sets

proof. (O1) is trivial

(O2) use N_3.

(O3) use N_2. $\square$



Def. A topology $\mathcal{T}$ is a collection $\mathcal{T} \subseteq P(X)$ that satisfies (O1) ~ (O3)

where elements are called open sets

Note $(X, \mathcal{T}) \Rightarrow$ we can define $\mathcal{N}(x) = \{N \mid \exists U \in \mathcal{T}$ s.t. $x \in U \subseteq N\}$.

Fact. neighborhood structure $\Leftrightarrow$ "open set" structure

$\Updownarrow$

"closed set" structure

closed set can be defined by the complement of openset.

which can induce similar "closed set axioms"

Example

① (X,d) $\mathcal{T}_d = \{U : \forall x\in U, \exists r>0, B(x,r)\subseteq U\}$.

② $\mathcal{T}_{dis} = P(X)$

③ $\mathcal{T}_{trivial} = \{X, \phi\}$.

④ $\mathcal{T}_{cofinite} = \{U : U^c \text{ is finite}\} \cup \{\phi\}$

(O1) trivial

(O2) ~~$U_1^c \cap U_2^c$~~ $(U_1\cap U_2)^c = U_1^c \cup U_2^c$

(O3) $(\bigcup U_\alpha)^c = \bigcap U_\alpha^c$

$\downarrow$

co countable . . . .

Lec 3. Topology : Convergence and continuity

For $(X,\mathcal{T})$, we say $x_n \xrightarrow{\mathcal{T}} x_0$ if

$\iff \forall U\in\mathcal{T}$ with $x\in U$. $\exists k>0$. s.t $x_n\in U$ for $n>k$.

$\forall N\in \mathcal{N}(x_0)$. $\exists k>0$. $x_n\in N$ for $\forall n>k$.

e.g. metric topology (X,d) we have $\mathcal{T}_d = \{U\subseteq X \mid \forall x_0\in U. \exists \varepsilon>0. B(x_0,\varepsilon)\subseteq U\}$

we have $x_n \xrightarrow{d} x_0 \iff x_n \xrightarrow{\mathcal{T}} x_0$.

the larger Topology the harder Convergence

In particular $x_n \xrightarrow{\mathcal{T}_{dis}} x_0 \iff x_n \xrightarrow{d_{dis}} x_0$

e.g. $x_n \xrightarrow{\mathcal{T}_{tri}} x_0$ the limit is not unique

e.g. cofinite topology : $x_n \xrightarrow{\mathcal{T}_{cofinite}} x_0$ consider $x_1, \cdots x_n \cdots \to x_0$ $x_i\neq x_j$ ununique

$x_1\ x_0\ x_2\ x_0 \cdots \to x_0$ unique

$x_1\ x_2,\ x_1.\ x_2$

$x_n \xrightarrow{\mathcal{T}} x_0$ $\forall x\neq x_0$. $\exists$ at most finite times

e.g. co countable U open if ϕ or $X\setminus U$ countable

if X is countable is the same as discrete topology

X is not $x_n \xrightarrow{\mathcal{T}} x_0 \iff x_n = x_0$ $n>k$

Example . consider $X = \mathcal{M}([0,1],\mathbb{R})$ $\left(=\mathbb{R}^{[0,1]}\right)$.

we have $f_n \to f$ pointwise convergence

Aim to find a topology $f_n \xrightarrow{\text{P.C.}} f \iff f_n \xrightarrow{\mathcal{J}} f$

"what are sets which must be open"

suppose $f_n \xrightarrow{\text{P.C.}}$ $x \in [0,1]$ then $f_n(x_0) \to f(x_0)$ $\forall \varepsilon > 0$. $\exists k$. $|f_n(x_0) - f(x_0)| < \varepsilon$. $\forall n > k$

$\longrightarrow$ Consider $\omega(f, x_0; \varepsilon) = \{ g \in X \mid |g(x_0) - f(x_0)| < \varepsilon \}$ "basic open sets"

$\Rightarrow \omega(f, x_1, \dots x_m; \varepsilon) = \{ g \in X \mid |g(x_i) - f(x_i)| < \varepsilon \}, i = 1, \dots m \}$.

Def. $\mathcal{J}_{p.c} = \{ U \subseteq X \mid \forall f \in U, \exists x_1, \dots x_m \in [0,1], \varepsilon > 0$. s.t. $\omega(f, x_1, \dots x_m; \varepsilon) \subseteq U \}$

prop. ① $\mathcal{J}_{p.c}$ is a topology

② $f_n \xrightarrow{\text{P.C}} f \iff f_n \xrightarrow{\mathcal{J}_{p.c}} f$.

③ $\mathcal{J}_{p.c.}$ is NOT a metric topology

proof. ① ϕ. X ✓

$U_1, U_2 \in \mathcal{J}_{p.c} \Rightarrow U_1 \cap U_2 \in \mathcal{J}_{p.c}$ since $\forall f \in U_1 \cap U_2 \dots$

$\omega(f, x_1, \dots x_m, y_1, \dots y_n, \min(\varepsilon_1, \varepsilon_2)) \subseteq U_1 \cap U_2$

$\bigcup U_\alpha$ trivial

② $(\Rightarrow)$ $f_n \xrightarrow{\text{P.C.}} f \Rightarrow f_n \xrightarrow{\mathcal{J}_{p.c.}} f$ is easy

just consider basic open sets.

$(\Leftarrow)$ For any fixed x. -- $\omega(f, x, \varepsilon) \Rightarrow \dots$

Def. $f: (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$

we say f is <u>sequentially continuous</u> at x_0 if

$x_n \xrightarrow{\mathcal{J}_X} x_0 \Rightarrow f(x_n) \xrightarrow{\mathcal{J}_Y} f(x_0)$ call f seq cts if f is cts at each $x \in X$.

we say f is <u>continuous</u> at x_0 if

$f^{-1}(N_{f(x_0)}) \in \mathcal{N}(x_0)$ $\dots$

e.g. seq cts $\not\Rightarrow$ cont

$f(x) = x$ $f(\mathbb{R}, \mathcal{J}_{cocou}) \to (\mathbb{R}, \mathcal{J}_{dis})$

f is seq cts

but $[-1,1]$ is not open in $\mathcal{J}_{cocountable}$

prop. f cts at $x_0 \Rightarrow f$ seq cts at x_0

proof. f is cts at x_0. $x_n \xrightarrow{\mathcal{T}_X} x_0$

(For any $N_{f(x_0)}$, we have $f^{-1}(N_{f(x_0)}) \overset{= N(x_0)}{\cancel{\in N(x_0)}}$

$\exists k$. s.t $x_n \in N_{x_0}$ $n > k$.

$\Rightarrow f(x_n) \in N_{f(x_0)}$ $n > k$.

$\Rightarrow f(x_n) \xrightarrow{\mathcal{T}_Y} f(x_0)$ □

prop. $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ $g: (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$

① f seq cts at x_0, g seq cts at $f(x_0)$

$\Rightarrow g \circ f$ seq cts at x_0

② .. cts --- cts --

-- cts --

proof. ① $x_n \xrightarrow{\mathcal{T}_X} x_0 \underset{\text{cts}}{\overset{\text{seq}}{\Rightarrow}} f(x_n) \xrightarrow{\mathcal{T}_Y} f(x_0) \Rightarrow g \circ f(x_n) \xrightarrow{\mathcal{T}_Z} g \circ f(x_0)$

② $g \circ f^{-1}(N_{g \circ f(x_0)}) = f^{-1} \circ g^{-1}(N_{g(f(x_0))}) \supseteq f^{-1}(N_{f(x_0)}) = N_{x_0}$ □

prop. $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is cts $\Leftrightarrow f^{-1}(\text{open set}) = \text{open set}$.

$\Leftrightarrow f^{-1}(\text{closed}) = \text{closed}$

proof. the same as before

Rmk. f^{-1} is nicer than f

$f^{-1}(\bigcup U_\alpha) = \bigcup f^{-1}(U_\alpha)$

$f^{-1}(\bigcap U_\alpha) = \bigcap f^{-1}(U_\alpha)$

$f^{-1}(U^c) = (f^{-1}(U))^c$

but

$f(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f(U_\alpha)$

$f(\bigcap U_\alpha) \subseteq \bigcap f(U_\alpha)$

$f(U^c) \supseteq (f(U))^c$

So $f(\text{open}) = \text{open} \not\Leftrightarrow f(\text{closed}) = \text{closed}$

eg $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$

$f \equiv y_0 \in Y$

easily check f is cts.

if $\mathcal{T}_Y$ is trivial

f is cts.

e.g. Q when a Id is cts $Id : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$

A: $\mathcal{T}_2 \subseteq \mathcal{T}_1$

Def. let $\mathcal{T}_1$, $\mathcal{T}_2$ ~~is~~ are two topologies. we say $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$
$\mathcal{T}_2$ is stronger than $\mathcal{T}_1$

if. $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Note. · Trivial topology is weakest ⋯ on X.
discrete topology is strongest on X.

NOT all the topologies can be comparable

e.g $(\mathbb{R}, \mathcal{T}_{standard})$ & $(\mathbb{R}, \mathcal{T}_{cocountable})$
$(0,1)$ in $\qquad \mathbb{R} \setminus \{\frac{1}{n}\}$ $n = 1, \ldots$

Note $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology

prop. $\mathcal{T}_\alpha$ on X $\bigcap_\alpha \mathcal{T}_\alpha \overset{= \mathcal{T}}{}$ is a topology

$\mathcal{T}$ is the <u>strongest</u> topology on X s.t $Id : (X, \mathcal{T}_\alpha) \Rightarrow (X, \mathcal{T})$ $\forall \alpha$.

pf if $\exists \mathcal{T}_0$ s.t ⋯

since $\mathcal{T}_0 \subseteq \mathcal{T}_\alpha$ $\forall \alpha$

$\Rightarrow \mathcal{T}_0 \subseteq \mathcal{T}$ $\qquad \square$

Lec4. New topology from old

① $A \subseteq (X, \mathcal{T})$. $\mathcal{T}_A = \{ U \cap A : U \in \mathcal{T} \}$ subspace topology

prop. the weakest topology on A s.t inclusion map is cts
$(\iota : A \hookrightarrow X)$.

$(\iota \circ g)^{-1}(O) = g^{-1} \circ \iota^{-1}(O)$
$= g^{-1}(O \cap A)$ ✓

prop. (1) $f : X \longrightarrow Y$ is cts $\Rightarrow$ $f \circ \iota = f|_A : A \longrightarrow Y$ is cts

(2) $g : Z \rightarrow A$ is cts $\Leftrightarrow$ $\iota \circ g : Z \rightarrow X$ is cts

$$\begin{tikzcd} Z \arrow[r, dashed, "\iota \circ g"] \arrow[d, "g"] & X \\ A \arrow[ur, "\iota"'] & \end{tikzcd}$$

② $(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$

$\mathcal{T}_{X \times Y} = \{ W \mid \forall (x,y) \in W.\ \exists U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y,\ (x,y) \in U \times V \subseteq W \}$

prop. $\mathcal{T}_{X\times Y}$ is the weakest topology s.t

$\pi_X : X\times Y \to X$, $\pi_Y : X\times Y \to Y$ are ~~is~~ cts

π_X, π_Y are projections.

proof ~~is trivial~~ cts is trivial

if π_X, π_Y are cts, then $U\times Y,\ X\times V$ are

open sets in $\mathcal{T} \Rightarrow U\times V \in \mathcal{T}$

NOTE we have $W = \bigcup_{(x,y)\in W} U_x \times V_Y$

prop. $f : (Z, \mathcal{T}_Z) \to (X\times Y, \mathcal{T}_{X\times Y})$ cts $\Leftrightarrow f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f$ are cts.

proof. ($\Rightarrow$) trivial

($\Leftarrow$) $f^{-1}(W) \ni z \Rightarrow f(z) = (x,y) \in W \Rightarrow (x,y) \in U\times V$

$f_X(z) = x,\ f_Y(z) = y \Rightarrow z \in f_X^{-1}(U)$
$z \in f_Y^{-1}(V)$

$z \in f_X^{-1}(U) \cap f_Y^{-1}(V) \subseteq f^{-1}(W)$

BUT What about $\mathcal{T}_\alpha = \prod_\alpha \mathcal{T}_\alpha$.

| the $\mathcal{T}$ is too strong

we call the $\mathcal{T}$ on the left "Box Topology"

Example $X_\alpha = \mathbb{R}$. $X = \mathbb{R}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathbb{R}$

$f : \mathbb{R} \to (X, \mathcal{T})$ $t \mapsto (t, \cdots t, \cdots)$

BUT $f^{-1}((-1,1)\times \cdots \times (-\frac{1}{n}, \frac{1}{n}) \times \cdots) = \{0\}$.

How to construct a topology on $\prod_\alpha X_\alpha$.

aim π_α is cts $\forall \alpha$. $\pi_\alpha : \prod_\alpha X_\alpha \to X_\alpha \Leftrightarrow \pi_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta\neq\alpha} X_\beta$.

Compare. T.p.c. $\mathcal{T}_d$, $\mathcal{T}_{X\times Y}$

under what condition on $\mathcal{B}$

| NOTE . $\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}}$

$\mathcal{T}_{\mathcal{B}} = \{U \mid \forall x \in U,\ \exists B \subseteq \mathcal{B},\ x \in B \subseteq U\}$

Check: ① ϕ ✓. $X \Leftrightarrow \boxed{\bigcup_{\mathcal{B}} B = X}$ (B1)

② $U_1 \cap U_2 \Rightarrow \forall x \in U_1 \cap U_2,\ \exists B \subseteq \mathcal{B}.\ x \in B \subseteq U_1 \cap U_2$

$\Updownarrow$

$\boxed{\forall x \in B_1 \cap B_2\ \ \exists B \subseteq \mathcal{B}\ \ x \in B \subseteq B_1 \cap B_2}$ B2

③ $\bigcup_\alpha U_\alpha$ trivial

Def. ① A family $\mathcal{B}$ satisfy (B1), (B2) is called a topological basis of the set X and call $\mathcal{T}_{\mathcal{B}}$ the topology generated by basis $\mathcal{B}$.

② If $(X, \mathcal{T})$ is a topological space. $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ we call $\mathcal{B}$ a basis of the topological space $(X, \mathcal{T})$

eg $\mathbb{R}^n$, $\{B(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_+\}$ are basis of $(\mathbb{R}^n, \mathcal{T})$

eg Box Topology generated by $\prod_{\alpha} U_\alpha$.

prop. If $\mathcal{T}$ is a topology and $\mathcal{B} \subseteq \mathcal{T}$ then $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$

trivial

Cor $\mathcal{T}_{\mathcal{B}} = \bigcap_{\mathcal{B} \subseteq \mathcal{T}} \mathcal{T}$

In general, $\mathcal{S} \subseteq \mathcal{P}(\mathcal{B})$, we can define $\mathcal{T}_{\mathcal{S}} = \bigcap_{\mathcal{S} \subseteq \mathcal{T}} \mathcal{T}$

weakest topology s.t. Any set in $\mathcal{S}$ is open

prop. suppose $\bigcup_{S \in \mathcal{S}} S = X$, $\mathcal{B} = \{B \mid \exists S_1 \cdots S_n \in \mathcal{S}, \text{ s.t. } B = S_1 \cap \cdots \cap S_n\}$ is a topological basis of and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{S}}$

$\mathcal{S} \subseteq \mathcal{B}$

proof ① (B1) ✓ (B2) ✓

② $\mathcal{T}_{\mathcal{B}}$, all sets in $\mathcal{B}$ are open

$\Updownarrow$

$\mathcal{T}_{\mathcal{S}}$ — $\forall s$ is open □

Def. ① $\bigcup_{S \in \mathcal{S}} S = X$. $\longrightarrow$ $\mathcal{S}$ is called subbasis of a topology on the set X

② $(X, \mathcal{T})$ if $\mathcal{T}_{\mathcal{S}} = \mathcal{T}$ we call $\mathcal{S}$ is a subbasis of $(X, \mathcal{T})$

rmk. what if $\bigcup S \neq X$. ① let $\mathcal{S}' = \mathcal{S} \cup \{X\}$

② $X' = \bigcup S$ $\quad \mathcal{T}_{\mathcal{S}'}$ on X' $\longrightarrow$ $\mathcal{T}_{\mathcal{S}}' \cup \{X\}$ is $\mathcal{T}_{\mathcal{S}}$

Def. $\mathcal{T}_{prod}$ on $\prod_{\alpha} U_\alpha$ = the topology generated by subbasis $\{\pi_\alpha^{-1}(U_\alpha)\}$

$\mathcal{T}_{prod} = \{W \mid \forall (x_\alpha) \in W, \exists \alpha_1, \cdots \alpha_m \text{ s.t. } (x_\alpha) \in \bigcap_{i=1}^{m} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \}$

$\downarrow$

finite open sets and others are all spaces.

lemma. let $\mathcal{B}$ be a basis of $(X, \mathcal{T})$

$\mathcal{S}$ sub – —

then $f: Y \to X$ cts

$\Longleftrightarrow f^{-1}(B)$ open $\forall B \in \mathcal{B}$

$\Longleftrightarrow f^{-1}(S)$ open $\forall S \in \mathcal{S}$

Thm (universality of $\mathcal{T}_{prod}$)

① $f: Z \longrightarrow (\prod X_\alpha, \mathcal{T}_{prod})$ cts

$\Longleftrightarrow \forall \alpha.\ f_\alpha = \pi_\alpha \circ f$ cts

② $\mathcal{T}_{prod}$ is the unique topology satisfy

$\mathcal{B}$ is a basis of $\mathcal{T}$ $\rightsquigarrow$ $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

(B1)
(B2)

$\mathcal{T} = \mathcal{T}_{\mathcal{B}} = \{U \mid \forall x \in U,\ \exists B \in \mathcal{B}, x \in B \subseteq U\}$

$\Downarrow$

· $\mathcal{B} \subseteq \mathcal{T}$

· $\{U \mid \forall x \in U,\ \exists B \in \mathcal{B}.\ x \in B \subseteq U\}$.

---

subbasis

$\mathcal{S}$ is a sub-basis $\Longleftrightarrow \mathcal{T} = \mathcal{T}_S = \bigcap\limits_{S \subseteq \mathcal{T}'} \mathcal{T}'$

$= \mathcal{T}_{\mathcal{B}}$ $\quad \mathcal{B}$: all finite intersection of S

Lec 5 generate ~~topo~~ topology by maps.

$\mathcal{F} = \{f_\alpha : X \to (Y, \mathcal{T}_\alpha)\}$

Def. the weakest topology $\mathcal{T}_{\mathcal{F}}$ on X s.t. each f_α cts

~~we call it as~~ is called the topology induced by maps.

rmk. weakest exists since $\bigcap\limits_\alpha \mathcal{T}_\alpha$ is also a topology.

Example. only one map: $f : X \longrightarrow (Y, \mathcal{T})$

f cts $\Rightarrow \forall\, V \in \mathcal{T}$. $f^{-1}(V)$ should be open $\in \mathcal{T}_f$

$$\Rightarrow \mathcal{T}_f = \{ f^{-1}(V) \mid V \in \mathcal{T} \}.$$

In general,

$$\cancel{\mathcal{T}_{\mathcal{F}} = \bigcap_{\alpha} \{ f_\alpha^{-1}(V) \mid V \in \mathcal{T}_\alpha \}}$$

$$\mathcal{T}_{\mathcal{F}} = \text{generate by } \underbrace{\{ f_\alpha^{-1}(V_\alpha) \mid \forall\, V_\alpha \in \mathcal{T}_\alpha \}}_{\mathcal{S}}$$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "g_\alpha = f_\alpha \circ g"'] & X \arrow[d, "f_\alpha"] \\ & Y_\alpha \end{tikzcd}$$

Universality of induced topology

① A map $g : (Z, \mathcal{T}_Z) \longrightarrow (X, \mathcal{T}_{\mathcal{F}})$ is cts iff g_α cts $\forall \alpha$

② $\mathcal{T}_{\mathcal{F}}$ is the unique topology on X with property ①

---

proof ① ($\Rightarrow$) trivial

($\Leftarrow$) $g^{-1}(\text{sub basis}) = g^{-1}(f_\alpha^{-1}(\quad)) = (f_\alpha \circ g)^{-1}(\cdots)$

② $\mathcal{T}_y$ on X with property ①

$\mathrm{Id} : (X, \mathcal{T}_{\mathcal{F}}) \longrightarrow (X, \mathcal{T})$ cts

$$\begin{tikzcd} (X, \mathcal{T}_{\mathcal{F}}) \arrow[d, "\mathrm{Id}"'] \arrow[dr, "f_\alpha \cancel{\circ \mathrm{Id}}"] & \\ (X, \mathcal{T}) \arrow[r, "f_\alpha"'] & (Y, \mathcal{T}_\alpha) \end{tikzcd}$$

$\mathrm{Id}' : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}_{\mathcal{F}})$ cts ?

we need $f_\alpha : (X, \mathcal{T}) \longrightarrow (Y_\alpha, \mathcal{T}_\alpha)$ cts

$$\begin{tikzcd} (X, \mathcal{T}) \arrow[d, "\mathrm{Id}'"'] \arrow[dr, dashed] & \\ (X, \mathcal{T}_{\mathcal{F}}) \arrow[r, "f_\alpha"'] & (Y, \mathcal{T}_\alpha) \end{tikzcd}$$

left for checking.

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① $\mathcal{T}_d \longleftrightarrow \mathcal{T}_{ball} \longleftrightarrow \mathcal{T}_{\mathcal{F}}$ $\quad \mathcal{F} = \{ d_x \mid x \in X \}$

② $\mathcal{M}([0,1], \mathbb{R}) = \mathbb{R}^{[0,1]}$

$\mathcal{T}_{p.c.} = \mathcal{T}_{prod} = \mathcal{T}_{\mathcal{F}}$ $\quad \mathcal{F} = \{ ev_x \mid x \in [0,1] \}$

$ev_x(f) = f(x)$.

③ $X^* = \{ L : X \to \mathbb{F} \mid L \text{ is cts} \}$

↑ topological vector space

Weak topology on X = $\mathcal{T}_{X^*}$

weak $*$-topology on X^* = $\mathcal{T}_{\mathcal{F}}$

$\mathcal{F} = \{ f_\alpha : (X_\alpha, \mathcal{T}_\alpha) \to Y \}$.

Def the strongest topology on Y s.t. all f_α cts is called co-induced topology

Example $f : (X, \mathcal{T}) \to Y$.

if $V \subseteq Y$, $f^{-1}(V) \notin \mathcal{T}$, then $V \notin \mathcal{T}_f$

$\mathcal{T}_f \subseteq \underbrace{\{ V \mid f^{-1}(V) \in \mathcal{T} \}}_{\text{topology}}$

↓

"=" then $\mathcal{T}_f$ is the strongest

In general $\mathcal{T}_{\mathcal{F}} = \bigcap_{\alpha} \{ V_\alpha \mid f_\alpha^{-1}(V_\alpha) \in \mathcal{T}_\alpha \}$

universality

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "g_\alpha"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

① g is cts $\Leftrightarrow$ $\forall \alpha$, g_α cts.

② $\mathcal{T}_{\mathcal{F}}$ is the unique one.

e.g ① $\bigcap_\alpha \mathcal{T}_\alpha$ is a co-induced topology

$\mathcal{F} : \{ Id_\alpha : (X, \mathcal{T}_\alpha) \longrightarrow X \}$

② $X = \bigcup_\alpha X_\alpha$. $(X_\alpha, \mathcal{T}_\alpha)$

Assume $X_\alpha \cap X_\beta$. $\mathcal{T}_{\alpha\, sub} = \mathcal{T}_{\beta,\, sub}$.

$\mathcal{F} = \{ i_\alpha : (X_\alpha, \mathcal{T}_\alpha) \to X \}$

Quotial space topology

X, Y are sets. $p : X \longrightarrow Y$ sur | is equivalent to $\sim$

Now, $(X, \mathcal{T})$ is topological space. $p : X \longrightarrow Y$ sur | $f^{-1}(y)$ fiber over y

$\rightsquigarrow$ Quotial topology on $Y = \{ V \mid p^{-1}(V)$ is open in $X \}$

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g = f \circ p"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

arbitrary

e.g. $S^1 = [0,1] / _{0 \sim 1}$

$S^1 = \mathbb{R} / \mathbb{Z}$

Group acts on X.

$g \rightsquigarrow T_g : X \xrightarrow{bi} X$

$T_{gh} = T_g \circ T_h$ | $\Rightarrow T_g^{-1} = T_{g^{-1}}$

$T_e = Id$

---

now $(X, \mathcal{T})$ topo space $\quad G \circlearrowright X$

$\forall g$. T_g is homeo

---

① $x \sim y \Longleftrightarrow \exists g \ T_g(x) = y$

$\mathbb{R}/\mathbb{Z} \qquad \mathbb{Z} \circlearrowright \mathbb{R}$. $\quad n \cdot x := n + x$.

② $\mathbb{Z}_n \circlearrowright S^1 \subseteq \mathbb{C} \qquad k \cdot z = e^{i\frac{2\pi k}{n}} z$

$\downarrow$

S^1

bad example

① $\mathbb{Q} \circlearrowright \mathbb{R}$

$p^{-1}(V)$ open $\Rightarrow$ $p^{-1}(V) = \mathbb{R}$.
in real line

② $\mathbb{R}_+ \circlearrowright \mathbb{R}$.
multi

$Y = \{-\ \ 0\ \ +\}$

$\mathcal{T} = \{\{-\}, \{+\}, \underline{\phi, Y}\}$
$\{-, +\}$.

Lee 6. points / sets in $(X, \mathcal{T})$

"clopen" : both open and close

for $(X, \mathcal{T}_{dis})$ $\#X > 1$ all the points all clopen

if $(X, \mathcal{T})$ only have ϕ X be clopen

$\Leftrightarrow X$ is connected

Example $\mathbb{R}$. U is open $\Leftrightarrow U = \bigcup_{i=1}^{N} (a_i, b_i)$ $(1 \leq N \leq +\infty)$

but no ~~st~~ such for closed set

eg Cantor set uncountable points / closed

we know most sets on $\mathbb{R}$ is not open or closed

counting number so that

Example $\mathbb{Q} \subseteq \mathbb{R}$.

$A = (\sqrt{2}, \pi) \cap \mathbb{Q}$ is ~~ole~~ open

$A^c = ((-\infty, \sqrt{2}] \cap \mathbb{Q}) \cup ([\pi, +\infty) \cap \mathbb{Q})$

$= ((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cup ((\pi, +\infty) \cap \mathbb{Q})$

$=$ ~~ele~~ open

A is closed

or if $x_n \in A \to x_0 \in \mathbb{Q} \Rightarrow x_0 \in A$.

Example $X = \mathcal{M}([0,1], \mathbb{R})$ $\mathcal{J}_{p.c.}$

Do we have A closed $\Longleftrightarrow$ $x_n \in A$ $\underline{\underline{x_n \to x_0}}^{\text{countable}} \in X \Rightarrow x_0 \in A$?

Consider $A = \{ f \mid f(x) \neq 0$ at most countable $x \in [0,1] \}$.

$f_n \in A$. $f_n \to f \in X$

since $x \notin \underbrace{\bigcup_{n=1}^{\infty} \{ f_n \neq 0 \}}_{\text{countable}} \Rightarrow f \in A. \Rightarrow \checkmark$

but A is not closed $\forall g \in A^c$ $\forall$ open $U \ni g$ $\exists \omega(g; x_0, \cdots x_n; \varepsilon) \subseteq U$.

consider $g_1 = \begin{cases} g & x = x_0, \cdots x_n \\ 0 & \text{o} \end{cases}$ $\in U$ $\in A$ $\Rightarrow U \not\subseteq A^c \Rightarrow A^c$ is not open $\Rightarrow A$ is not closed.

prop. $F \subseteq (X, d)$ is closed $\Longleftrightarrow$ $x_n \in F$. $x_n \to x_0 \in X \Rightarrow x_0 \in F$

proof $(\Rightarrow)$ F is closed by contradiction, suffice $x_0 \in F^c$

F^c is open $\Rightarrow \exists \varepsilon > 0$. $B(x_0, \varepsilon) \subseteq F^c$ but $\exists x_n \in B(x_0, \varepsilon)$

$\Rightarrow x_0 \in F$

$(\Leftarrow)$ ~~$\forall \varepsilon > 0$. $\exists k$. $x_n \in B(x_0, \varepsilon)$ $\forall n > k$.~~

consider F^c. if F^c is not a open set

$\underline{\underline{\exists x_0 \in F^c}}$ $\forall$ open set contains x_0 is not in F^c

$\Rightarrow \underset{\substack{\Psi \\ x_n}}{B(x_0, \tfrac{1}{n})} \cap F \neq \phi$ $\Rightarrow$ give a sequence $x_n \to x_0$

$\Rightarrow x_0 \in F$. ↯ $\square$

Def We say $(X, \mathcal{J})$ is first countable (or (A1) space) if every point $x \in X$ has a countable neighborhood basis, i.e. $\forall x \in X$, $\exists$ open U_n^x s.t. $\forall$ open $U^x \ni x$ $\exists n$ s.t. $U^x \supseteq U_n^x$.

Rmk. we can always take $U_1^x \supseteq U_2^x \cdots \supseteq U_n^x \supseteq \cdots$

since if not let $\tilde{U}_n^x = \bigcap_{k=1}^{n} U_k^x$.

Compare the proof of the previous proposition, we have

prop If $(X, \mathcal{J})$ is A1. then $\underset{\substack{\cap \\ F}}{x_n} \to x_0 \in X \Rightarrow x_0 \in F$" lead "$F$ is closed". ~~X is~~

cor $\mathcal{J}_{p.c}$ is not $\mathcal{J}_d$ $\left(\mathcal{J}_d \text{ is A1 } \cancel{\text{space}} \right)$.

cor $\mathcal{J}_{p.c}$ is not A1 ~~space~~

18.

Def. (1) $x_n \in A$ $x_n \to x_0$ call x_0 is a sequence limit point of A

(2) $\overset{A \subseteq X}{x_0} \in X$ if $\forall$ $x \in U$ is open we have $U \cap (A - \{x_0\}) \neq \phi$

we call x_0 is a limit point of A.

Notation A' = all the limit point of

property ① $\phi' = \phi$

② $a \in A' \Rightarrow a \in (A \setminus \{a\})'$

③ $A \subseteq B \Rightarrow A' \subseteq B'$

④ $(A \cup B)' = A' \cup B'$

left for check ⑤ $(A')' \subseteq A \cup A'$

e.g. in $\mathcal{J}_{p.c}$ & A. $A' = X$, $\forall q \in X$, $q_1 = \begin{cases} q_0(x) & x_1, \dots x_n \\ 0 & \end{cases}$ $\in A$

prop. $A \overset{\subseteq (X, \mathcal{J})}{\text{is closed}} \Leftrightarrow A' \subseteq A$ $A^c \subseteq (A')^c$

($\Rightarrow$) A is closed $\Rightarrow A^c$ is open

$\forall x \in A^c$. $\exists U \Rightarrow A^c \cap (A \setminus \{x_0\}) = \phi$

$\Rightarrow x \notin A' \Rightarrow x \in (A')^c$

$\Rightarrow A^c \subseteq (A')^c$

($\Leftarrow$) if $A' \subseteq A \Rightarrow$

$\forall x \in A^c \Rightarrow x \notin A' \Rightarrow \exists U$ is open

$U \cap (A \setminus \{x_0\}) = \phi \Rightarrow U \subseteq A^c$

$\Rightarrow A^c$ is open $\Rightarrow$ A is closed.

Def $D: \mathcal{P}(X) \to \mathcal{P}(X)$

$A \mapsto A'$

characterist the topology

rmk. sequence limit point

$\not\Uparrow$ $\not\Downarrow$.

limit point

But what's A'?

① $(X, d) \Rightarrow$ "A' is closed".

② $\exists (X, \mathcal{J})$ st "A' is not closed" prop. $A \cup A'$ is closed

proof $(A\cup A')' = A' \cup \underline{A''} \subseteq (A\cup A') \cup (A \cup A') = A \cup A'$

$\Rightarrow A \cup A'$ is closed $\square$

Note. if F is closed & $F \supseteq A$ then $F \supseteq (A \cup A')$

~~pf~~ since $F \supseteq F' \supseteq A' \Rightarrow F \supseteq (A\cup A')$

i.e. $A\cup A'$ is the smallest closed set which contains A.

$\Downarrow$

cor $A \cup A' = \bigcap_{\substack{F \text{ closed} \\ F \supseteq A}} F$.

LHS $\subseteq$ RHS trivial
RHS $\subseteq$ LHS ~~left for checking~~
since $A\cup A'$ is closed

def $\bar{A} := A \cup A$ call it the closure of A

New operation

$\mathcal{C} : P(X) \to P(X)$

$A \mapsto \bar{A}$

prop.
① $A \subseteq \bar{A}$
② $\overline{A\cup B} = \bar{A} \cup \bar{B}$
③ $\bar{\bar{A}} = \bar{A}$
④ $\bar{\emptyset} = \emptyset$

closure Axiom

⑤ A is closed $\iff A = \bar{A}$
⑥ $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
⑦ $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
⑧ $\overline{A\times B} = \bar{A} \times \bar{B}$

prop. $x_0 \in \bar{A} \iff \forall U \in \mathcal{U}_x$ $U \cap A \neq \emptyset$.

proof ($\Rightarrow$) $x_0 \in A$ or $x \in A'$

$\Downarrow$ $\forall U \in \mathcal{U}_x$ $x_0 \in U \cap A$

$\Downarrow$ $\forall U \in \mathcal{U}_x$ $\emptyset \neq U \cap (A - \{x\}) \subseteq U \cap A$

($\Leftarrow$) if $x_0 \in A$ trivial

if $x_0 \notin A \Rightarrow \forall U \in \mathcal{U}_x$ $U \cap A = U \cap (A - \{x_0\}) \neq \emptyset$

$\Rightarrow x \in A'$ $\square$

prop. A map $f: X \to Y$ is cts $\iff f(\bar{A}) \subseteq \overline{f(A)}$

proof ($\Rightarrow$) f is cts $\iff f^{-1}(\text{closed}) = \text{closed}$

so $f^{-1}(\overline{f(A)})$ is closed $\supseteq A$ $\Rightarrow f^{-1}(\overline{f(A)}) \supseteq \bar{A}$

$\Rightarrow \overline{f(A)} \supseteq f(\bar{A})$

($\Leftarrow$) $f(\bar{A}) \subseteq \overline{f(A)}$

$f^{-1}(B)$, $B \subseteq Y$ closed

$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} = \bar{B} = B$

$\Downarrow$ $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$

$\Downarrow$ $f^{-1}(B)$ is closed

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the 3rd operation

$I : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$

$A \longrightarrow \mathring{A} := \mathrm{Int}(A)$

where $\mathring{A} := \{ x \in A \mid \exists \underbrace{U}_{\text{open}} \in \mathcal{U}_x \text{ s.t. } U \subseteq A \}$

Note $U \subseteq \mathring{A}$, $\forall y \in U$. we have $y \in U \subseteq A$

$\Rightarrow \mathring{A}$ is open

if U is open. $U \subseteq A$. $U \subseteq \mathring{A}$

Cor $\mathring{A} = \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$ LHS $\supseteq$ RHS trivial RHS $\subseteq$ LHS $\mathring{A} \subseteq A$.

i.e. $\mathring{A}$ is the largest open set which is contained by A

In general, $\nexists$ the largest closed in A

$\nexists$ the smallest open containing A

prop. $\mathring{A} = \overline{A^c}^{\,c}$

proof ① $(\overline{A^c})^c$ is open

$(\overline{A^c})^c \subseteq A \Leftrightarrow \overline{A^c} \supseteq A \supseteq \mathring{A}$

$\Rightarrow \overline{A^c}^{\,c} \subseteq A^\circ$

② $x \in A^\circ$ $\exists U \in \mathcal{U}_x$. $x \in U \subseteq A$

$\Rightarrow U \cap A^c = \phi$

$\Rightarrow$ ~~$U \subseteq A$~~ $x \notin \overline{A^c}$

$\Rightarrow x \in \overline{A^c}^{\,c}$

$\Rightarrow \mathring{A} \subseteq \overline{A^c}^{\,c}$ □

prop. I satisfy

① $\mathring{A} \subseteq A$

② $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$

③ $\mathring{\mathring{A}} = \mathring{A}$

④ $\mathring{X} = X$

⑤ A open $\Leftrightarrow A = \mathring{A}$

⑥ $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}$

⑦ $\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ$

⑧ $(A \times B)^\circ = \mathring{A} \times \mathring{B}$

Def. $(X, \mathcal{T})$ ① A set is ~~called~~ <u>dense</u> if $\overline{A} = X$

② A is ~~nondense~~ nowhere dense if $\mathring{\overline{A}} = \phi$

e.g. $\overline{\mathbb{Q}} = \mathbb{R}$. $\overline{\mathbb{Q}^c} = \mathbb{R}$

$(\mathbb{R}, \mathcal{T}_{standard})$ $\overline{\mathbb{N}} = \mathbb{N}$ but $\mathring{\overline{\mathbb{N}}} = \phi$.

$(\mathbb{Q}, \mathcal{T}_{cofinite})$: $\overline{\mathbb{N}} = X$

P = all the polynomials on $[0,1]$ on $C([0,1])$

$\Downarrow$ weierstrass thm

$\overline{P} = C([0,1])$ in d_∞

$\mathcal{C}$ cantor set $\overline{\mathcal{C}} = \mathcal{C}$ and it has not int points $\Rightarrow$ $\mathcal{C}$ is nowhere dense

Def (Boundary) $\partial A = \overline{A} \setminus \mathring{A} \boxed{= \overline{A} \cap (\mathring{A})^c = \overline{A} \cap \overline{A^c}}$

In particular. ∂A is always closed.

Cor. $x \in \partial A \iff \forall$ ~~open~~ U open, $x \in U$, $U \cap A \neq \phi$. $U \cap A^c \neq \phi$.

prop. ① ∂A closed

② $\partial A = \partial A^c$

③ $\partial \mathring{A} \subseteq \partial A$. $\partial \overline{A} \subseteq \partial A$ $\quad \overline{\odot} \rightsquigarrow \bigcirc$

④ $\partial\partial A = \partial A$ "=" if A open or closed

⑤ $\partial(A \cup B) \subseteq \partial A \cup \partial B$

$X = \mathring{A} \sqcup \partial A \sqcup (\overline{A})^c$

Rmk. Another def of Boundary "on the manifolds"

different ways to understand topology

"class"

Category objects $ob(\mathcal{C})$

$\mathcal{C}$ set with special structure

Morphism

$(X, Y \in ob(\mathcal{C})) \longrightarrow Mor(X, Y)$

can be ϕ

$f \in Mor(X, Y)$

$g \in Mor(Y, Z)$

$\longrightarrow g \circ f \in Mor(X, Z)$

① associated

$h \circ (g \circ f) = (h \circ g) \circ f$

② Id $\exists\ Id_Y \in Mor(Y, Y)$

s.t $Id_Y \circ f = f$

$g \circ Id_Y = g$

objects: topology spaces

methods (axioms)

open (O1) (O2) (O3)

closed (C1 - C3)

neighborhood (N1 ~ N4)

closure (K1 - K4)

Int (I1 - I4)

e.g. Lin objects = linear spaces

Morphism = linear maps

↓ sub

Morphism = linear ~~homorphism~~ isomorphism

e.g. Group objects = groups

morphism = group homomorphism

So in topology category

we can consider "Morphism" = continuous map.

rmk:

Morphism need not be map

e.g. Set category

object sets

Morphism: relation

a relation R is $R \subseteq X \times Y$

if $R_1 \subseteq X \times Y$

$R_2 \subseteq Y \times Z$

$R_2 \circ R_1 \subseteq X \times Z$ defined by

$R_2 \circ R_1 = \{ (x, z) \mid \exists y \in Y. \text{ s.t } (x, y) \in R_1, (y, z) \in R_2 \}$