lecture 8. Compactness. finiteness on topology ⟶ local to field

$X=$ / objects	finite set	~~compact~~ $[0,1]$	$(0,1]$
$f: X \to \mathbb{R}$, cts	✓	bounded, attains min/max	×
$x_1, \cdots x_n \cdots \in X$	✓	BW admit convergent subseq	×
infinite subset	——	$A' \neq \phi$	$\exists\ A' = \phi$
$X = \cup U_\alpha$	✓	H.B finite sub covering	×
$F_1 \supseteq F_2 \cdots \supseteq X$, ϕ	✓	Cantor $\cap F_n \neq \phi$	

⟶ sequential compact

⟶ compact

Def. $(X, \mathcal{T})$ (1) We say $(X,\mathcal{T})$ is compact, if for any open covering of X admits a finite subcovering

$\mathcal{U} = \{U_\alpha\}$ open, $X = \bigcup_\alpha U_\alpha$

(2) We say $(X,\mathcal{T})$ is seq compact, if for any sequence $\{x_n\}$ admits a subsequence $\underline{\underline{x_{n_k}}} \to x_0 \in X$

$n_1 < n_2 < \cdots$

Def. if $A \subseteq (X, \mathcal{T})$ (1) We say A is cpt if $(A, \mathcal{T}_A)$ is cpt (subspace topology)

(2) seq cpt seq cpt.

prop. A is cpt $\iff$ For any open covering $\underline{\underline{\{U_\alpha\}}}$ of A (by open sets in X). $A \subseteq \bigcup_\alpha U_\alpha$.

there exists a finite sub covering.

Example (1) $\mathbb{R}^n$ standard not cpt, not seq cpt

(2) $A \subseteq \mathbb{R}$ cpt $\iff$ seq cpt $\iff$ bounded & closed

NOT True for $(\mathbb{N}, \mathcal{T}_{dis})$

(2) cofinite cpt ✓

~~cpt~~ seq cpt ✓

X cpt $\iff$ "$X = \bigcup_\alpha U_\alpha \Rightarrow \exists\ X = \bigcup_{i=1}^n U_{\alpha_i}$" $\iff$ "$\phi = \bigcap_\alpha F_\alpha \Rightarrow \exists \bigcap_{i=1}^n F_{\alpha_i} = \phi$"

$\Updownarrow$

"$\forall \bigcap_{i=1}^n F_{\alpha_i} \neq \phi \Rightarrow \bigcap_\alpha F_\alpha \neq \phi$"

Finite intersection "FIP"

Cor. $(X, \mathcal{T})$ is cpt if $F_1 \supseteq F_2 \supseteq \cdots$ $F_k \neq \phi$

then $\bigcap_{i=1}^\infty F_n \neq \phi$.

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Prop. Let $\mathcal{B}$ be a basis of $(X, \mathcal{T})$

Then X is cpt $\Leftrightarrow$ Any open covering of $\mathcal{B}$ admits a finite covering

$\mathcal{U} \subseteq \mathcal{B}$

proof ($\Leftarrow$). let $\mathcal{U} \subseteq \mathcal{T}$. $\forall x.\ \exists U^x \in \mathcal{U}$

$\Rightarrow \exists U_x$ s.t $x \in U_x \subseteq U^x$. $U_x \in \mathcal{B}$

$\Rightarrow$ U_x is a basis covering

$\Rightarrow$ $X = \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup^{n} U^{x_i}$

$\Downarrow_{\Leftarrow}$ $=$

($\Rightarrow$) basis covering is also open covering

Alexander subbasis thm: let $\mathcal{C}$ is a subbasis of $(X, \mathcal{T})$

then " X is cpt $\Leftrightarrow$ Any subbasis covering $\mathcal{U} \subseteq \mathcal{C}$ admits finite subcovering

Def (Hausdorff) $(X, \mathcal{T})$ is hausdorff if $\forall x \neq y \in X$. $\exists U, V \in \mathcal{T}$

T_2 s.t $x \in U$. $y \in V$. $\underline{U \cap V = \phi}$

Duality between cpt & T_2

(1) (a) $(X, \mathcal{T})$ cpt, "$A \subseteq X$ is closed $\Rightarrow$ A is cpt"

(b) $(X, \mathcal{T})$ cpt $\mathcal{T}' \subsetneq \mathcal{T} \Rightarrow (X, \mathcal{T}')$ is cpt

(c) $(X, \mathcal{T})_{\text{tri}}$ is cpt

(2) (a) $(X, \mathcal{T}\ \text{dis})$ is Hausdorff

(b) $(X, \mathcal{T})$ T_2 $\mathcal{T}' \supseteq \mathcal{T} \Rightarrow (X, \mathcal{T}')$ T_2

(c) $(X, \mathcal{T})$ T_2 "$A \subseteq X$ is cpt $\Rightarrow$ A is closed"

(1) (a). If A is closed U_α is an open covering

$\Rightarrow A^c \cup (\bigcup_\alpha U_\alpha)$ is an open covering

$\Rightarrow$ $U_{\alpha_1}, \ldots U_{\alpha_k}$, A^c is ~~an~~ ~~open~~ sub covering

$\Rightarrow \bigcup_{n=1}^{k} U_{\alpha_n} \supseteq A$

(2) (c) $\forall x \in A^c \Rightarrow$ For any $y \in A$

$\exists U_y^x, V_y$ $U_y \cap V_y = \phi$

$\Rightarrow \{V_y\}$ is open covering of

V_{y_i} $i = 1, \ldots n$ is finite subcovering

$\Rightarrow \bigcap_{i=1}^{n} U_{y_i}$ is open
$x \in$ $\overset{\shortparallel}{U}$

and $U \cap A = \phi$

$\Rightarrow A^c$ is open

Prop. $f: X \to Y$. cts.

(1) if $A \subseteq X$ is cts. then $f(A) \subseteq Y$ is cpt

(2) seq cts seq cts.

proof (1) let $V = \{V_\alpha\}$ is an open covering of $f(A)$

$\Rightarrow f^{-1}(V_\alpha)$ is an open covering of A

$\Rightarrow f^{-1}(V_i)$ is a sub covering of A

$\Rightarrow V_i$ is finite $\because A$

(2) f persist seq convergence

Cor $f: X \to \mathbb{R}$. X is cpt $\Rightarrow f$ is bounded, attains min/max

Cor Quotient space of compact space is cpt.

like $\mathbb{RP}^n$

Cor X cpt. Y Hausdorff, $f: X \to Y$ cts $\Rightarrow f$ is closed map

Cor X cpt Y T_2. f cts bijective $\Rightarrow f$ is homomorphism.

$X \xrightarrow{f} Y$

$X \xleftarrow{g = f^{-1}} Y$

Cor $(X, \mathcal{T})$ is $\underline{\underline{\text{cpt hausdorff}}}$ "CH space."

if $\mathcal{T}_1 \subsetneq \mathcal{T}_0 \subsetneq \mathcal{T}_2$

then NOT Hausdorff NOT cpt.

proof $\mathcal{T}_1 \xleftarrow{Id} \mathcal{T} \xleftarrow{Id} \mathcal{T}_2$

rmk cpt $\nearrow$ week

Hausdorff $\searrow$ fine

CH : "the balance."

Rmk. In general $f: X \to Y$

$f^{-1}(\text{cpt}) \neq \text{cpt}$

Def f is proper if $f^{-1}(\text{cpt}) = \text{cpt}$.

we can proof f is proper

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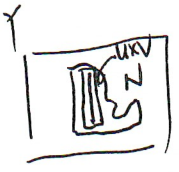
Lecture 9. compactness of product space

finite infinite and $\overset{\text{cpt}}{\cancel{\text{com}}} \not\Rightarrow$ seq cpt.

lem 1. (tube lemma). Suppose $X_0 \in X$, B is cpt in Y

$\{X_0\} \times B \subseteq N$, N is open in $X \times Y$. then we can

find open sets $x \in U$, $B \subseteq V$, s.t. $\{X_0\} \times B \subseteq U \times V \subseteq N$.



i.e. we can find an open set with "good shape".

proof. for $\forall$ ~~any~~ $y \in B$, $\Rightarrow$ we can find $x \in U_y$, $y \in V_y$

s.t. $(X_0, y) \in U_y V_y \subseteq N$

By openness of B, $\bigcup_{y \in B} V_y \supseteq B \Rightarrow N \supseteq \bigcup_{i=1}^{m} V_y \supseteq B$.

$\Rightarrow U = \bigcap_{i=1}^{m} U_{y_i}$, $V = \bigcup_{i=1}^{m} V_{y_i}$

$\Rightarrow U \times V \subseteq N$. $\square$

cor. Suppose $A \subseteq X$, $B \subseteq Y$ is cpt. $A \times B \subseteq N$ ~~then $\exists$ U.~~

then $\exists$ open $U \supseteq A$, $V \supseteq B$. $U \times V \subseteq N$

proof. $\forall x \in A$. $\{x\} \times B \subseteq U_x \times V_x \subseteq N$

since A is cpt

$\Rightarrow \exists$ $x_1, \cdots x_n$. $A \subseteq \bigcup_{i=1}^{n} U_{x_i} =: U$

$V = \bigcap_{i=1}^{n} V_{x_i}$ contains B & open

$\Rightarrow A \times B \subseteq U \times V \subseteq N$. $\square$

Thm. if A, B is cpt. $\Rightarrow A \times B$ is cpt.

proof. Let $\mathcal{W} = \{W_\alpha\}$ is an open covering of $A \times B$

$\forall x$, then $\{x\} \times B$ is cpt. since $\{x\} \times B$ is the image of $\begin{array}{l} i_x: Y \to X \times Y \\ \quad y \mapsto (x, y) \end{array}$ is cts

So $\exists W_1^x, \cdots W_m^x$ s.t. $\{x\} \times B \subseteq W_1^x \cup \cdots \cup W_m^x$

$\Rightarrow \exists U_x$ s.t. $\{x\} \times B \subseteq U_x \times B \subseteq W_1^x \cup \cdots \cup W_m^x$

Since A is cpt $\exists x_1, \cdots x_n$. $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$

So $A \times B \subseteq \left(\bigcup_{i=1}^{n} U_{x_i} \right) \times B \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} W_i^{x_j}$ $\square$

We ~~&~~ use subbasis to ~~structure~~ solve the thm in fact.

Thm (Tychonoff) $(X_\alpha, \mathcal{T}_\alpha)$ cpt $\Longleftrightarrow$ $\left(\prod_\alpha X_\alpha, \mathcal{T}_{prod}\right)$ cpt.

Example $(X^{\mathbb{N}}, \mathcal{T}_{prod}) = (\mathcal{U}(\mathbb{N}, X), \mathcal{T}_{p.c.})$

① $X = \{0, 2\}$ $\rightsquigarrow$ $(\mathcal{C}, \mathcal{T}_{scan})$ is cpt.

② $X = [0,1]$ is seq cpt

a^1 a^2 ---- a seq of seq.

$a = (a_1, a_2, \dots)$

take $n(1,1)$ $n(1,2)$ ... $a_1^{n(1,i)} \to a_1^\infty$ a_1 is convergent

~~take~~

$$\begin{pmatrix} a^{n(1,1)} & a^{n(1,2)} & \cdots \\ a^{n(2,1)} & & \\ \vdots & & \end{pmatrix}$$

seq. cpt. $\overset{\text{metric.}}{\Longleftrightarrow}$ cpt

thm (Alexander) Any subbasis covering has finite subcovering $\Rightarrow$ X is cpt.

proof is later

~~thm (Tyo~~

the proof of Tychonoff.

Let $\mathcal{U} = \{\pi_\alpha^{-1}(U) \mid U \in \mathcal{A}_\alpha\}$ be a subbasis covering.

where $\mathcal{A}_\alpha \subseteq \mathcal{T}_\alpha$

Claim $\exists \alpha_0$ s.t. $\mathcal{A}_{\alpha_0}$ is open covering of X_{α_0}

otherwise. $\forall \alpha$. $X_\alpha \setminus \bigcup_{U \in \mathcal{A}_\alpha} U \neq \emptyset \Rightarrow \prod_\alpha \left(X_\alpha \setminus \bigcup_{U \in \mathcal{A}_\alpha} U\right) \neq \emptyset$. Axiom of Choice

$\Rightarrow$ $\mathcal{U}$ is not a covering

$\exists U_1, \dots, U_m$ on $\mathcal{A}_{\alpha_0}$ s.t. $X_{\alpha_0} = U_1 \cup \dots \cup U_m$

$\Rightarrow$ $\pi_{\alpha_0}^{-1}(U_1), \dots, \pi_{\alpha_0}^{-1}(U_m)$ is finite subcovering $\square$

Example cpt $\not\Rightarrow$ seq cpt

- $X = [0,1]$ $\quad [0,1]^{[0,1]} = (\mathcal{U}[0,1], [0,1], \mathcal{T}_{p.c})$ is cpt by Tychonoff
- consider $f_n : [0,1] \to [0,1]$ $\quad x \mapsto$ nth digit of its binary expression of x.

For Any (f_{n_k}) take $x_0 = (\underset{n_{2k}}{\overset{0}{\uparrow}} \quad \underset{n_{2k+1}}{\overset{1}{\uparrow}} \quad - -)$

$\Rightarrow f_{n_k}$ is not convergent at x_0. $\#$

(2) seq cpt $\not\Rightarrow$ cpt

$A = \{ f : [0,1] \to [0,1] \mid f(x) \neq 0$ for at most countable times$\}$

- A is seq cpt

Let $f_n \in A$. Then $S = \{ t \mid \exists n \in \text{s.t. } f_n(t) \neq 0 \}$ is countable

" the convergence of f_n "

$\Updownarrow$ $\quad [0,1]^S \sim [0,1]^{\mathbb{N}}$

— — —

$\Downarrow$

seq cpt.

But A is not cpt.

$A_t = \pi_t^{-1}(1) = \{ f \mid f(t) = 1 \}$

$\bigcap_{[0,1]} A_t = \phi$ $\quad$ but $\bigcap_{i=1}^{n} A_{t_i} \neq \phi$.

the proof of Alexander subbasis theorem

Axiom of Choice

For any $\mathcal{A} \subseteq P(X) \setminus \{\phi\}$ $\exists$ a function $f : \mathcal{A} \to X$ s.t. $f(A) \in A$. $\forall A \in \mathcal{A}$

$\Updownarrow$

Zorn lem

if X is not cpt.

$\mathcal{W}oke = \{\mathcal{A} \subseteq \mathcal{T} \mid \mathcal{A}$ is open covering, but has no finite subcovering$\}$

since X is not $\Rightarrow$ $\mathcal{W}oke$ is not ϕ.

"$\subseteq$" is a partial order on $\mathcal{W}oke$.

take a totally ordered subset of $\mathcal{W}oke$ — $\mathcal{W}ike$

then ① $\mathcal{E} = \bigcup\limits_{\mathcal{A} \in \mathcal{W}ike} \mathcal{A} \subseteq \mathcal{T}$

② $\mathcal{E}$ is an open covering of X.

③ $\mathcal{E}$ is a upper bond of $\mathcal{W}ike$

Fact. $\mathcal{E}$ has no finite subcovering.

$U_1 \cdots U_n$ in $\mathcal{E}$

$\Rightarrow U_i \in \mathcal{A}_i$

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have a max $\cdots$ $\mathcal{A} \in \mathcal{W}oke$

30 Lecture 10. the compactness of metric space

For metric space.

"countability"

A1. "countable neighborhood basis" — F is closed set $\Leftrightarrow$ F contains ~~All the~~ all ~~the~~ its seq limit point.

— $f:(X,d)\to(Y,\mathcal{T})$ cts $\Leftrightarrow$ seq cts

the second countability --

"separable"

T_2 (Hausdorff) $\forall x\neq y$, $\exists U, V\in\mathcal{T}$, $x\in U$, $y\in V$, $U\cap V=\phi$.

metric space is T_2. ~~to~~ prove by triangle inequality.

$\Downarrow$

① the limit is unique

② cpt subset is closed

T_4 (normed) two closed sets can be ~~sp~~ separated by open sets

prove by uryson lemma.

metric space is T_4.

e.g. bounded and closed set need not to be cpt

$(\mathbb{N}, d_{dis})$ ~~both~~ not cpt. Not seq cpt

$[0,1]\subseteq\mathbb{Q}(0,+\infty)$ is not cpt, not seq cpt

"Compactness"

Thm For $A\subseteq(X,d)$ TFAE.

① A is cpt

② A is seq cpt

③ A is totally bounded (3a) and absolute closed (3b)

$\Updownarrow$ complete

(3') totally bounded & Lebesgue number property

④ A is limit cpt

⑤ A is countable cpt

⑥ A is pseudo cpt

Def. We say a metric space is totally bounded if $\forall\varepsilon>0$, $\exists x_1,\cdots x_m$ ($m=m(\varepsilon)$) $X=\bigcup_{i=1}^{m}B(x_i,\varepsilon)$ $\Longleftrightarrow$ finite ε-net

prop (X,d) cpt or seq cpt $\Rightarrow$ (X,d) is total ~~cpt~~ bounded

proof. (cpt) take $B(x,\varepsilon)$ $\longrightarrow$ finite covering

(seq cpt) if $\exists\varepsilon>0$. s.t. there exists no finite ε-net

take $\forall x_1$ $\Rightarrow$ $X\setminus B(x_1,\varepsilon)\neq\phi$

take x_2 $\Rightarrow$ $X\setminus(B(x_1,\varepsilon)\cup B(x_2,\varepsilon))\neq\phi$

$\Rightarrow X \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon) \neq \phi$

but $d(x_i, x_j) > \varepsilon$. $\Rightarrow$ not seq cpt.

prop. (Lebesgue Number Property) If (X, d) is seq cpt. then the set satisfy

" ____ ", i.e. for any open covering $\mathcal{U}$. $\exists\, \delta \overset{>0}{.}$ (dep on $\mathcal{U}$)

s.t any subset $A \subseteq X$. diam $A < \delta$, then ~~A~~ $\exists\, U \in \mathcal{U}$.

s.t $A \subseteq U$. "δ. is called ~~the~~ lebesgue number of the open covering"

proof. if not, $\exists \mathcal{U}$ has no lebesgue number

$\exists \{C_n \neq \phi\}$ diam $C_n < \frac{1}{n}$. $C_n \not\subseteq \forall\, U \in \mathcal{U}$

take $x_n \in C_n$, $x_{n_k} \to x_0$ $\exists\, U \ni x_0$ when n_k big enough

~~$B(x_0$~~ $C_{n_k} \subseteq U$. ↯

seq cpt $\Rightarrow$ cpt

seq cpt $\Rightarrow$ LN δ. finite

$\mathcal{U} \Rightarrow$ $U_i \supseteq B(x_i, \frac{\delta}{3})$ $\exists\, \frac{\delta}{3}$-net $X = \bigcup_{i=1}^{n} B(x_i, \frac{\delta}{3}) \Rightarrow$ cpt.

(1) $\Rightarrow$ (3a)

(2) $\Rightarrow$ (3a)

(2) $\Rightarrow$ (3b')

(2) $\Rightarrow$ (3') $\Rightarrow$ (1)

Def. (X, d) is complete if any cauchy seq converges

See in hw.

Def. We say $(\hat{X}, \hat{d})$ is a completion of (X, d) if

(1) $\exists$ isometric embedding $(X, d) \hookrightarrow (\hat{X}, \hat{d})$

(2) $\overline{\iota(X)} = \hat{X}$

prop. ~~Any~~ Any (X, d) admits a completion

$B(X, Y) = \{ f : X \to Y \mid f(X) \text{ is bounded} \}$

pf 1 $(X, d) \hookrightarrow C(X, \mathbb{R})$

pf 2 " cauchy seq "

as $\mathbb{Q} \rightsquigarrow \mathbb{R}$

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prop. (X,d) complete $A \subseteq X$ closed $\Rightarrow (A,d)$ complete

def ~~prop~~. We say (X,d) absolutely closed, if. For any isometric embedding $f:(X,d) \to (Y,d_Y)$ $f(X)$ is closed in Y

prop A metric space (X,d) is complete $\Leftrightarrow$ it's absolutely closed.

proof. $(\Rightarrow)$ isometric embedding persist its cauchy seq

$\Rightarrow f(X)$ is always closed.

$(\Leftarrow)$ consider $(X,d) \hookrightarrow (\hat{X}, \hat{d})$ $\iota(X)$ is closed ~~in $\hat{X}$~~

$\Rightarrow X$ is complete

cor (X,d) is cpt or seq cpt $\Rightarrow (X,d)$ is complete

pf. iso f ~~(X)~~ $: X \hookrightarrow Y$

f is cts. $\Rightarrow f(X)$ is cpt or seq cpt $\Rightarrow$ closed $\square$

now. $(1),(2) \Rightarrow (3)$

$(2) \Rightarrow (1) \Rightarrow (3)$

$\Downarrow$ $\Downarrow$

$(3')$

Finally

prop. (X,d) is totally bounded and absolutely closed $\Rightarrow X,d$ is seq cpt

proof. If (x_n) totally bounded $\Rightarrow \exists$ finite 1-net

one ball B_1 contains infinite points. $J_1 := \{n \mid x_n \in B_1\}$

$\downarrow$

finite $\frac{1}{2}$-net $J_2 = \{n \in J_1 \mid x_n \in B_2\}$

$\vdots$

$J_1 \supseteq J_2 \supseteq \cdots$

$\forall m,n \in J_k$ $d(x_m, x_n) < \frac{2}{k}$

take $n_1 \in J_1$, $n_2 > n_1 \in J_2$ $\cdots$

$\Downarrow$ cauchy seq. $\xRightarrow{\text{complete}}$ limit $\square$

recall

In (X,d) ~~cts~~ compact $\Rightarrow$ totally bounded + complete $\Rightarrow$ seq compact $\Rightarrow$ totally bounded, lebesgue property $\Rightarrow$ compact

in A: $\longrightarrow$, $\not\longleftarrow$

Lecture 11 $\quad X^Y = \underline{\mathcal{M}(X,Y)} \supseteq \mathcal{C}(X,Y)$

can be a simple set $\qquad$ with topology

Setting 1. Y should be $(Y,\mathcal{T}) \longrightarrow \mathcal{T}_{prod} = \mathcal{T}_{p.c.}$, $\mathcal{T}_{box}$.

Setting 2. $(Y,d) \longrightarrow d_{uni}(f,g) = \sup_{x\in X} \dfrac{d(f(x),g(x))}{1+d(f(x),g(x))}$ ~~on~~ metric on $\mathcal{M}(X,Y)$.

we have " $f_n \to f$ on $X \Leftrightarrow f_n \to f$ w.r.t $\underline{d_u}$ " $\longrightarrow \mathcal{T}_{u.c.}$

$f_n \to f$ w.r.t $\mathcal{T}_{u.c.}$

prop. if (Y,d) complete, then $(\mathcal{M}(X,Y), d_u)$ complete.

If we don't know the limit and the ~~se~~ space is complete we can finally use the cauchy seq to test a seq is convergent or not.

Setting 3. $(X,\mathcal{T}), (Y,d) \longrightarrow \mathcal{C}(X,Y)$ NOT closed in $\mathcal{T}_{prod}$, $\mathcal{T}_{box}$.

Fact. $\mathcal{C}(X,Y)$ is closed in $(\mathcal{M}(X,Y), \mathcal{T}_{u.c})$. | the uniform limit is still cts.

$\Rightarrow$ Cor. ~~$\mathcal{C}(X,Y)$~~ if (Y,d) complete, then $(\mathcal{C}(X,Y), d_u)$ is complete.

Example. $X = Y = \mathbb{R}$.

① $f_n(x) = e^{-nx^2/n} \xrightarrow{\text{pointwise}} \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$ NOT cts.

② $f_n(x) = \dfrac{x^2}{n} \xrightarrow{u.c.} \times$

$\xrightarrow{p.c.} f \equiv 0 \in C(X,Y)$.

| $\mathcal{T}_{p.c}$ too weak.
| $\mathcal{T}_{u.c.}$ too strong
| Continuity is local property.

Compare $\mathcal{T}_{u.c.} \ni B(f, x, \varepsilon) = \{g \in \mathcal{M}(X,Y) \mid d(g(x), f(x)) < \varepsilon.\ \forall x \in X\}$. global.

$\mathcal{T}_{p.c.} \ni B(f, x_1, \cdots x_n, \varepsilon) = \{g \in \mathcal{M}(X,Y) \mid d(g(x_i), f(x_i)) < \varepsilon.\ i=1,\cdots,n\}$ point

compact convergent topology $\mathcal{T}_{c.c} \ni B(f, K, \varepsilon) = \{g \in \mathcal{M}(X,Y) \mid \sup_{x\in K} d(g(x), f(x)) < \varepsilon.\ \}$.

$\mathcal{B}_{c.c}$

$\mathcal{T}_{c.c.}$ is generated by $\mathcal{B}_{c.c}$.

prop. ① $\mathcal{B}_{c.c.}$ is a basis. ② $f_n \to f$ w.r.t $\mathcal{T}_{c.c.} \Leftrightarrow f_n \to f$ unif on each K.

proof of ②. $f_n \to f$ unif $\Leftrightarrow \forall K.\ \forall \varepsilon>0.\ \exists N>0,\ n\geq N\ \sup_K (d(f(x), g(x))) < \varepsilon$.

$\Leftrightarrow f_n \in B\cdots \Leftrightarrow f_n \to f$ is ~~$\mathcal{T}_{c.c}$~~ $\mathcal{T}_{c.c}$

So. if $f_n \to f$ in $\mathcal{T}_{c.c.} \Rightarrow f$ cts on each K. Q: f cts on X?

Example. $(\mathbb{R}, \mathcal{T}_{\text{countable}})$ — "K cpt $\Leftrightarrow$ K is finite" | K is too few.

$(K, \mathcal{T}_{c.c}) = (K, \mathcal{T}_{dis})$

Def. (locally cpt) We say X is locally cpt. if $\forall x \in X$, $\exists$ open neigh U of x s.t $\bar{U}$ is cpt. it should be called strong locally cpt. the definition is true for LCH space.

$\Downarrow$ with paste lemma

prop. if X is lc, (Y, d) and $f_n \in C(X, Y)$. $f_n \xrightarrow{\mathcal{T}_{c.c.}} f$.

then f is cts.

In analysis. $\rightsquigarrow$ locally compact Hausdorff space $=$ LCH space

locally Euclidean $\forall x \in X$, $\exists U$ open contains x. s.t $U \simeq \mathbb{R}^n$ ~~or~~ B

prop. Let X be LCH. K cpt. U open $K \subseteq U$, then $\exists$ open set V. s.t $\bar{V}$ is cpt and $K \subseteq V \subseteq \bar{V} \subseteq U$

Rmk. We have infinite "partition" of U.

proof. assume $K = \{x\}$. By definition. $\exists W$ open. $\bar{W}$ is cpt.

let $\underset{\text{open}}{U_1} = U \cap W$ then $\bar{U_1} \subseteq \bar{W} \Rightarrow \bar{U_1}$ is cpt

if $U_1 = \bar{U_1}$ trivial

if $U_1 \neq \bar{U_1}$ $\bar{U_1} \setminus U_1$ is closed $\Rightarrow$ cpt

since T_2. $\forall y \in \bar{U_1} \setminus U_1$ $\exists$ U_y, $x \in V_y$. $V_y \subseteq U$. | We can always

$\Rightarrow \exists y_1, \dots y_m$. s.t $\underline{U_{y_1} \cup \dots \cup U_{y_m} \supseteq \bar{U_1} \setminus U_1}$

consider $V = V_{y_1} \cap \dots \cap V_{y_m} \subseteq U_1$

$\bar{V} \subseteq \bar{V_{y_1}} \cap \dots \cap \bar{V_{y_m}} \subseteq \bar{U_1}$ closed & cpt

$\Rightarrow \bar{V}$ is cpt.

$\bar{V} \overset{?}{\subseteq} U$: $V_y \subseteq U_y^c$

$\bar{V} \subseteq \bigcap U_y^c = (\bigcup U_y)^c \subseteq U_1$

$\forall x \in V \subseteq \underset{\text{cpt}}{\bar{V}} \subseteq U$

case 2. general K. $\{V_x\} \supseteq K$. $\Rightarrow V_{x_1} \cdots V_{x_m} \supseteq K$.

$\Rightarrow K \subseteq \bigcup^m V_{x_m} \subseteq \bigcup^m \bar{V}_{x_m} \subseteq U$.

② Composition

$\Phi: \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$

$f, g \mapsto g \circ f$

prop. Φ is cts (Y is LCH)

$K \subseteq X$ cpt $W \subseteq Z$ open

$g \circ f \in S(K, W)$ $f(K) \subseteq Y$ cpt, $g^{-1}(W)$ is open

$\Rightarrow f(K) \subseteq g^{-1}(W) \Rightarrow f(K) \subseteq V \subseteq \bar{V} \subseteq g^{-1}(W)$

$\Rightarrow S(K, V) \times S(\bar{V}, W) \subseteq \Phi^{-1}(K, W)$

cor X is LCH $\Rightarrow$ evaluation is cts.

$\mathcal{M} = \mathcal{C}$. $\supseteq Y$ $\not\Rightarrow$ $(\mathcal{C}, \mathcal{T}_{c.o}) \simeq (Y, \mathcal{T}_Y)$

Example $X = \{x\}$. $\mathcal{T}_{c.o} = \mathcal{T}_Y$ | $\mathcal{T}_{c.c}$ dep on metric but not so much.

prop. ~~If~~ Y, d. $\mathcal{T}_{c.o} = \mathcal{T}_{c.c.}$

Setting 4. $(X, \mathcal{T})$. $(Y, \mathcal{T})$

Def (Compact Open topology) on $\mathcal{M}(X,Y)$ or $\mathcal{C}(X,Y)$ generated by $\mathcal{S} = \{S(K,V) \mid K \subseteq X \text{ cpt. } V \subseteq Y\}$

$S(K,V) = \{f \mid f(K) \subseteq V\}$

Natural maps betw mapping space

① $A \subseteq X$. $r_A: \mathcal{M}(X,Y) \to \mathcal{M}(A,Y)$ $f \mapsto f|_A$

prop. endow $\mathcal{T}_{c.o.}$ ① r_A cts ② $r_A(\mathcal{C}(X,Y)) \subseteq \mathcal{C}(A,Y)$ and $r_A|_{\mathcal{C}(X,Y)}$ cts.

Lecture 12. Arzelà–Ascoli Thm

Classic version. $\{f_n\}$. $f_n \in \mathcal{C}([0,1], \mathbb{R})$

① if. f_n uniformly bounded and equicontinuous, the (f_n) has a convergent seq

- uniformly bounded. ~~$\exists M>0$~~. $\exists M>0$. s.t. $|f(x)| \le M$. indep of $f \in \mathcal{F}$ | $\mathcal{F} \subseteq \mathcal{C}([0,1], \mathbb{R})$.
- equicontinuous. $\forall x_0.\ \forall \varepsilon>0.\ \exists \delta = \delta(x_0, \varepsilon)$.

$$|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)| < \varepsilon.\ \forall f \in \mathcal{F}$$

e.g. $f_n \equiv n$. NOT uniformly bounded

$f_n = x^n$ on $[0,1]$ NOT equicontinuous at $x_0=1$

② If any subseq of (f_n) has convergent subseq, then (f_n) ~~has~~ uni bounded and equicontinuous.

take away the bad ones.

Today's basic setting $(X, \mathcal{T})$ (Y, d). metric notation

Def. $\mathcal{F} \subseteq \mathcal{C}(\cancel{[0,1]}, \cancel{\mathbb{R}})$ [X, Y] is equicontinuous if

$\forall x_0.\ \forall \varepsilon>0.\ \exists U \ni x_0$. [depend on x_0, but ~~not~~ indep of $f \in \mathcal{F}$]

s.t $d(f(x), f(x_0)) < \varepsilon$ for $\forall x \in U.\ \forall f \in \mathcal{F}$)

Rmk. the definition has not to do with the topology on $\mathcal{C}(X, Y)$.

prop. Any totally bounded subset $\mathcal{F} \subseteq (\mathcal{C}(X,Y), d_u)$ is equicontinuous.

proof. [use finite ε-net $\longrightarrow$ finite f
finite $\mathcal{F} \Rightarrow$ must equicontinuous.]

$\forall \varepsilon>0$. take finite ε_1-net $\{f_1, \cdots f_n\}$ of $\mathcal{F}$ in d_u metric

$$d_u \triangleq \sup_{x\in X} \frac{d_Y(f(x), g(x))}{1+d_Y(f(x), g(x))}$$

$\Downarrow$ ε_1 small enough

$d_u(f, g) < \varepsilon_1 \Rightarrow d(f(x), g(x)) < \varepsilon$ $\forall x \in X$.

Fix x_0, take $U = \bigcap_{k=1}^{n} f_k^{-1}(B(f_k(x_0), \varepsilon/3))$

then for $\forall x \in U.\ \forall f \in \mathcal{F}$, take k s.t $d_u(f_k, f) < \varepsilon_1$

$$d(f(x), f(x_0)) \le \underbrace{d(f(x), f_k(x))}_{\varepsilon/3} + \underbrace{d(f_k(x), f_k(x_0))}_{\varepsilon/3} + \underbrace{d(f_k(x_0), f(x_0))}_{\varepsilon/3}$$

totally bounded. i.e. finite, good enough, approximate f.

What does equicontinuous say?

understand $\mathcal{F}$ on finite points (compare with cpt)

prop. $\mathcal{F}$ equicontinuous $\Rightarrow (\mathcal{F}, \mathcal{T}_{p.c}) = (\mathcal{F}, \mathcal{T}_{c.c})$

$\mathcal{T}_{p.c} \subseteq \mathcal{T}_{c.c}$. we need to prove $\mathcal{T}_{c.c} \subseteq \mathcal{T}_{p.c}$

proof. ~~$\forall f \in \mathcal{F}$. want $\exists U$ s.t.~~

By def. $\forall \varepsilon>0$, $x_0 \in K$. $\exists x_0 \in U_0$. s.t $d(f(x), f(x_0)) < \varepsilon/3$

since K is cpt. finite $U_1, \cdots U_n$.

$d(f(x), f(x_i)) < \varepsilon/3$. $\forall x \in U_i$ $\forall f \in \mathcal{F}$

$U = \omega(f_0, x_1, \cdots x_n, \varepsilon)$ we can check $\forall f \in U \cap \mathcal{F}$, $\forall x \in K$. $d(f(x), f_0(x)) \le \varepsilon$.

36. prop. $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ equicont $\Rightarrow \overline{\mathcal{F}}^{p.c.}$ equicont.
$\mathcal{K}=$

[In particular, $\mathcal{K} \subseteq \mathcal{C}(X,Y)$]

proof. ($\forall x_0, \forall \varepsilon>0$, want U. $d(g(x),g(x_0))<\varepsilon$. $\forall x\in U$. $g\in \mathcal{K}$).

By def. $\forall \varepsilon>0$. $U \ni x_0$ s.t. $d(f(x), f(x_0))<\varepsilon$. $\forall f\in\mathcal{F}$ $\forall x\in X$,

(We ~~want~~ with the U works out).

~~fix g, x_0~~ $\boxed{\forall x \in U. \ \forall g\in K}$ what we fixed. take $f\in W(g, x, x_0, \varepsilon/3)\cap \mathcal{F}$

$d(g(x), \cancel{f(x)}\, g(x_0)) \le d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < \varepsilon$

general ~~case~~ version Arzelà–Ascoli Thm.

① if $\mathcal{F}$ equicontinuous and pointwise precompact

then $\overline{\mathcal{F}}^{c.c.} \subseteq (\mathcal{C}(X,Y), \mathcal{T}_{c.c.})$ is cpt.

$\boxed{A\subseteq X \text{ is precompact means } \bar{A} \text{ is cpt}}$

$\boxed{\text{pointwise precompact}: \forall a\in X.\ \mathcal{F}_a = \{f(a) \mid f\in \mathcal{F}\} \text{ is precompact}}$

② if X is LCH, then the converse is true.

Rmk 1 ① $\mathcal{T}_{c.c.}$ is not metric ~~topo~~ topology in ~~gene~~ general

then cpt $\not\Rightarrow$ Seq ~~seq~~ cpt

② if X is cpt, then $\mathcal{T}_{c.c.} = \underline{\mathcal{T}_{u.c}}$ is metric topology.

$\Rightarrow$ cpt = seq cpt.

③ $Y=(\mathbb{R}^n, \mathcal{T}_E)$ pointwise ~~seq~~ precpt = pointwise bounded.

Rmk. For noncompact X

Thm [AA for LC + σ-compact]

$(X=\bigcup_{i=1}^{\infty} K_i, K_i \text{ cpt})$

if X is LC+σ-cpt, (Y,d)

(f_n) equicont + pointwise pre cpt

then (f_n) has convergent subseq $\to f$ uniformly on Any cpt set

① Denote $R = \overline{\mathcal{F}}^{p.c.}$ $\underbrace{K_a = \overline{\mathcal{F}_a} \subseteq Y}_{\text{by condition } K_a \text{ closed \& cpt.}}$

$\xRightarrow{\text{Tychonoff}} \prod_a K_a$ cpt ~~cp cpt~~

$$\prod_a K_a = \bigcap_{a\in X} \underbrace{\pi_a^{-1}(K_a)}_{\text{closed}} \subseteq (\mathcal{U}(X, Y), \mathcal{T}_{p.c.})$$

By def. $\mathcal{F} \subseteq \underbrace{\prod_{a\in X} K_a}_{\text{closed}} \Rightarrow R$ is cpt in $\mathcal{T}_{p.c.}$

But R is equicontinuous. $\Rightarrow (R, \mathcal{T}_{p.c}) = (R, \mathcal{T}_{c.c})$

$\Rightarrow \overline{\mathcal{F}}^{c.c} = R$ is cpt in $\mathcal{T}_{c.c.}$

---

②. "Let $R = \overline{\mathcal{F}}^{c.c.}$ is cpt $\underset{\text{ought to prove.}}{\Rightarrow} R$ equicont + pointwise"

(i) K_a cpt. since it's the ~~its map~~ image of K

$$\mathcal{C}(X, Y) \xhookrightarrow{j_a} X \times \mathcal{C}(X, Y) \xrightarrow[\text{LCH}]{ev} Y$$
$$f \longmapsto (a, f) \longmapsto f(a)$$

(ii) $\forall x$ ~~that~~ take A cpt $x \in A$

$r_A : \mathcal{C}(X, Y) \to \mathcal{C}(A, Y)$ cts.

$R \rightsquigarrow r_A(R)$ cpt in $(\mathcal{C}(A, Y), \mathcal{T}_{c.c})$ $\overset{\text{cpt}}{=}$ $\mathcal{T}_{u.c.}$

$\Downarrow$

totally bounded

$\Downarrow$

equicont

Lecture 13. Stone - Weierstrass Theorem

Basic Setting X is CH. (Y, d) is $\mathbb{R}/\mathbb{C}$.

$\mathcal{C}(X, \mathbb{R})$, $\mathcal{T}_u = \mathcal{T}_{d_u}$ $\quad d_u \simeq d_\infty$ topologically equivalent.

$\quad\quad\quad\quad\quad\quad \| \quad\quad$ since X is Cpt.

$\quad\quad\quad\quad\quad\quad \mathcal{T}_{d_\infty}$

Classic Version Any $f \in \mathcal{C}([0,1], \mathbb{R})$. can be approximated $\underset{\text{unif}}{}$ by a collection of polynomials.

$P_n \rightrightarrows f$.

Rmk. Can take $a_0 = f(0)$. Make the ~~zeroth~~ term be const.

$P_n(0) \to f(0)$ use. $\tilde{P}_n = P_n(x) - P_n(0) + f(0) \rightrightarrows f$

Observation: $Poly([0,1], \mathbb{R})$ is "closed" under "+" & "·" $\leftarrow$ algebraically

We have used Berstein Polynomial to prove the theorem above.

~~pf~~ Def. (1) "Algebra" $(\mathcal{A}, +, \cdot)$ s.t. ① $(\mathcal{A}, +)$ is a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ ...

② $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfy

- distribution law (to +)
- compatibility $(\lambda a) \cdot (\mu b) = (\lambda\mu) a \cdot b$

Warning. not mentioned associated law. $\Leftarrow$

let alone commutive law

(2) Sub algebra. sub-vector space with "·" closed.

(3) unitary $\exists\, 1 \in \mathcal{A}$. s.t. $1 \cdot a = a \cdot 1 = a$.

(4) topological algebra. $(\mathcal{A}, +, \cdot, \mathcal{T})$ s.t

① $(\mathcal{A}, +, \mathcal{T})$ is a topological vector space

② $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is cts.

(5) closed sub algebra "closed" in topology

$\mathcal{C}(X, \mathbb{R})$ is a topological algebra.

e.g. Weierstrass: $\overline{P([0,1], \mathbb{R})} = \mathcal{C}([0,1], \mathbb{R})$.

prop. $\mathcal{A}$ topological algebra. $\mathcal{A}_1$ is a subalgebra $\Rightarrow$ $\overline{\mathcal{A}_1}$ is closed sub algebra

prop. X is cpt. $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$. closed subalgebra.

(1) $f \in \mathcal{A}$. $|f| \in \mathcal{A}$

(2) $f_1, \dots f_n \in \mathcal{A}$. $\min\{f_1, \dots f_n\}$, $\max\{f_1, \dots f_n\} \in \mathcal{A}$.

proof. (1) By weierstrass. $\exists P_n$ $\underline{P_n(0) = 0}$ s.t. $P_n \rightrightarrows f(x) = \sqrt{x}$. on $[0, \|f\|_\infty^2]$

$P_n(f^2) \in \mathcal{A}$

We don't know if we have const term. $\boxed{\mathcal{A} \text{ has unit?}}$

and $p_n(f^2) \rightrightarrows \sqrt{f^2} = |f| \in \mathcal{A}$

(2) prove by induction #

~~Ce~~ Certain order ~~set~~ structure ? "Lattice".

example (1) $\mathcal{A} = \{ \sum_{k=1}^{n} a_k x^k \} \subseteq \mathcal{C}([0,1], \mathbb{R})$. Not dense.

(2) $\mathcal{A} = \{ \sum_{k=1}^{n} a_k \sin kx + b_k \cos kx \}$

Def. $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$, subalgebra.

(1) $\mathcal{A}$ is <u>nowhere vanishing</u> if $\forall x \in X$, $\exists f \in \mathcal{A}$. $f(x) \neq 0$.

(2) $\mathcal{A}$ is <u>point separating</u> if $\forall x \neq y$, $\exists f$ s.t. $f(x) \neq f(y)$.

Rmk. $\mathcal{C}(X, \mathbb{R})$ point separating $\Rightarrow$ X Hausdorff

$\Rightarrow$ use $f(x) < f(z) < f(y)$ construct open sets.

$\boxed{\text{"CH} \Rightarrow \text{point separating" ?}}$

Prop X is cpt. $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ nowhere vanishing $\Leftrightarrow 1 \in \bar{\mathcal{A}}$

($\Leftarrow$) $(1-\varepsilon, 1+\varepsilon) \ni f \rightrightarrows 1$.

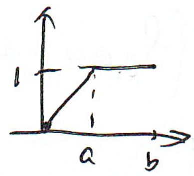
($\Rightarrow$). $\forall x \in X$. $\exists f_x \in \mathcal{A}$ s.t. $f_x(x) \neq 0$.

$x \in U_x = \{ y \mid f_x(y) \neq 0 \}$ open

X is cpt. $X = \bigcup_{i=1}^{m} U_{x_i} \rightsquigarrow f_1(x) = f_{x_1}^2 + \cdots + f_{x_m}^2 \in \mathcal{A}$.

$f_1(x) > 0$. everywhere g

$\Rightarrow 0 < a \leq f_1(x) \leq b$. on X.

use Weierstrass theorem. $p_n \rightrightarrows g$. $g = $ 

$p_n(0) = 0$.

$p_n \circ f_1 \in \mathcal{A}$. $\subseteq (1-\varepsilon, 1+\varepsilon)$.

$\searrow g \circ f \equiv 1$. #

fo Stone-Weierstrass, X CH. $\mathcal{A}$ subalgebra satisfy

nowhere vanishing & ~~s~~ point separating

then $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$.

~~proof~~.

Version 2 X CH. $\mathcal{A}$ unit closed subalgebra and point separating. then $\mathcal{A} = \mathcal{C}$.

$2 \Rightarrow 1$ $\overline{\mathcal{A}}$ is closed sub. nowhere $1 \in \overline{\mathcal{A}} \Rightarrow \overline{\mathcal{A}} = \mathcal{C}$.

$1 \Rightarrow 2$. $\mathcal{A}$ has $1 \Rightarrow$ nowhere ..

Want.

proof of (V2). $\forall f \in \mathcal{C}(X, \mathbb{R})$, $\boxed{\forall \varepsilon > 0.\ \exists f_\varepsilon \in \mathcal{A}\ \text{s.t.}\ d_\infty(f_\varepsilon, f) < \varepsilon}$

$\forall a \neq b \in X$. $\exists g \in \mathcal{A}$. $g(a) \neq g(b)$

$$f_{a,b,\varepsilon} = f(a) + \frac{g(x) - g(a)}{g(b) - g(a)} (f(b) - f(a)) \in \mathcal{A}.$$

$\boxed{\begin{array}{l} f_{a,b,\varepsilon}(a) = f(a) \\ f_{a,b,\varepsilon}(b) = f(b) \end{array}}$ Consider $U_{a,b,\varepsilon} = \{ f_{a,b,\varepsilon} < f + \varepsilon \} \ni a, b$. open.

fix b. change a.

$\Rightarrow \{ U_{a,b,\varepsilon} \}$ is an open covering.

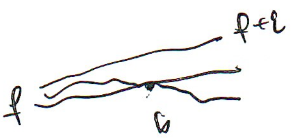
$\Rightarrow \bigcup_{i=1}^{n} U_{a_i(b,\varepsilon), b, \varepsilon} \supseteq X$.

$\rightarrow f_b^\varepsilon := \min \{ f_{a_1(b,\varepsilon), b, \varepsilon}, \cdots f_{a_n(b,\varepsilon), b, \varepsilon} \} \in \mathcal{A}$

$\Rightarrow f_b^\varepsilon < f + \varepsilon$. & $f_b^\varepsilon(b) = f(b)$

change b. $V_{b,\varepsilon} = \{ f_b^\varepsilon > f - \varepsilon \}$.

$f_\varepsilon = \max \{ f_{b_i}^\varepsilon \} \in \mathcal{A}$.



$\Rightarrow f + \varepsilon > f_\varepsilon > f - \varepsilon$

#

V3. X CH. $\mathcal{A}$ subalgebra point separating

if $\overline{\mathcal{A}} \notin \mathcal{C}(X, \mathbb{R})$. then $\exists!$ x_0 s.t. $\overline{\mathcal{A}} = \{f \mid f(x_0) = 0\}$

proof. x_0 $\exists!$ $\Rightarrow \overline{\mathcal{A}} \subseteq \{f \mid f(x_0) = 0\} = \mathcal{A}_1$

suppose $\mathcal{A}_2 = \langle \mathcal{A}_1, 1 \rangle$ $\overline{\mathcal{A}} = \mathcal{C}$.

$f_n \in \mathcal{A}_2$ $f_n \rightrightarrows f$. $f_n(x_0) \rightrightarrows 0$.

$\tilde{f}_n(x) = f_n(x) - f_n(x_0)$ so $\tilde{f}_n \rightrightarrows f$

Rmk. ① LCH version

② $\mathcal{C}(X, \mathbb{C})$. the third condition $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$

$$\begin{tikzcd} X \arrow[r, squiggly] & \mathcal{C}(X, \mathbb{R}) \\ & \mathcal{A} \arrow[u, phantom, "\cup" rotate=90] \arrow[ul, bend left] \end{tikzcd}$$

Lec 14. Countability & separability

Recall A1: $(X, \mathcal{T})$ is A1 if $\forall x \in X$ has a countable neigh basis

$\Downarrow$ "F is closed $\Leftrightarrow$ F contains all its seq limit pts"

(A1) $\Rightarrow$

We can choose $\Downarrow$ neigh basis s.t. $U_x^1 \supseteq U_x^2 \supseteq \cdots$

(A1) $\Rightarrow$ "f cts $\Leftrightarrow$ f seq cts".

Def (A2) | $(X, \mathcal{T})$ is called <u>second countable (A2)</u> if it admits a countable basis.

i.e. $\exists$ $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ topological basis.

eg (X, d) is A1 : $\mathcal{B}_x = \{B(x, \frac{1}{n}) \mid \frac{1}{n} \in \mathbb{N}\}$.

Question: A topological space can be metrization?

prop || (X, d) is totally bounded $\Rightarrow (X, d)$ is A2.

proof. By definition, $\exists$ $x_i^{(n)}$ $1 \le i \le k(n)$ s.t $\bigcup_{i=1}^{k(n)} B(x_i^{(n)}, \frac{1}{n}) = X$.

claim $\mathcal{B} = \{B(x_i^{(n)}, \frac{1}{n}) \mid n = 1, \cdots\}$ is a countable basis.

reason: $U \subseteq X$ $\forall x \in U$. $\exists$ $B(x, \varepsilon) \subseteq U$. take $\frac{1}{n} < \varepsilon/2$.

$x \in B(x_i^{(n)}, \frac{1}{n}) \Rightarrow B(x_i^{(n)}, \frac{1}{n}) \subseteq U$. #

42.
Cor ~~推~~ cpt (X,d) is A_2.

Example ~~推~~ ① $\left([0,1]^{\mathbb{N}}, \mathcal{T}_{prod}\right)$ is $\underline{\underline{cpt}}$ $(X, \underline{\underline{d}})$
Tychonoff $d = \sum_{i=1}^{\infty} \frac{1}{2^n} \frac{d(x_i, y_i)}{1+d(x_i,y_i)}$.

② $\mathbb{R}^n$, $\mathcal{B} = \{ B(q,r) \mid q \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0} \}$.
$\mathbb{Q}^n$ is dense in $\mathbb{R}^n$

Def (separable) | $(X, \mathcal{T})$ is separable if $\exists$ a countable dense subset

prop "$(A_2) \Longrightarrow$ separable"

proof. $\mathcal{B}$ is countable basis. $\leadsto A = \{ x_n \mid n \in \mathbb{N} \}$.
$= \{U_n\}$

claim $\overline{A}$ is X. ; reason $\forall x \in X$. , $\forall x \in \underline{\underline{U}}$. there is $U_{n_0} \subseteq U$
$\Longrightarrow A \cap U \ni x_{n_0} \Longrightarrow x \in \overline{A}$

Counterexample. $(\mathbb{R}, \mathcal{T}_{Sorgenfrey})$.
$\mathcal{B} = \{ [a,b) \}$.

it's separable. Since $\overline{\mathbb{Q}} = \mathbb{R}$.
it's A1. $\mathcal{B}_x = \{ [x, x+\frac{1}{n}) \}$.
But it's not A2. "for Any $\mathcal{B}$. it's not countable".
$\forall x \in \mathbb{R}$. $x \in B_x \in \mathcal{B}$. $B_x \subseteq [x, x+1)$.
$\inf B_x = x$.
$\mathcal{B} \longrightarrow \mathbb{R}$ $\boxed{sur}$ #

prop (X,d). $(A_2) \Longleftrightarrow$ Separable.

proof $\Longrightarrow$ ✓
$\Longleftarrow$ $A = \{x_n\}$ $\overline{A} = X$.
$\mathcal{B} = \{ B(x_n, \frac{1}{m}) \mid n, m \in \mathbb{N} \}$. check it is a basis.

Cor. $\mathcal{T}_{Sorgenfrey}$ is not a metric topology

Thm. Any compact metric space is homeomorphic to a closed subset in $([0,1]^{\mathbb{N}}, \mathcal{T}_{pro})$. (Hilbert Cube)

proof WLOG, assume $\mathrm{diam}(X) \leq 1$

let $\{x_n\}$ be dense subset.

Consider $F: X \longrightarrow [0,1]^{\mathbb{N}}$

$$x_0 \longmapsto \left(d(x, x_1), d(x, x_2), \cdots\right).$$

claim. $F: X \to F(X)$ is a homeomorphism.

① F is cts. Universality + d metric is cts.

X cpt $\Rightarrow$ $F(X)$ is cpt in Hausdorff $\Rightarrow$ $F(X)$ closed.

② F is injective. If ~~F~~ $F(x) = F(y) \Rightarrow d(x, x_i) = d(y, x_i)$ $i \in \mathbb{N}$.

take $x_{n_k} \to x$. $d(x,y) = \lim\limits_{k \to +\infty} d(x_{n_k}, y) = \lim\limits_{k \to +\infty} d(x_{n_k}, x) = 0$.

So $\underset{\text{cpt}}{F: X} \to \underset{\text{Hausdorff}}{F(X)}$. $\Rightarrow$ F is homeo

---

~~(T2)~~. T_1 $\underset{\neq}{\subset}$ T_2 Hausdorff T_3 regular T_4 normal

$\mathbb{R}, \mathcal{T}_{cofinite}$

$(\mathbb{R}, {}_{u.s.}\mathcal{T}_c)$ T_4 ✓

$(a, +\infty)$ T_2, T_3 ✗

rmk. In different literature regular, normal, T_3, T_4 may have different meanings.

prop. (T1) $\Longleftrightarrow$ Any $\{x\}$ is closed

(T2) $\Longleftrightarrow$ $\Delta = \{(x,x) \mid x\in X\}$ is closed in $X\times X$.

(T3) $\Longleftrightarrow$ $\forall x \in X$, $x\in U$, $\exists V$. $x \in V \subseteq \bar{V} \subseteq U$

(T4) $\Longleftrightarrow$ $\forall A \subseteq X$, A is closed $\exists V$ s.t. $A \subseteq V \subseteq \bar{V} \subseteq U$.

Thm ~~prop~~ CH $\Rightarrow$ T4.

| Compactness argument

prop. 1 ~~Com~~ CH $\Rightarrow$ T3 , prop. C+T3 $\Rightarrow$ T4.

in LCH

$x \in \cancel{\ } \underbrace{V}_{\text{open}} \subseteq \underbrace{\bar{V}}_{\text{cpt}} \subseteq U$

proof. $A\cap B = \phi$

T3, $\forall x\in A. \Rightarrow \exists U_x, V_x$. $U_x \cap V_x = \phi$

$x\in U_x$ $B \subseteq V_x$

A cpt $A \subseteq \bigcup_{i=1}^{n} U_{x_i} \Rightarrow B \subseteq \bigcap_{i=1}^{n} V_{x_i}$

$\Rightarrow$ T4 #

---

prop. A_2 + T3 $\Rightarrow$ T4

rmk. A_2 + T2 $\not\Rightarrow$ T3

LC + T3 $\not\Rightarrow$ T4.

proof $\mathcal{B} = \{U_n\}$ basis.

A, B closed.

$\forall x\in A$. $x\in B^c$. $\overset{T3}{\Rightarrow}$ $x\in V \subseteq \bar{V} \subseteq B^c$

$\{V_x \mid x\in A\}$ is a open covering

$\Downarrow A_2$

it admits countable subcovering

$V_{x_1}, \cdots V_{x_n}, \cdots$

$V_i \subseteq \bar{V_i} \subseteq B^c$

In fact we use lindelöf property.

similarly $U_i \subseteq \bar{U_i} \subseteq A^c$

take $U = \bigcup_{n=1}^{\infty} \left(U_n \setminus \bigcup_{i=1}^{n} \bar{V_i}\right)$

$V = \bigcup_{m=1}^{\infty} \left(V_m \setminus \bigcup_{i=1}^{m} \bar{U_i}\right)$ #

Lecture 16 Urysohn theorem.

Last time: countability — A1, A2 (basis), separable (set), lindelöf (open covering)

Separability — T_2 point & point, T_3 point & closed set, T_4 closed set & closed set

T_1 single point closed.

$T_2 \xrightarrow{+\text{cpt}} T_3 \xleftarrow{+\text{cpt}} T_4$ $\qquad T_2 \overset{T_1}{<} T_3 \overset{T_1}{<} T_4$.

recall (X, d) $\underline{(A1)}$ $\quad (T_2)$
$(A2)$ (need totally bounded). $\quad (T_4)$ $\sim \dfrac{d_A(x)}{d_A(x)+d_B(x)}$ is cts.
$\Updownarrow$ separable

$(\mathbb{R}, \mathcal{T}_{\text{sorgenfrey}})$ $(A1)$ (T_2) (T_4) $\underline{\text{separable}}$ but not $A2$.

so that $\boxed{\mathcal{T}_{\text{sor}} \neq \mathcal{T}_d}$.

Def. We say $(X, \mathcal{T})$ is <u>metrizable</u> if $\exists$ metric d on X s.t. $\mathcal{T} = \mathcal{T}_d$.

Thm. (Urysohn metrization thm) Let $(X, \mathcal{T})$ be A_2 space.
Then it's metrizable $\Leftrightarrow$ it's T_2 & T_4.

Rmk. Can't replace (A_2) by separable. CE. $\mathcal{T}_{\text{sorgenfrey}}$.

Cor. CH space is metrizable $\Leftrightarrow$ it's (A_2)

($\Leftarrow$) Urysohn Thm
($\Rightarrow$) cpt + $\mathcal{T}_d \Rightarrow A_2$. (cpt in $\mathcal{T}_d$ = TB + AC). #

Idea. cpt metric space $\hookrightarrow [0,1]^{\mathbb{N}}$ topological embedding

$F: X \to F(X) \subseteq [0,1]^{\mathbb{N}}$
$\quad \| $
$(d_1(x,x_1), \cdots)$.

Want to find a topological embedding

$F: X \xrightarrow{\text{Homeo}} F(X) \subseteq [0,1]^{\mathbb{N}}$ sub metric topo is metric Td's sub topo.
$\quad \varphi \ \| ?$
$(f_1, \cdots f_n, \cdots)$

How to construct func to separate sth.

Urysohn Lemma. $(X, \mathcal{T})$ is T4 if and only if $\forall$ disjoint closed sets $A, B \subseteq X$
$\exists f \in \mathcal{C}(X, [0,1])$. s.t. $f(A) = 0$. $f(B) = 0$

proof of Urysohn theorem

suppose $(X, \mathcal{T})$ is (A2), (T2), (T4)
let $\mathcal{B} = \{B_n\}$ be countable basis.

Step 1. suppose $x \in U$. then $\exists$ m, n. s.t. $x \in B_n \subseteq \overline{B_n} \subseteq B_m \subseteq U$

Reason, $x \in U. \Rightarrow x \in B_m \subseteq U.$
$T2 + T4 \Rightarrow T3$
$T2 \Rightarrow T1$
$\Rightarrow x \in V \subseteq \overline{V} \subseteq B_m$
$\Rightarrow x \in B_n \subseteq \overline{B_n} \subseteq \overline{V} \subseteq B_m \subseteq U$.

Step 2. ~~$\forall x \in U$~~. $\exists f_1, \cdots f_n, \cdots \in C(X, [0,1])$. s.t.

$\forall x \in U$, $\exists$ n. s.t $f_n(x) = 1$. $f(U^c) = 0$

Reason let $J = \{(m,n) \mid \overline{B_n} \subseteq B_m\} \neq \emptyset$ (step 1). J is countable.

For any $(m,n) \in J$. $\exists g_{m,n}: X \to [0,1]$ s.t $g_{m,n}(\overline{B_n}) = 1$
Urysohn lemma. $g_{m,n}(B_m^c) = 0$.

Step. 3. $F: X \to F(X) \subseteq [0,1]^{\mathbb{N}}$
$x \mapsto (f_1(x), \cdots f_n(x), \cdots)$ is a topological embedding.

reason. ① $F: X \to [0,1]^{\mathbb{N}}$ is cts since f_i is cts. $\forall i \in \mathbb{N}$.

② F is injective

$\forall x \neq y. \Rightarrow x \in \underbrace{\{y\}^c}_{\text{open}}$ $\exists n.$ $f_n(x)=1$. $f(y)=0$.

$\Rightarrow F(x) \neq F(y)$.

$F: X \to F(X) \subseteq [0,1]^{\mathbb{N}}$ is bijective $\underset{\text{universality}}{\text{cts}}$

③ $F: X \to F(X)$ is open map.

Let $U \subseteq X$. be open $F(U) \subseteq F(X)$ is open ?

take $z_0 = F(x_0) \in F(U) \Rightarrow \exists n$ s.t. $f_n(x_0)=1$. $f_n(U^c)=0$

Let $W = \pi_n^{-1}((0,+\infty))$ where $\pi_n: [0,1]^{\mathbb{N}} \to [0,1]$.

Claim. $z_0 \in W \cap F(X) \subseteq F(U)$.

$z_0 = F(x_0)$

$\pi_n(z) = 1 \in W$

$\pi_n(F(x_0)) = f_n(x_0)$

$W \subseteq F(U)$

$\forall z \in W.$ $\exists x$ s.t. $F(x)=z$

$f_n(x) > 0$

$x \notin U^c$

$x \in U$.

$z \in F(U)$.

Step 4. Metrization.

By step 2.3. $F(X, \mathcal{T}) \to (F(X). \underline{\mathcal{T}_{prod}})$ homom

$\mathcal{T}_{d_0}$

Let $d(x,y) = d_0(F(x), F(y))$

$F: (X, d) \to (F(X), d_0)$ isometry

$\Rightarrow (X, \mathcal{T}_d) \xrightarrow[\text{homom}]{F} (F(X), \mathcal{T}_{prod})$

rmk. Step 2 + Step 3 $\Rightarrow$ Topological embedding

$$x \hookrightarrow [0,1]^{\mathbb{N}}$$

need $x \in U. \longrightarrow \exists f_n \quad f_n(x) = 1 \quad f_n(U^c) = 0$

repeat above process, we can prove:

Prop. Suppose X is (T1) and satisfies: completely regular (T3.5) $\Rightarrow$ T3

$\boxed{\forall x \in U.\ \exists f \in \mathcal{C}(X, [0,1]).\ \text{s.t } f(x) = 1.\ f(U^c) = 0.}$

then one can have a topogical embedding $X \hookrightarrow [0,1]^{\mathcal{C}(X,[0,1])}$.

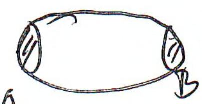
$$x \mapsto \left(ev_x . \mathcal{C}(X,[0,1]) \to [0,1]\right)$$

the proof of Urysohn lemma

Idea. use level set, dense ~

|| T4 $\Leftarrow$. trivial

|| T4 $\Rightarrow$ (X, T) is T4. $\underset{A_0}{\underset{\parallel}{A}} \subseteq \underset{U_1}{\underset{\parallel}{U}} = B^c$



A B

By T4, $\exists \underset{\text{closed}}{A_{\frac{1}{2}}}.\ \underset{\text{open}}{U_{\frac{1}{2}}} \Rightarrow A_0 \subseteq U_{\frac{1}{2}} \subseteq A_{\frac{1}{2}} \subseteq U_1$

$$\Downarrow$$

$$A_0 \subseteq U_{\frac{1}{4}} \subseteq A_{\frac{1}{4}} \subseteq U_{\frac{1}{2}} \subseteq A_{\frac{1}{2}} \subseteq U_{\frac{3}{4}} \subseteq A_{\frac{3}{4}} \subseteq U_1$$

$$r \in \left\{ \frac{m}{2^n} \,\middle|\, n \in \mathbb{N}.\ 1 \le m \le 2^n \right\}$$

① $U_r \subseteq A_r$

② $A_r \subseteq U_{r'}, \forall r' > r$

Let. $f(x) = \inf \{ r \mid x \in A_r \} = \inf \{ r \mid x \in U_r \}$

$\inf \emptyset = 1$

To prove f is cts. Note $\left\{ [0, \alpha).\ (\alpha, 1] \,\middle|\, \alpha = \frac{m}{2^n} \right\}$ is a subbase.

But $f^{-1}([0,\alpha)) = \bigcup_{r<\alpha} U_r \quad f^{-1}((\alpha, 1]) = \bigcup_{r>\alpha} A_r^c$

NOTE. $f(A)=0$. $f(B)=1$ $\not\Leftrightarrow$ $\underline{f^{-1}(0)=A.\ f^{-1}(1)=B}$

Q. Which sets can be realized as $f^{-1}(0)$ When

- A is closed.
- $\{0\}=\bigcap_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n}) \Rightarrow f^{-1}(0)=\bigcap_{n=1}^{\infty} f^{-1}(-\frac{1}{n},\frac{1}{n}) = G_\delta$.

Example $X=\{0,1\}^{\mathbb{R}}$. T_{prod}. Hausdorff. cpt.

CH space. $\Rightarrow \{x\}$ is closed.

$G=\bigcap_{n=1}^{\infty} U_n$ is uncountable.

$\pi_r^{-1}(U_n)=\{0,1\}$ for all but finite many r.

$\pi_r^{-1}(G) = \cdots$ countable.

prop. (X,T).(T4). then. $A=f^{-1}(0) \Leftrightarrow A$ is closed G_δ. set.

($\Rightarrow$)✓ ($\Leftarrow$) $A=\bigcap_n U_n$ $A\subseteq U_n$. Urysohn $f_n\ X\to[0,1]$

$f_n(A)=0$. $f(U_n^c)=1$.

$f(x)=\sum\frac{1}{2^n}f_n(x)$

$f(A)=0$. $\Rightarrow f^{-1}(0)=A$.

Cor. $(X,T)(T4)$. A.B disjoint $G_\delta \Rightarrow \exists f\in\mathcal{C}(X,[0,1])$
$f^{-1}(0)=A$. $f^{-1}(1)=B$.

Cor. (LCH)

$\Downarrow$

$K \subseteq U$

$K \subseteq V \subseteq \bar{V} \subseteq U$. $\boxed{\bar{V} \text{ is } T4.}$

$\uparrow$ cpt.

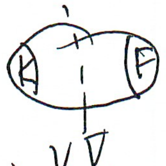
$\exists f \in \mathcal{C}(X, [0,1])$. $f(K) = 1$. $f(U^c) = 0$

$\overline{\{f \neq 0\}}$ is cpt.

Last time

Urysohn lemma : (T4) $\iff$ $A \cap B = \phi$. A, B is closed. $\exists f \in \mathcal{C}(X, [0,1])$.
s.t $f(A) = 0$, $f(B) = 1$

- Not true for T3
- closed G_δ set.
- LCH : Cpt set & closed set



cpt Hausdorff

$\Downarrow$ cpt

Metric space $\hookrightarrow$ Hilbert space.

separable.

Urysohn metrization thm

A2. " metrizable $\iff$ $T_2 + T_4$."

Lecture 17. tietze extension theorem $(X, \mathcal{T})$ is T4. $A \subseteq X$ is closed then Any $f \in \mathcal{C}(A, [-1,1])$ can be ~~embedded~~ extended to $\tilde{f}(X, [-1,1])$.

Def// for $S \subseteq X$, $f: S \to Y$ is a map . if $\tilde{f}: X \to Y$, satisfies $\tilde{f}(s) = f(s)$ $\forall s \in S$, then we call $\tilde{f}$ as extension of f.

e.g. Zero extension: if $f: S \to \mathbb{R}$. $\tilde{f}(x) = \begin{cases} f(x) & S \\ 0 & S^c \end{cases}$.

<u>We always focus on cts extension</u> $f \in \mathcal{C}(S, Y) \rightsquigarrow \tilde{f} \in \mathcal{C}(X, Y)$

existence & uniqueness ?

$A \subseteq \mathbb{R}$. not closed $\Rightarrow$ $a \in A' \setminus A$. let $f(x) = \sin\frac{1}{x - a}$ $x \in A$

Rmk. In general, existence is Not unique

prop// let Y be T_2. $S \subseteq X$ be dense . then any $f \in \mathcal{C}(S, Y)$ admits at most one cts extension. $\tilde{f} \in \mathcal{C}(X, Y)$.

Rmk. Can't replace $[0,1]$ by more general Y.

$S = \{0, 1\} \subseteq [0,1] = X$. $(Y, \mathcal{T})$

Any f is cts , but $f \rightsquigarrow \tilde{f} \Rightarrow f(0), f(1)$ lies in the same path

($\because$ cts map preserve connectness).

eg. $Id: S^1 \to S^1 = Y$, $X = D$ } not exist brouwer fixed point tietze thm $\Longrightarrow$ Urysohn lemma

$A_1 \cap B_1 = \phi$

$A = A_1 \cup B_1$ is closed

$f(A_1) = 1$. $f(B_1) = 0$

paste

$f : A \to [0,1]$ is cts

$\Downarrow$

$\tilde{f} : X \to [0,1]$. is cts.

proof of tietze extension

($T4 \Longleftrightarrow f \to \tilde{f}$, tietze $\Rightarrow$ urysohn lemma $\Rightarrow T4$)

Idea. $r_A : \mathcal{C}(X, [0,1]) \to \mathcal{C}(~~X~~A, [0,1])$

$f \longmapsto r_A(f)$

We want to solve $\boxed{r_A(\tilde{f}) = f}$.

① approximate solution $\quad r_A(g) \simeq f \quad |f - r_A(g)| \leq 1$

$\downarrow$

② iteration $\quad r_A(\tilde{g}) \approx f - r_A(g)$

$\downarrow$

③ prove $\quad \cdots$

① consider $\bar{f} : A \to [-1, 1]$ $\quad A_1$ closed

$$\bar{f}(x) = \begin{cases} \frac{1}{3} & \boxed{f(x) \geq \frac{1}{3}} \\ f(x) & \\ -\frac{1}{3} & \boxed{f(x) \leq -\frac{1}{3}} \end{cases}$$

B_1 closed

then $\bar{f} \in \mathcal{C}(A, [-1,1])$

paste lemma

but we wouldn't use $\bar{f}$

$|\bar{f} - f| \leq \frac{2}{3}$. use urysohn. $\exists g : X \to [-\frac{1}{3}, \frac{1}{3}]$

$g(A_1) = \frac{1}{3}$. $g(B_1) = -\frac{1}{3}$

$|f - r_A(g)| \leq \frac{2}{3} \quad x \in A$

② Denote $f_1 = f$. $|f_1 - g_1| \leq \frac{2}{3}$.

$f_2 = f_1 - g_1 \qquad g_2 : X \to [\frac{2}{3} \cdot \frac{-1}{3}, \frac{2}{3} \cdot \frac{1}{3}]$

$|f_2 - g_2| \leq (\frac{2}{3})^2$

$\vdots$

$|f_n - g_n| \leq (\frac{2}{3})^n$

③ let $\tilde{f} := \sum_{n=1}^{\infty} g_n(x)$ cts on X : Weierstrass M-test.

$$|\tilde{f}| \le \frac{1}{3} \sum \left(\frac{2}{3}\right) = \frac{1}{3} \cdot \frac{1}{\frac{1}{3}} = 1$$

$$\left| f - \sum_{n=1}^{N} g_n(x) \right| = \left| f_N - g_N \right| \le \left(\frac{2}{3}\right)^N \to 0. \text{ on } A.$$

So $\tilde{f}$ is f's extension on X. □.

$\boxed{\text{Obviously. one can replace } [-1,1] \text{ by any } [a,b].}$

Variant of tietze: one can replace $[-1,1]$ by $\mathbb{R}$.

let $g = \arctan \circ f \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $f : A \to \mathbb{R}$

$\xRightarrow{\text{tietze}} \tilde{g} \in \mathcal{C}(X, [-\frac{\pi}{2}, \frac{\pi}{2}])$.

Note $B = \tilde{g}^{-1}(\pm\frac{\pi}{2})$ and $B \cap A = \phi$

$\xRightarrow{\text{Urysohn}} h \in \mathcal{C}(X, [0,1])$. $h(A) = 1$. $h(B) = 0$.

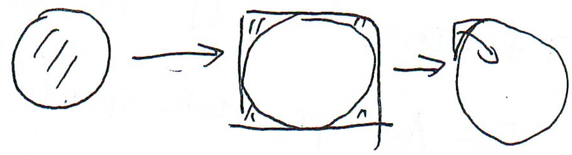
let $\tilde{f} = \tan(\tilde{g} \cdot h)$

Rmk. one can also ~~extended~~ $f : A \to [-1,1]^n$

or $A \to \cancel{[-1,1]} \prod_{i=1}^{\infty} [a_i, b_i]$ prod topo

"pullback the bad part"

- $\mathbb{C}$-value. preserve norm.
- lip. C^{∞}



LCH version. X is LCH. $K \subseteq X$ cpt. $f \in \mathcal{C}(K, [-1,1])$.

$$\rightsquigarrow \tilde{f} \in \mathcal{C}_c(X, [-1,1]).$$

cpt supp

Idea. $K \subseteq V \subseteq \bar{U} \subseteq X$ ($\bar{U}$ cpt.)

$\tilde{f} = 0$ on $\bar{U} \setminus V$

$$f_1 = \begin{cases} f(x) & K \\ 0 & \bar{U} \setminus V \end{cases} \quad f_1 \rightsquigarrow \tilde{f}_1 \xrightarrow{\text{zero}} \tilde{f}$$

□

Application

prop. $(X,d) \Rightarrow$ "$\underbrace{\text{Any } f \in \mathcal{C}(A,\mathbb{R}) \text{ is bounded} \Leftrightarrow A \text{ cpt}}_{\text{pseudo cpt}}$"

proof. ($\Rightarrow$) For (X,d) A cpt $\Leftrightarrow$ A limit point cpt

suppose A is not cpt $\Rightarrow$ A is not limit point cpt

$\exists$ Set $S = \{x_1, \cdots x_n\}$, $S' = \phi$ (S is closed)

let $f(x_n) = n$ $\Rightarrow f$ is cts on S

use variation of tietze. f is not bounded! $\square$

Cantor set

- $g: \{0,1\}^{\mathbb{N}} \xrightarrow{\sim} C$ homeo
- $h: \{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1]^2$ sur

$\rightsquigarrow h \circ g^{-1}: C \twoheadrightarrow [0,1]^2$ cts sur

$\exists f: [0,1] \to [0,1]^2$ cts sur

We can even construct

$h: \{0,1\}^{\mathbb{N}} \to \underbrace{[0,1]^{\mathbb{N}}}_{\text{Hilbert Cube}}$

prop. For Any cpt metric space (Y,d), $\exists$ cts sur $f: C \twoheadrightarrow Y$

Lecture 17 Partition of unity.

Def. We say $\{A_\alpha\}$ is <u>locally finite</u> if $\forall x \in X$. $\exists$ $x \in U_x$. s.t. $U_x \cap A_\alpha = \phi$ for only finitely many α.

Thm (Simple P.O.U.). X is T4. $\{F_\alpha\}$ is ~~a locally finite~~ closed covering. and $F_\alpha \subseteq U_\alpha$

$\{U_\alpha\}$ is locally finite, Then. $\exists \{\rho_\alpha\}: \rho_\alpha: X \to [0,1]$, satisfies

① $\rho_\alpha|F_\alpha > 0$. ② $\rho_\alpha(U_\alpha^c) = 0$ ($\operatorname{supp} \rho_\alpha \subseteq \overline{U_\alpha}$). ③ $\sum_\alpha \rho_\alpha(x) = 1$. $\forall x \in X$.

Rmk. By local finiteness and $\rho_\alpha(U_\alpha^c) = 0 \Rightarrow \sum_\alpha \rho_\alpha(x)$ is a finite sum $\Rightarrow \sum \rho_\alpha$ is cts (in U_α).

the reason why we use locally finite but pointwise finite

$\xRightarrow{\text{paste lemma}} \sum \rho_\alpha$ is cts on X.

more generally. we can "glue $\underset{\text{defined locally in } U_\alpha}{F_\alpha\text{'s}}$" with $\sum_\alpha \rho_\alpha F_\alpha$

proof. Use Urysohn. $\Rightarrow g_\alpha \in \mathcal{C}(X,[0,1])$. $g_\alpha = \begin{cases} 1 & F_\alpha \\ 0 & U_\alpha^c \end{cases}$

$\Rightarrow g(x) = \sum_\alpha g_\alpha(x)$ is a well-defined cts ~~conti~~ function. & $g(x) \geq 1$ since $\{F_\alpha\}$ is a covering

$\Rightarrow 1 = \sum_\alpha \frac{g_\alpha(x)}{g(x)} = \sum_\alpha \rho_\alpha(x)$.

Def (a) We call $\{\rho_\alpha\}$ is a partition of unity if.

locally finite in Analysis

① $\rho_\alpha \in \mathcal{C}(X,[0,1])$ ② $\{\text{supp}\,\rho_\alpha\}$ is locally finite ③ $\sum_\alpha \rho_\alpha(x) = 1$. $\forall x \in X$.

(b) We say P.O.U. $\{\rho_\alpha\}$ is a P.O.U. subordinate to an open covering $\{U_\alpha\} = \underline{\underline{\mathcal{U}}}$

if it also satisfies ④ $\forall\alpha$ $\text{supp}\,\rho_\alpha \subseteq U_\alpha$.

We Do NOT assume it's locally finite.

Suppose $\exists \{\rho_\alpha\}$ subordinate to $\mathcal{U} = \{U_\alpha\}$.

$\rightsquigarrow \underline{\{x \mid \rho_\alpha(x) > 0\}}$ is open covering a locally finite refinement of $\{U_\alpha\}$

Def. In general, We say $\mathcal{V} = \{V_\beta\}$ open covering, is a refinement of $\mathcal{U}$. if $\forall V_\beta$. $\exists U_\alpha$ s.t $V_\beta \subseteq U_\alpha$

Def. We say $(X,\mathcal{T})$ is a paracompact if any open covering $\{U_\alpha\}$ admits a locally finite open refinement

compact $\leftrightsquigarrow$ finite $\boxed{\text{Paracompact} \leftrightsquigarrow \text{locally finite.}}$ in topology

Example. ① $\mathbb{R}^n$ $\mathcal{U} = \{B(0,n) \mid n \in \mathbb{N}\}$ NOT locally finite.

$\mathcal{V} = \{V_k \mid V_k = B(0,k) \setminus \overline{B(0,k-1)}, k = 1\}$ NOT open covering BUT We can adjust easily

② In general. $\mathcal{U}_1$ $\{B(x,r_x)\}$ is a refinement

$\overline{B(0,k+1)} \setminus B(0,k)$ can be covered finitely in $\mathcal{U}_1$ $\rightsquigarrow \boxed{\tilde{\mathcal{U}}}$ locally finite.

Rmk. Any metric space is paracompact.

Thm. if $(X,\mathcal{T})$ is paracompact & (T_2). then for Any $\mathcal{U} = \{U_\alpha\}$ $\exists$ P.O.U. $\{\rho_\alpha\}$ ~~subordinate~~ Subordinate to $\mathcal{U}$.

We need prop A. Paracompact $+ T_2 \Rightarrow T_3, T_4$

prop B ~~open~~ "for Any open covering $\mathcal{U}$. $\exists$ locally finite refinement $\mathcal{V} = \{V_\alpha\}$ s.t $\overline{V_\alpha} \subseteq U_\alpha$.

paracompact $+ T_2 \Rightarrow$

We need ~~locally~~ finiteness

proof: $\overline{Z_\alpha} \subseteq W_\alpha \subseteq \overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. (lf)

Apply simple P.O.U. $\{\overline{Z_\alpha}\}$. $\{W_\alpha\}$

$\rightsquigarrow \rho_\alpha \in \mathcal{C}(X,[0,1])$ $\text{supp}\,\rho_\alpha \subseteq \overline{W_\alpha} \subseteq \underset{\text{locally finite}}{\underbrace{U_\alpha}} \subseteq U_\alpha$

Rmk. LCH version. $(X,\mathcal{T})$ LCH $+ \sigma$-cpt

Then for Any $\mathcal{U}$. $\exists\{\rho_n\}$ P.O.U. s.t ① $\forall n$. $\exists\alpha$ $\text{supp}\,\rho_n \subseteq U_\alpha$

② $\rho_n \in \mathcal{C}_c(X,[0,1])$.

prop. $(X,\mathcal{T})$ paracompact. $A \subseteq X$. A is closed $\Rightarrow$ A paracpt.

$\mathcal{U}$ of $A \to \mathcal{U} \cup \{A^c\} \longrightarrow \mathcal{U}_1 \cup \{A^c\} \longrightarrow \tilde{\mathcal{U}}$ $U \in \tilde{\mathcal{U}}$ $U \cap A \neq \phi$. #

prop. LCH + A_2 $\Rightarrow$ para cpt.

proof. $\forall \mathcal{U}$. $\forall x$, $\exists U_{\alpha_x} \ni x \xRightarrow{LCH.} x \in V_x \subseteq \overline{V_x} \subseteq W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$

$\mathcal{V} = \{V_x\}$ is a refinement of $\mathcal{U}$. $\xRightarrow{A_2 \text{ (or lindelöf)}}$ $\tilde{\mathcal{V}} = \{V_1, \cdots V_n, \cdots\}$

$\Rightarrow \tilde{\mathcal{W}} = \{W_1, \cdots W_n, \cdots\}$

let $R_1 = W_1$, $R_2 = W_2 \setminus \overline{V_1}$, $\cdots R_n = W_n \setminus (\overline{V_1} \cup \cdots \cup \overline{V_n})$

$\mathcal{R}$: ① $R_n \subseteq W_n \subseteq U_\alpha$

② $\forall x$, $\exists$ smallest n s.t. $x \in W_n$. $x \notin \overline{V_1}, \cdots \overline{V_{n-1}} \Rightarrow x \in R_n$

③ $\forall x$, $\exists n$ s.t. $x \in V_n \Rightarrow x \notin R_{n+1}, \cdots$ $\Rightarrow$ locally finite.

Def. Topological manifold of dim n is a topological space s.t. ① A_2 ② T_2 ③ $\underbrace{\text{locally Euclidean}}$ $\Rightarrow$ LC

$\forall x\ \exists U_x \subseteq M$. & $V_x \subseteq \mathbb{R}^n$

$\varphi_x : U_x \xrightarrow{homeo} V_x$.

cor. Manifolds ~~is~~ are always paracompact

cor. Manifolds ~~is~~ are T_3, T_4.

An application of P.O.U.

Thm. Any compact dim n topological manifold can be embedded into a $\mathbb{R}^N$

proof. $\forall x$. $\varphi_x : U_x \longrightarrow V_x \subseteq \mathbb{R}^n$

compactness : $\varphi_i : U_i \to V_i$ s.t. $\{U_i\}$ is an open covering of M. $1 \le i \le m$

let $\{\rho_i\}$ ~~for~~ P.O.U. for U_i

let $h_i : M \to \mathbb{R}^n$ $h_i(x) = \begin{cases} \rho_i(x) h_i(x) & x \in U_i \\ (0, \cdots 0) & x \in (\operatorname{supp} \rho_i)^c \end{cases}$

$\Rightarrow h_i$ is cts on M

$\Phi : M \to \mathbb{R}^{mn+m}$ $x \mapsto (h_1, \cdots h_m, \rho_1, \cdots \rho_m)$.

Now. Φ is injective.

$\boxed{\rho_i(x) = \rho_i(y) \Rightarrow h_i(x) = h_i(y) \Rightarrow x = y}$

$\Rightarrow \Phi$ is homeo.