Algebraic Topology (X) Topological invariants (V)

Connectness ~~lies~~ lies between topological property and topological invariants
What we deal with before / May we calculate sth.

Lecture 18 Connectness

Def ① $(X, \mathcal{T})$ X (nonempty) is disconnected if $\exists A, B \neq \phi$. $A \cap B = \phi$. $A \cup B = X$. s.t. $\bar{A} \cap B = \phi$. $A \cap \bar{B} = \phi$

② We say X is connected if X is not disconnected.

eg. $(X, \mathcal{T}_{trivial})$ is connected since "$A \neq \phi \Rightarrow \bar{A} = X$"

$(X, \mathcal{T}_{discrete})$
- $|X| = 0, 1 \Rightarrow X$ is connected
- $|X| \geq 2 \Rightarrow X$ is disconnected.

prop. the followings are equivalent.
1. $(X, \mathcal{T})$ is disconnected
2. $\exists$ closed $A, B \neq \phi$. $A \cap B = \phi$ $A \cup B = X$
3. $\exists$ open $A, B \neq \phi$ $A \cap B = \phi$ $A \cup B = X$
4. $\exists$ clopen $A \neq \phi, X$
5. $\exists$ cts sur $f: X \to \{0, 1\}_{dis}$

Pf. $(2) \Leftrightarrow (3) \Leftrightarrow (4)$

$(1) \Rightarrow (2)$: ~~$A \cap \bar{B} = \phi$~~ By (1). $\exists A, B \neq \phi$. $\bar{A} \cap B = A \cap \bar{B} = \phi$ $X = A \cup B$

$\Rightarrow A^c = B$, $\bar{B} \subseteq A^c = B \Rightarrow B$ is closed

similarly $\Rightarrow A$ is closed.

$(2) \Rightarrow (1)$ definition

$(3) \Rightarrow (5)$ $f(A) = 0$ $f(B) = 1$

$(5) \Rightarrow (3)$ $f^{-1}(0), f^{-1}(1) \neq \phi$ open #

Def. $Y \subseteq (X, \mathcal{T})$ is disconnected / connected if $(Y, \mathcal{T}_{subspace})$ is disconnected / connected

prop. T.F.A.E.
1. $Y \subseteq (X, \mathcal{T})$ is disconnected
2.
3. $\exists A, B \neq \phi$ open in X, s.t. $A \cap Y \neq \phi$. $B \cap Y \neq \phi$. $\boxed{A \cap B \cap Y = \phi. \ Y \subseteq A \cup B}$

Def We say $(X, \mathcal{T})$ is totally disconnected. if for any $Y \subseteq X$. with $|Y| \geq 2$. Y is disconnected

① $(X, \mathcal{T}_{trivial})$ ② $\mathbb{Q} \subseteq \mathbb{R}$. $Y \subseteq \mathbb{Q}$ ③ $(\mathbb{R}, \mathcal{T}_{sorgenfrey})$

Thm. $S \subseteq \mathbb{R}$ is connected if and only if S is a interval

$\forall x, y \in I \quad x \le z \le y \Rightarrow z \in I$.

pf (⇒ if) $S \ne I$. i.e. $\exists\, x < z < y$. $z \notin S$

let $S_1 = (-\infty, z) \cap S$. $S_2 = (z, +\infty) \cap S$.

(⇐) By contradiction suppose $\exists\, U, V \subseteq \mathbb{R}$

$\underbrace{U \cap I \ne \phi}_{a \in}$. $\underbrace{V \cap I \ne \phi}_{b \in}$. $U \cap V \cap I = \phi$ $U \cup V \supseteq I$.

WLoG. $\Rightarrow (a, b) \subseteq I$

$A = \{ x \in U \cap I \mid x \le b \} \ne \phi$
$a \in$

$\exists\, c = \sup A < +\infty$. $c \ne a$ since U is open

Claim 1. $c \notin U$ if so, $\Rightarrow c < b$. $\Rightarrow \underset{\substack{\cap \\ U}}{c+\varepsilon} < \underset{\substack{\cap \\ (a,b) \subseteq I}}{b}$ $\Rightarrow c \ne \sup A$

Claim 2. $c \notin V$ if so $\Rightarrow a < c$. $\Rightarrow \underset{\substack{\cap \\ V}}{c - \varepsilon} > \underset{\substack{\cap \\ (a,b) \subseteq I}}{a}$ $\Rightarrow (c-\varepsilon, c) \subseteq V$

$\Rightarrow c \ne \sup A$

Contradiction! #

Rmk Dedekind Completeness & Dense + totally order ...

Connectness argument. To prove $P(t)$ $t \in I$.

① $\exists\, t_0 \in I$. s.t. $P(t_0)$ ✓

② $S = \{ t \in I \mid P(t) \}$ open

③ $S = \{ t \in I \mid P(t) \}$ closed

prop. $f: \mathbb{R} \to \mathbb{R}$ analytic if $\exists\, x_0$ s.t. $f^{(n)}(x_0) = 0$ $\forall n$. then $f = 0$.

prop. $f: X \to Y$ cts $A \subseteq X$ connected $\Rightarrow$ $f(A)$ connected.

prop. $f: X \to Y$ $f(A)$ is disconnected $\longrightarrow$ $f(A) \subseteq V_1 \cup V_2$ open in Y

proof. suppose $f(A)$ is $V_1 \cap V_2 \cap f(A) = \phi$

$V_1 \cap f(A)$, $V_2 \cap f(A) \ne \phi$

$U_i = f^{-1}(V_i)$ open in X. ... #

~~$U_i \cap A \ne \phi$~~

prop. If $f: X \to Y$ homeo. then $\overset{\subseteq X}{A}$ connected $\Leftrightarrow f(A)$ ...

Define a equivalent relation on (X, T)

$x \sim y \Leftrightarrow \exists$ connected subset $A \ni x, y$.

$X/\sim \ \ni [x]$. — connected component

well defined

f cts Then ① $\bar{f}: X/\sim \to Y/\sim$

② $f: X \to Y$ homeo $\Rightarrow \bar{f}$ is bi $\Rightarrow |X/\sim| = |Y/\sim|$.

prop. (I.V.T). $f: X \to \mathbb{R}$ X connected $f(x_1) = a < b = f(x_2)$

$\Rightarrow \forall c \in (a,b)$. $\exists x \in X$, s.t. $f(x) = c$.

prop. suppose A is connected. $A \subseteq B \subseteq \bar{A}$. $\Rightarrow B$ connected.

proof. let. $f: B \to \{0,1\}$ cts.

$f|_A : A \to \{0,1\}$ cts is not a cts sur

$\overset{\text{WLOG}}{\Longrightarrow} f(A) = 0$ $f(B) = f(\bar{A}) \subseteq \overline{f(A)} = \{0\} \Rightarrow f$ is not sur.

closure as subset of B

so $\bar{A} = B$ here

#

In particular, A connected $\Rightarrow \bar{A}$ connected.

Example. topologist's sine curve.

$y = \sin\frac{1}{x}$ $x > 0$. $t \to (t, \sin\frac{1}{t})$

$(0,\infty) \to \mathbb{R}^2$

$S = (\{0\} \times [-1,1]) \cup \{(t, \sin\frac{1}{t}) \mid t > 0\}$.

Cor. S is connected.

Review. X is disconnected $\Longleftrightarrow$ $X = A \cup B$. $\bar{A} \cap B = A \cap \bar{B} = \phi$ $A \cap B = \phi$

Set theory $\Longleftrightarrow$ open --

$\Longleftrightarrow$ closed --

$\Longleftrightarrow$ $A \neq \phi$. X clopen

or $f: X \to \{0, 1\}$ is continuous surjective.

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connectness

$\underset{\text{connected.}}{\underline{A}} \subseteq B \subseteq \bar{A} \Rightarrow B$ is connected

connectness is a topological invariant

Fact (topologist's sine curve is not path-connected).

$s = \sup\{t \in [0,1] \mid f(t) = 0\} < 1$ $\quad f(t) = (f_1(t), f_2(t))$

$f_1(s) = 0$. $f_1(t) > 0$ $\forall t > s$.

$s_n \searrow s$ $\quad f_2(s_n) = \sin \frac{1}{f_1(s_n)} = (-1)^n$ 矛

$s_n = \frac{2}{(2n+1)\pi}$

Def. $(X, \mathcal{T})$ $x, y \in X$. We call $\gamma: [0,1] \to X$ cts $\gamma(0) = x$, $\gamma(1) = y$ a path from x to y

rmk. path is a map, not just a subset in X. We have the parametrization information in path

rmk. ① if $\left.\begin{array}{l}\gamma_1 \text{ path from } x \text{ to } y \\ \gamma_2 \text{ path from } y \text{ to } z\end{array}\right] \longrightarrow \gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

is a path from x to z

② $\gamma: x \to y$ $\bar{\gamma}(t) = \gamma(1-t) : y \to x$.

Def. We say $(X, \mathcal{T})$ is path-connected if $\forall x, y \in X$, $\exists$ a path γ from x to y

So. ① Topologist's sine curve is not path connected.

② $\{(t, \sin\frac{1}{t}) \mid t \in (0,1)\}$ is path connected.

— A set is path-connected $\not\Rightarrow$ its closure is path connected

— A space is path connected $\Rightarrow$ it's connected.

pf: if not connected, $\exists f \ X \to \{0,1\}$ sur

$\exists f(x) = 0$. $f(y) = 1$

By def, $\gamma(0) = x$, $\gamma(1) = y$

$\underbrace{f \circ \gamma}_{\text{cts}} : [0,1] \to \{0,1\}$. X

prop. A path connected, $f: X \to Y$ is path connected.

pf: $f(x), f(y) \in f(A)$

$f \circ \gamma$: ~~[0,1] →~~ is a path from $f(x)$ to $f(y)$. #

Def We say $(X, \mathcal{T})$ is locally path-connected if $\forall x$. $\forall U_x \in \mathcal{N}(x)$.
$\exists V \in \mathcal{T}$ s.t V is path-connected, $x \in V \subseteq U$.

e.g. locally euclidean

prop. $(X, \mathcal{T})$ is connected & locally path connected $\Rightarrow$ path-connected.

pf: (connectness argument) fix $x_0 \in X$, Consider $S = \{y \mid \exists$ path from x to $y\}$.

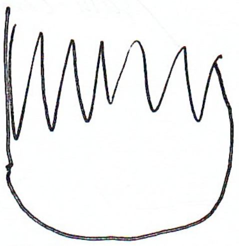
① S is not ϕ. since $x_0 \in S$.

② S is open $\forall y \in S$. by locally path connectness. $y \in V_y \subseteq S$.

③ S^c is open ...

$\Rightarrow S = X$ since X is connected. #

rmk ~~pa~~ path-connected $\not\Rightarrow$ locally path connected



now it's path connected but still locally path-connected.

Cor. Since locally-euclidean $\Rightarrow$ locally path connected

$\Rightarrow$ " LE + connected $\Rightarrow$ path-connected.

$\Downarrow$

connected subsets in topological manifolds are path-connected

~~remark~~ rmk. In general. Path need not to be injective.

"arc" $\gamma : [0,1] \longrightarrow X$, s.t. $[0,1] \simeq \gamma([0,1])$.

We can prove In (T2) space, any path can be replaced by an arc.

prop. (star-shaped union) $\bigcap_\alpha X_\alpha \neq \phi$. $X = \bigcup_\alpha X_\alpha$

(1) each X_α is connected $\Rightarrow$ $\bigcup_\alpha X_\alpha$ is connected.

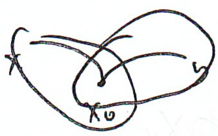
(2) path - - path

Pf. (1) $f: X \to \{0,1\}$, then $f: X_\alpha \to \{0,1\}$ cts.

take $x_0 \in \bigcap_\alpha X_\alpha$

$\Rightarrow f(X_\alpha) = f(x_0) \Rightarrow$ it's not sur.

(2) trivial.



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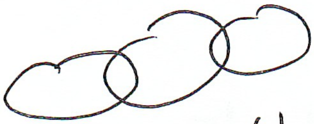


prop (chain-like union) $X_1, \cdots X_N$ $(N \le +\infty)$.

$X_k \cap X_{k+1} \neq \phi$. then (1) each X_n is connected $\Longrightarrow X = \bigcup_{n=1}^{N} X_n$ is connected

(2) path .. path..

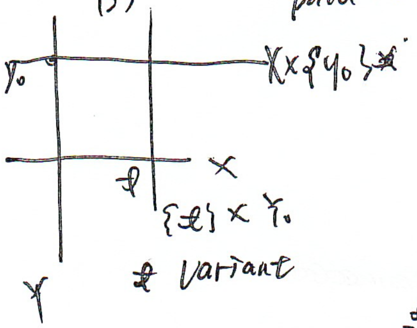
pf: $Y_n = X_1 \cup \cdots \cup X_n$ use last Prop (star) by induction. Y_n is connected.

Note $\bigcap_{1 \le n \le N} Y_n \neq \phi \Longrightarrow X = \bigcup X_N = \bigcup Y_N$ is connected / path connected

/ path-connected. #

prop. (1) X, Y is connected $\Longrightarrow X \times Y$ is connected

(2) path -- path --



$Y_x = \left(X \times \{y_0\}\right) \cup \left(\{x\} \times Y_0\right)$ / path connected.

connected, connected

$(x, y_0) \in \cdots$

$\bigcap_x Y_x = X \times \{y_0\} \neq \phi$

$$X \times Y = \bigcup_x \left(\{x\} \times Y) \cup (X \times \{y_0\}\right) \text{ is connected.}$$ #

prop. (1) X_α is connected $\Longrightarrow \prod_\alpha X_\alpha$ connected (prod)

(2) p.c. $\Longrightarrow$ p.c.

pf. (1) For Any finite $K \subseteq \Lambda$ the set.

Fix $a_\alpha \in X_\alpha$ $\quad X_K^{(a_\alpha)} = \left(\prod_{\alpha \in K} X_\alpha\right) \times \prod_{\alpha \notin K} \{a_\alpha\} \simeq \left(\prod_{\alpha \in K} X_\alpha\right)$ is connected.

$\bigcap_{\text{Finite } K} X_K^{(a_\alpha)} \ni (a_\alpha) \Longrightarrow \tilde{X} = \bigcup_{\text{finite } K \subseteq \Lambda} X_K^{(a_\alpha)}$ is connected.

claim $\overline{\tilde{X}} = \prod_\alpha X_\alpha \Longleftrightarrow \tilde{X} \cap U^{\in \mathcal{T}} \neq \phi$

$\prod_{\alpha \in K} X_\alpha \times \prod_{\alpha \notin K} \{a_\alpha\}$ contain

$\prod_{\alpha \in K} U_\alpha \times \prod_{\alpha \notin K} X_\alpha$

the same K.

(2)
$\forall (x_\alpha), (y_\alpha) \in \prod X_\alpha$

$\exists \gamma_\alpha : [0,1] \to X_\alpha$ s.t. $\gamma_\alpha(0) = x_\alpha$, $\gamma_\alpha(1) = y_\alpha$.

$\gamma = (\gamma_\alpha) : [0,1] \to \prod_\alpha X_\alpha$ (the universal property)

$\gamma(0) = (x_\alpha)$ $\gamma(1) = (y_\alpha)$. #

recall. $(X, \mathcal{T})$ $x \sim y \Leftrightarrow \exists$ connected set $A \ni x, y$.

Now $x \underset{p}{\sim} y \Leftrightarrow \exists$ path from x to y

prop. $\sim$, $\underset{p}{\sim}$ are both equivalence relation.

pf: (1) $x \underset{S_1}{\sim} y$, $y \underset{S_2}{\sim} z \Rightarrow x \sim z$ in $S_1 \cup S_2$ #

(2) trivial.

Def. equivalence classes in $X/\sim$ are called connected component

$X/\underset{p}{\sim}$ path connected component.

$\pi_c(X) = X/\sim$, $\boxed{\pi_0(X) = X/\underset{p}{\sim}}$

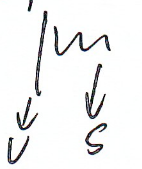
rmk. $(\pi_c(X), \mathcal{T}_{quotient})$ is T1. & totally disconnected.

$(\pi_0(X), \mathcal{T}_{quotient})$ could be non T1

non path connected

So, when we study $\pi_0(X)$ ~~like~~ Sine

We usually regard it as a set.



$\mathcal{T} = \{\phi, \{S\}, \{S, V\}\}$.

Not T1.

it's p.c.

$|\pi_0(X)|$ ~ topological quantity

Category $\mathcal{C}$ $ob(\mathcal{C})$, $Mor(\mathcal{C})$ is too hard to understand.

functor $\downarrow$

Catagory $\mathcal{D}$ $ob(\mathcal{D})$, $Mor(\mathcal{D})$

$$\underset{Top}{X} \longrightarrow \underset{Set}{\Pi_0(X)} \qquad \pi_1$$

$$f: X \to Y \longrightarrow \Pi_0(f): \Pi_0(X) \to \Pi_0(Y).$$

$$[x] \mapsto [f(x)].$$

Last time — Path connectness. A path is a map. but ~~a~~ a set of Topological space

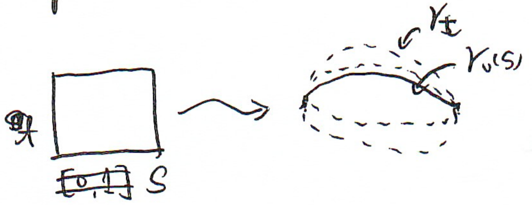
$\text{p.c} \Rightarrow \text{C.}$ $\qquad \prod C = C$

$\text{C} + \text{L.p.c} \Rightarrow \text{p.c.}$ $\qquad \prod \text{p.c.} = \text{p.c.}$

$f(\text{p.c.}) = \text{p.c.}$ $\qquad \overline{C} = C$

$f(C) = C$ $\qquad \overline{\text{p.c.}} \neq C$

Today continuous deformation of path



Generally setting. for continuous deformation of $f \in \mathcal{C}(X, Y)$ topological space

with perrameter space T is a continuous map.

$$F : T \longrightarrow \mathcal{C}(X, Y) \quad \text{with } F(t_0) = f$$

$$(\mathcal{C}(X, Y), \mathcal{T}_{c.o.})$$

~~path is a cts defor~~ Path is a cts deformation ~~from~~ for a point. ?

Rmk. In most cases. we take $T = [0,1]$. $\mathbb{R}$. or S^1.

e.g. cts deformation of perth.

① $F : [0,1] \longrightarrow \mathcal{C}([0,1], X)$ with $F(0) = \gamma$.

(Not fix the two side pts.)

② $F : [0,1] \longrightarrow \mathcal{C}([0,1], X)$ with $F(0) = \gamma$

$F(t)(0) = \gamma(0)$

$F(t)(1) = \gamma(1)$.

~~Thus~~, consider the bijective correpondenee.

$$\mathcal{M}(T, \mathcal{C}(X, Y)) \longrightarrow \mathcal{M}(T \times X, Y).$$

~~$F(t)(x)$~~ $F(t)(x) = G(t, x)$. $\longleftarrow$ $G(t, x) = F(t)(x)$.

Thm. Let T, X, Y be topological space.

(1) $G \in \mathcal{C}(T\times X, Y) \Rightarrow F \in \mathcal{C}(T, \underbrace{\mathcal{C}(X,Y)}_{\mathcal{T}_{c.o.}})$

(2) $\underbrace{X \text{ is LCH}}_{\text{like } [0,1],\ \mathbb{R} \cdots} \Rightarrow$ "(1) $\Leftarrow$"

proof: (1) Suppose $G \in \mathcal{C}(T\times X, Y)$

$X \xrightarrow{j_t} T\times X \xrightarrow{G} Y$

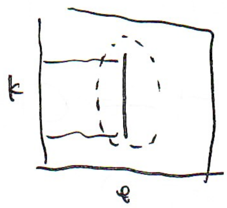
$x \longmapsto (t,x) \longmapsto G(t,x)$.

For $t \in T$. $\underbrace{F(t) = G\circ j_t}_{\text{is cts}}$

to prove F is cts, it suffice to prove

$\forall$ cpt set K, open set U.

$\boxed{F^{-1}(\underbrace{S(K,U)}_{(f(K)\subseteq U)}) \text{ is open in } T}$



For any $t \in F^{-1}(S(K,U))$

$F(t) \in S(K,U) \Rightarrow G(\{t\}\times K) \subseteq U$

$\{t\}\times K \subseteq \underbrace{G^{-1}(U)}_{\text{open}}$. use tube lemma

$\Rightarrow \exists V \subseteq T$. s.t. $\{t\}\times K \subseteq V\times K \subseteq G^{-1}(U)$

$\Rightarrow G(V\times K) \subseteq U. \Rightarrow F(V) \subseteq S(K,U) \Rightarrow V \subseteq F^{-1}(S(K,U))$

$\Rightarrow F^{-1}(S(K,U))$ is open

(2) suppose X is LCH, $F \in \mathcal{C}(T, \mathcal{C}(X,Y))$

To prove G is cts. it suffice to prove $\forall$ $U \subseteq Y$ (open). $G^{-1}(U)$ is open in $T\times X$

For $(t,x) \in G^{-1}(U)$ $G(t,x) \in U \Rightarrow F(t)(x) \in U$

$F(t) \in S(\underbrace{\{x\}}_{\text{cpt}}, U)$, since X is LCH. space

So $S(\{x\}, U) = \bigcup_{x\in W_x} S(\overline{W_x}, U)$ | it's natural. $x \in W_x \subseteq \overline{W_x} \subseteq K$.

$\exists$ $x \in W_x \subseteq \overline{W_x}$ s.t. $F(t) \in S(\overline{W_x}, U)$

$t \in F^{-1}(S(\overline{W_x}, U))$ $\exists$ $t \in V \subseteq F^{-1}(\ )$

$\Rightarrow G(V\times W_x) \subseteq G(V, \overline{W_x}) \subseteq U$

$\Rightarrow V\times W_x \subseteq G^{-1}(U)$ #

Def. Let $f_0, f_1 \in \mathcal{C}(X,Y)$ if $\exists$ cts $F:[0,1]\times X \to Y$ $F(0)=f_0$, $F(1)=f_1$. We say f_0 & f_1 are homotopic and call F a homotopy between f_0 & f_1.

Notation $f_0 \sim f_1$

Rmk. (1) $\exists$ homotopy $f_0 \overset{F}{\sim} f_1 \Longrightarrow \exists$ cts deformation (1-parameter) from f_0 to f_1

(2) X is LCH " $\Longleftrightarrow$ " $\exists$ 1-parameter deformation $\Updownarrow$ $f_0, f_1 \in (\mathcal{C}(X,Y), \mathcal{T}_{c.o})$ can be connected by a path $\Updownarrow$ f_0, f_1 lie in the path-connected component in map space.

In particular

(3) X is $\{x\}$

$f \in \mathcal{C}(X,Y) \longleftrightarrow f(pt) \in Y$

$\Downarrow$

$\mathcal{C}(X,Y) = Y$

(4) one can prove homotopy is an equivalent relation.

$\rightsquigarrow [f]$. homotopy class of f.

$[X,Y] = \mathcal{C}(X,Y)/\sim$

prop. The following are well-defined.

(1) composition. $[X,Y]\times[Y,Z] \longrightarrow [X,Z]$

$([f],[g]) \longmapsto [g\circ f]$

(2) pull-back. $\varphi \in \mathcal{C}(X_0, X_1) \rightsquigarrow \varphi^*: [X_1, Y] \to [X_0, Y]$.

$[f] \mapsto [f\circ\varphi]$

(3) push-forward $\psi \in \mathcal{C}(Y_0, Y_1) \rightsquigarrow \psi_*: [X, Y_0] \to [X, Y_1]$

$[f] \mapsto [\psi\circ f]$.

Now back to path.

the multiply Not possess associate principle.

$(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3)$ since they use different parameter.

Def. We say $\tilde{\gamma}$ is a reparametrization of γ.

$\exists f$ cts, $f:[0,1]\to[0,1]$, $f(0)=0$, $f(1)=1$, $\tilde{\gamma} = \gamma\circ f$.

(don't require f to be injective)

prop. if $\tilde{\gamma}$ is a repa~ of γ. $\Rightarrow \tilde{\gamma}\sim\gamma$.

Pf ($[0,1]$ is convex) $\gamma = \gamma\circ \mathrm{Id}$, $\tilde{\gamma} = \gamma\circ f$.

($t\,\mathrm{Id} + (1-t)f$) $\Rightarrow$ $F(t,s) = \gamma(tx + (1-t)f(x))$. #

cor (1) $[\gamma_1 * (\gamma_2 * \gamma_3)] = [(\gamma_1 * \gamma_2) * \gamma_3]$

(2) $[\gamma_{x_1} * \gamma] = [\gamma] = [\gamma * \gamma_{x_2}]$

(3) $[\gamma * \bar{\gamma}] = [\gamma_{x_1}]$ is not repara.

proof of (3) $F(t,s) = \begin{cases} \gamma(2s(1-t)) & s \le \frac{1}{2} \\ \bar{\gamma}(\cdots) & s \ge \frac{1}{2} \end{cases}$ #

Def. (Null-homotopic) We say f is <u>null-homotopic</u> if $f \sim f_{y_0}$ $\left(\begin{array}{l} f: X \to Y \\ f(X) \equiv y_0 \end{array} \right)$.

Rmk. $Y \subseteq \mathbb{R}^n$ is convex or star-shaped

$\Rightarrow$ (1) Any $f \in \mathcal{C}(X, Y)$ is null-homotopic

(2) Any $f \in \mathcal{C}(Y, Z)$ is null-homotopic

i.e. it's p-c.

rmk. $f_n : S^1 \to S^1$ $z \mapsto z^n$

we will prove $f_n \not\sim f_m$

Def. If Id_X is null-homotopic, then X is contractible.

Def. $X \sim Y \iff \exists f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, X)$. s.t. $f \circ g \sim Id_Y$, $g \circ f \sim Id_X$

we call it space homotopy

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last time. Homotopy $f_0 \sim f_1$ "almost" (continuous deformation)

map

$f_1 \sim f_2 \iff \exists F \in \mathcal{C}([0,1] \times X, Y)$

$\Downarrow$ $\Uparrow$ X is LCH

$\mathcal{C}([0,1], \mathcal{C}(X, Y))$

$\downarrow$

Space Homotopy $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, X)$

$\Downarrow$ $f \circ g \sim Id_Y$ $g \circ f \sim Id_X$

$\downarrow$

path. $\gamma_1 \sim \gamma_2$, $H : [0,1] \times [0,1] \to X$

Aim: define operations on Path.

— multiply $\gamma_1 * \gamma_2$

— inverse $\bar{\gamma}_1$

— unit γ_{x_0}

But we found that $(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3)$. generally.

"$\sim$" reparametrization.

We use homotopy class

but we have trouble in the endpoints.

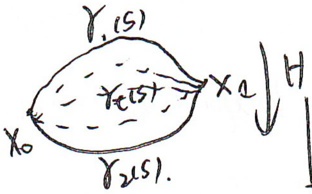
so ...

Def (Path homotopy) We say $\gamma_1, \gamma_2 \in \Omega(X, x_0, x_1) = \left\{\gamma \in \mathcal{C}([0,1], X) \;\middle|\; \begin{array}{l}\gamma(0)=x_0 \\ \gamma(1)=x_1\end{array}\right\}$

are path homotopic if $\exists\ H: [0,1]\times[0,1] \to X$. s.t.

$H(0,s) = \gamma_1(s)$, $H(1,s) = \gamma_2(s)$. ~~$\gamma_t(0)=x_0$, $\gamma_t(1)=x_1$~~

$H(t,0) = x_0$. $H(t,1) = x_1$

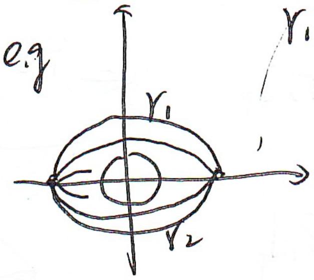


$\boxed{\text{Notation. } \gamma_1 \underset{p}{\sim} \gamma_2.}$

Note "$\sim$" $\neq$ "$\underset{p}{\sim}$"

$\uparrow$ $\mathcal{C}([0,1]; X)$ $\quad$ $\uparrow$ $\Omega(X, x_0, x_1)$.

e.g



$\gamma_1 \sim \gamma_2$ but $\gamma_1 \not\underset{p}{\sim} \gamma_2$

prop. We say

(1) $\gamma_i, \gamma_i' \in \Omega(X, x_i, x_{i+1})$. $\gamma_i \underset{p}{\sim} \gamma_i' \Rightarrow \gamma_1 * \gamma_2 \underset{p}{\sim} \gamma_1' * \gamma_2'$

(2) $(\gamma_1 * \gamma_2) * \gamma_3 \underset{p}{\sim} \gamma_1 * (\gamma_2 * \gamma_3)$ $\;\|$ reparametrization. though kind of differene from (1).

(3) $\gamma_{x_1} * \gamma_1 \underset{p}{\sim} \gamma_1 \sim \gamma_1 * \gamma_{x_1}$

(4) $\gamma_1 * \bar{\gamma}_1 \underset{p}{\sim} \gamma_{x_1}$ $\quad \bar{\gamma}_1 * \gamma_1 \sim \gamma_{x_2}$

(5) $f \in \mathcal{C}(X, Y)$, $\gamma_1 \underset{p}{\sim} \gamma_2 \Rightarrow f\circ\gamma_1 \underset{p}{\sim} f\circ\gamma_2$

(6) reparametrization.

admits $[\gamma]_p$.

So we can define multiplication on $\boxed{\pi(X, x_1, x_2) = \Omega(X, x_1, x_2)/\underset{p}{\sim}}$

$$m: \pi(X, x_1, x_2) \times \pi(X, x_2, x_3) \longrightarrow \pi(X, x_1, x_3).$$

$$([\gamma_1]_p, [\gamma_2]_p) \longmapsto [\gamma_1]_p * [\gamma_2]_p = [\gamma_1 * \gamma_2]_p.$$

and inverse (though in different space).

$$i: \pi(X, x_1, x_2) \longrightarrow \pi(X, x_2, x_1)$$

$$[\gamma_1]_p \longmapsto [\bar{\gamma}_1]_p.$$

Well-defineness can be check easily

Cor. $([\gamma_1]_p [\gamma_2]_p)[\gamma_3]_p = [\gamma_1]_p([\gamma_2]_p[\gamma_3]_p)$.

- $[\gamma_{x_1}]_p[\gamma_1]_p = [\gamma_1]_p$. $\quad [\gamma_1]_p[\gamma_{x_2}]_p = [\gamma_1]_p$

- $[\gamma_1]_p[\bar{\gamma}_1]_p = [\gamma_{x_1}]_p \quad [\bar{\gamma}_1]_p[\gamma_1]_p = [\gamma_{x_2}]_p$.

"groupoid" structure on $\pi(X) = \bigcup_{x,y\in X} \pi(X, x, y)$

"partially defined multiplication" and "inverse" and "left, right unit"

What we have now is called "fundamental groupoid"

Fundamental group. $\pi_1(X, x_0) = \Omega(X, x_0)/\underset{p}{\sim}$

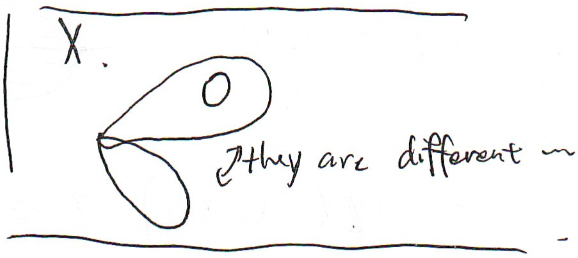
$\Omega(X, x_0) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = \gamma(1) = x_0\}$. "loop space with basepoint x_0"

with multiplication

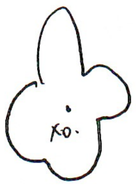
$[\gamma_1]_p [\gamma_2]_p = [\gamma_1 * \gamma_2]_p$ ← its multiplication is associative.

and identity $[\gamma_{x_0}]_p = e$

inverse. $[\gamma_1]_p^{-1} = [\bar{\gamma}_1]_p$



Example. X = star-like in $\mathbb{R}^n$



Then $\pi_1(X, x_0) = \{[\gamma_{x_0}]_p\}$.

proof. $\forall \gamma \in \Omega(X, x_0)$

Want $\gamma \underset{p}{\sim} \gamma_{x_0}$

$H(t, s) = t x_0 + (1-t)\gamma(s)$

Rmk. By def $\mathcal{C}([0,1], \mathcal{C}([0,1], X)) = \mathcal{C}([0,1] \times [0,1], X)$

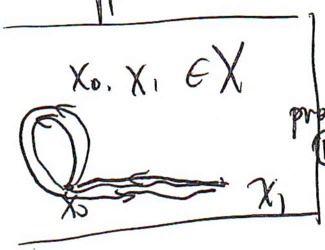
$\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$.

Dependence of $\pi(X, x_1)$ with x_1 ?

① let X_1 = path component contains x_0.

$\pi_1(X, x_1) = \pi_1(X_1, x_0)$

② Suppose X is path-connected $\quad \lambda(0) = x_0, \ \lambda(1) = x_1$.



prop. The map $\Gamma_\lambda : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$[\gamma]_p \longmapsto [\bar{\lambda} * \gamma * \lambda]_p$ is a group homomorphism.

proof. ① $\Gamma_\lambda([\gamma_1]_p [\gamma_2]_p) = \Gamma_\lambda([\gamma_1 * \gamma_2]_p) = [\bar{\lambda} * \gamma_1 * \gamma_2 * \lambda]_p$

$= [\bar{\lambda} * \gamma_1 * \underbrace{\lambda * \bar{\lambda}}_{\gamma_{x_1}} * \gamma_2 * \lambda] = \Gamma_\lambda([\gamma_1]_p)\, \Gamma_\lambda([\gamma_2]_p)$

② $(\Gamma_\lambda)^{-1} = \Gamma_{\bar{\lambda}}$

$\Gamma_{\bar{\lambda}} \circ \Gamma_\lambda([\gamma]_p) = \Gamma_{\bar{\lambda}}([\bar{\lambda} * \gamma * \lambda]_p) = \cdots = [\gamma]_p$ #

~~rmk~~ Notation. $\pi_1(X)$ = the isomorphism class of $\pi_1(X, x_0)$ ← when assume X is path-connected.

rmk. $\pi_1(X, x_0)$ is a concrete group whose elements have geometry meaning.

$\pi_1(X)$ is an abstract group ... No ...

$\pi_1(X, x_0) \overset{[\lambda]}{\cong} \pi_1(X, x_1)$ ~~dependence of~~ on $\boxed{\lambda}$

Cor $\pi_1(\text{Star-like}) = \{e\}$.

Def. We say X is <u>simply connected</u> if $\pi_1(X) = \{e\}$ (imply path-connected).

π_1 is a functor

$$\pi_1 : \underbrace{P\,Top}_{\text{with base pair}} \rightsquigarrow Group.$$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

morphism. $f \in \mathcal{C}((X, x_0), (Y, y_0)) \longmapsto f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

where f_* is defined as

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

$$[\gamma_p] \longmapsto [f \circ \gamma]_p.$$

prop. f_* is a group homomorphism, and ① $Id_* = Id_{\pi_1(X, x_0)}$ ② $(\underset{\text{Topo}}{g \circ f})_* = \underset{\text{group}}{g_* \circ f_*}$

proof. $f_*([\gamma_1]_p [\gamma_2]_p) = f_*([\gamma_1 * \gamma_2]_p) = [f \circ (\gamma_1 \underset{\text{in } X}{*} \gamma_2)]_p$.

$\overset{\text{check}}{=} [(f \circ \gamma_1) \overset{\text{in } Y}{*} (f \circ \gamma_2)]_p = [f \circ \gamma_1]_p * [f \circ \gamma_2]_p = f_*([\gamma_1]_p) f_*([\gamma_2]_p)$.

Cor $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$.

$f: X \to Y$, $g: Y \to X$ $\quad f \circ g = Id \quad g \circ f = Id$

let $y_0 = f(x_0)$. then $g_* \circ f_* = Id : \pi_1(X, x_0) \to \pi(X, x_0)$.

$f_* \circ g_* = Id : \pi(Y, y_0) \to \pi(Y, y_0)$.

$\Rightarrow f_*, g_*$ are group homomorphism. □

$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Suppose $f, g \in \mathcal{C}(X, Y)$ $f \sim g$

What's the relation between f_* & g_*.

fix $x_0 \in X$, let $y_0 = f(x_0)$. $y_1 = g(x_0)$

then ~~$\pi(X, x_0) \xrightarrow{f_*} \pi(Y, y_0)$~~

~~$\pi(X, x_0) \xrightarrow{g_*} \pi(Y, y_1)$~~.

$$\begin{tikzcd} \pi(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi(Y, y_0) \arrow[d, "T_\lambda"] \\ & \pi(Y, y_1) \end{tikzcd}$$

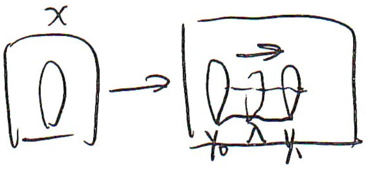
$F(t, x) \Rightarrow F(0,x) = f(x)$

$F(1, x) = g(x)$

use $\lambda = F(t, x_0)$

prop. $g_* = T_\lambda \circ f_*$

Given any $\gamma \in \Omega(X, x_0)$. want: $g \circ \gamma \underset{p}{\simeq} \bar{\lambda} * (f \circ \gamma) * \lambda$.



Cor $X \sim Y \Rightarrow \pi_1(X) \simeq \pi_2(Y)$.

I need to check it.

Some details ~~of~~ last time lecture.

~~conti~~ Idea: continuous deformation $\longrightarrow$ homotopy

It's an equivalent relation

we can do mapping operators on the class.

homotopy is a little stronger than deformation but when X is LCH they are the same. Infact, it's not a big deal.

$\Downarrow$

When we define some Algebraic Operators on Path.

it even not satisfies Associative principle.

~~it's look~~ In fact, they look similar in "geometry"

but the "velocity" in different segments is different

$\hookrightarrow$ reparametrization ~

We have a great observation.

reparametrization $\rightsquigarrow$ homotopy. since $\gamma_2 = \gamma_1 \circ f_0$ $\quad f(0)=0$, $f(1)=1$

~~$F(t,x) = (1-f(t))\gamma_1(x) + f(t)\gamma_2(x)$~~

$s \in I$. $f(s) \in [0,1] \Rightarrow (1-t)s + t f(s) \in I$.

$\Rightarrow \gamma_1((1-t)s + t f(s))$ is a homotopy

since $t=0$. $\gamma_1(s)$

$t=1$ $\gamma_1 \circ f(s) = \gamma_2(s)$.

#

Now, different paths with the "multiplication" is homotopy so we can define right multiplication on the "homotopy class" Since we have proved the homotopic relation is preserved under "composition", "pull-back" "push-forward".

~~So. Corollary : $[\gamma_1]$.~~

~~We also need a fact $\sim$ γ_{x_1} is null-homotopic.~~

We need some examples to get further proposition.

~~1) $\gamma \quad \gamma * \gamma_{x_1}$ $\quad \gamma_{x_1} * \gamma = \begin{cases} \gamma_1(2s) & s \in [0,\frac{1}{2}] \\ \gamma(2s-1) & s \in [\frac{1}{2},1] \end{cases}$~~

~~$f = \begin{cases} 0 & s \in [0,\frac{1}{2}] \\ 2s-1 & s \in [\frac{1}{2},1] \end{cases}$ $\quad \gamma \circ f = \begin{cases} \gamma(0) & s \in [0,\frac{1}{2}] \\ \gamma(2s-1) \end{cases}$~~

well, we ~~diff~~ define $\gamma_x \equiv x$, using concrete calculation we know

$(\gamma_1 * \gamma_2) * \gamma_3$ and $\gamma_1 * (\gamma_2 * \gamma_3)$ rep.

$\gamma_{x_1} * \gamma$ & $\gamma * \gamma_{x_2}$ rep

$\gamma_1 * \bar{\gamma}_1$ & γ_{x_1} rep ? it's not so trivial

$\gamma * \bar{\gamma} = \gamma * \lambda \sim \gamma * 0 = \gamma_{x_1}$

$\lambda \leftleftarrows$

In some Geometry Examples. we saw that homotopy relation is so flexible, so this leads the concept of Path-Homotopy.

We have notation "$\underset{p}{\sim}$"

path-homotopy is a "sub-relation" under the previous homotopic relation.
So we need some check of What we've proved as facts.

~~(1) $(\gamma_1 * \gamma_2) * \gamma_3 \underset{p}{\simeq} \gamma_1 * (\gamma_2 * \gamma_3)$~~

Maybe we need a lemma. "the reparametrization of path is path-homotopy
the proof is the same as before #

with the lemma. we immediately have

(1) $\gamma_1 * (\gamma_2 * \gamma_3) \underset{p}{\simeq} (\gamma_1 * \gamma_2) * \gamma_3$

(2) $\gamma_{x_1} * \gamma_1 \underset{p}{\simeq} \gamma_1 \underset{p}{\simeq} \gamma_1 * \gamma_{x_2}$

(3) $\gamma_1 * \bar{\gamma}_1 \underset{p}{\simeq} \gamma_{x_1}$

(4) $\gamma_1 * (\gamma_2 * \bar{\gamma}_2) \sim \gamma_1$

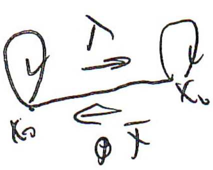
(5) $\gamma_1 \underset{p}{\simeq} \gamma_2 \Rightarrow f \circ \gamma_1 \underset{p}{\simeq} f \circ \gamma_2$.

$f \circ H$ ( )

What's more, We can check the well-definess of our multiplication and inverse
so we properly define multiplication and inverse on the homotopic class

Now, we consider the class with a basepoint. and in fact we have construct a group structure on our class $\pi_1(X, x_0) = \Omega(X, x_0) / \underset{p}{\simeq}$
if x_0, x_1 lies in the same path-component. then the Group is not dependent the choice of basepoint.

λ is a path from x_0 to x_1



we construct the Group ~~homo~~ isomorphism P_λ below

$[\gamma] \longrightarrow [\bar{\lambda} * \gamma * \lambda]$

~~first~~. Γ_λ is a ~~iso~~morphism.
homo

$$\Gamma_\lambda([\gamma_1]_p[\gamma_2]_p) = \Gamma_\lambda([\gamma_1 * \gamma_2]_p) = [\bar{\lambda} * \gamma_1 * \gamma_2 * \lambda]_p$$

$$= [\bar{\lambda} * \gamma_1 * \lambda * \bar{\lambda} * \gamma_2 * \lambda]_p = [\bar{\lambda} * \gamma_1 * \lambda]_p [\bar{\lambda} * \gamma_2 * \lambda]_p = \Gamma_\lambda([\gamma_1]_p)\,\Gamma([\gamma_2]_p)_q$$

suppose $\Gamma_{\bar{\lambda}}([\tilde{\gamma}]_p) = [\bar{\bar{\lambda}} * \tilde{\gamma} * \bar{\lambda}]_p = [\lambda * \tilde{\gamma}\,\bar{\lambda}]_p = (\Gamma_\lambda)^{-1}([\tilde{\gamma}]_p)$. #

the functor property of induced homomorphism.

$f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$

~~content~~ calculate: $\gamma_1 * \gamma_2 = \gamma(s) \begin{cases} \gamma_1(2s) & s \in [0, \frac{1}{2}] \\ \gamma_2(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$.

$$f \circ \gamma = \begin{cases} f \circ \gamma_1(2s) & s \in [0, \frac{1}{2}] \\ f \circ \gamma_2(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} = (f \circ \gamma_1) * (f \circ \gamma_2).$$

so $f_*([\gamma_1]_p[\gamma_2]_p) = f_*([\gamma_1 * \gamma_2]_p) = [f \circ (\gamma_1 * \gamma_2)]_p$

$= [(f \circ \gamma_1) * (f \circ \gamma_2)]_p = [f \circ \gamma_1]_p \cdot [f \circ \gamma_2]_p$

$= f_*([\gamma_1]_p) \cdot f([\gamma_2]_p)$.

We call "low-star" a functor since

(a) $(Id_X)_*$ — —

Group

(b). $(g f)_* = g_* \circ f_*$ #

p-topo

last time path homotopy $\gamma_1 \underset{p}{\simeq} \gamma_2$

$\Downarrow$

groupoid $\rightsquigarrow$ fundamental group $\pi_1(X, x_0) = \Omega(X, x_0)/\underset{p}{\sim}$

partial defined multiplication $\longrightarrow$ multiplication

Basic Setting: X is path connected.

— $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ (depend on $\underline{\text{the path}}$ from x_0 to x_1) $\lambda \longrightarrow \Gamma_\lambda$.

isomorphism class $\rightsquigarrow \pi_1(X)$

— $X \sim Y \Rightarrow \pi_1(X) \simeq \pi_1(Y)$

functor π_1: $\mathcal{T}op \longrightarrow \mathcal{G}roup$

$(X, x_0) \longrightarrow \pi_1(X, x_0)$

$f: (X, x_0) \to (Y, y_0) \rightsquigarrow f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

check: "functorial" $[\gamma]_p \to [f\circ\gamma]_p$.

Today $\pi_1(S^n) \simeq \begin{cases} \{e\} & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases} \rightsquigarrow$ Cor. $\pi_1(\mathbb{R}^{n+1} \setminus \{0\}) \simeq \begin{cases} \{e\} & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$

since $\mathbb{R}^{n+1} \setminus \{0\} \sim S^n$.

$n \geq 2$. prop‖ Suppose $X = U \cup V$. $U, V, U \cap V$ are path connected if U, V are "simply connected" $\Longrightarrow$ X is simply connected.

Cor. $\pi_1(S^n) \simeq \{e\}$, $n \geq 2$ since $S^n = (S^n - \{0, \cdots 1\}) \cup (S^n - \{0, \cdots -1\})$.

proof. take $x_0 \in U \cap V$, fix $\gamma: [0,1] \to U \cup V$

$\gamma \in \Omega(X, x_0)$

$\Rightarrow \{\gamma^{-1}(U), \gamma^{-1}(V)\}$ is open covering of $U \cup V$.

since $[0,1]$ is cpt metric space $\longrightarrow [0,1]$ has lebesgue property

$\exists \delta > 0$. $\Rightarrow \exists\, 0 = t_0 < t_1 < \cdots < t_n = 1$ s.t.

$\gamma([t_i, t_{i+1}]) \subseteq U$ or V. choose λ_i is a path from x_0 to $\gamma(t_i)$.

if $\gamma(t_i) \in U \cap V$, $\lambda_i: [0,1] \to U \cap V$.

$\gamma(t_i) \in \underset{V}{U}$ $\lambda_i: [0,1] \longrightarrow \underset{V.}{U}$

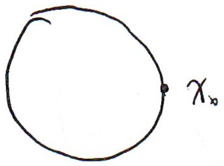
$\Rightarrow \gamma \underset{p}{\simeq} \gamma_1 * \cdots * \gamma_n \underset{p}{\simeq} \gamma_1 * \bar{\lambda}_1 * \lambda_1 \cdots \underset{p}{\simeq} \gamma_{x_0}$ #

② $\pi_1(S^1) \simeq \mathbb{Z}$ $\boxed{S^1 \subseteq \mathbb{C}}$ $\mathbb{R}/\mathbb{Z} \simeq S^1$

$$\boxed{\begin{aligned} p: \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto e^{i2\pi t} \end{aligned}}$$

$$\begin{aligned} \gamma_n = [0,1] &\longrightarrow S^1 \\ t &\longmapsto e^{i2\pi n t} \end{aligned}$$

take $x_0 = 1$



consider
$$\boxed{\begin{aligned} \Phi: (\mathbb{Z}, +) &\longrightarrow \pi_1(S^1, x_0) \\ n &\longmapsto [\gamma_n]_p \end{aligned}}$$

prop. Φ is an isomorphism

Cor $\pi_1(X, x_0) \simeq \mathbb{Z}$ and it has two generators $[\gamma_1]_p$, $[\gamma_{-1}]_p$

Step 1. Φ is a group homomorphism.

i.e. $\underset{[\gamma_{m+n}]_p}{\Phi(m+n)} = \underset{[\gamma_m * \gamma_n]_p}{\Phi(m) \cdot \Phi(n)}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\exists \tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

$\gamma = p \circ \tilde{\gamma}$

$\tilde{\gamma}$ is called "a lifting of γ"

prop (path lifting property)

Any path $\gamma: [0,1] \to S^1$ with $\gamma(0) = x_0$, has a unique lifting $\tilde{\gamma}: [0,1] \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$.

Ex. $\boxed{\tilde{\gamma}_n(t) = nt}$

Consider transform T_m $x \longmapsto x+m$

$\Rightarrow \tilde{\gamma}_{m+n} \underset{p}{\sim} \tilde{\gamma}_m * (T_m \circ \tilde{\gamma}_n)$

$\downarrow p$

$\gamma_{m+n} \underset{p}{\sim} \gamma_m * \gamma_n \Rightarrow \Phi(m+n) = \Phi(m) \cdot \Phi(n)$

Step 2. Φ is surjective

take Any $\gamma \in \Omega(X, x_0)$. By lifting lemma, $\exists!\ \tilde{\gamma}: [0,1] \to \mathbb{R}$, $p \circ \tilde{\gamma} = \gamma$, $\tilde{\gamma}(0) = 0$

$\Rightarrow p \circ \tilde{\gamma}(1) = \gamma(1) = x_0$

$\Rightarrow \tilde{\gamma}(1) \in p^{-1}(x_0) \Rightarrow \tilde{\gamma}(1) \overset{=n}{\in} \mathbb{Z}$

since $\tilde{\gamma} \underset{p}{\sim} \tilde{\gamma}_n$ in $\mathbb{R}$

$\downarrow p$

$\gamma \underset{p}{\sim} p \circ \tilde{\gamma}_n = \gamma_n$

$\Rightarrow [\gamma]_p = [\gamma_n]_p = \Phi(n)$

Step 3. Φ is injective

prop. (Homotopy lifting) (1) Any homotopy $F: [0,1]\times[0,1] \longrightarrow S^1$ with $F(s,0)\equiv x_0$ has a unique lifting $\tilde{F}: [0,1]\times[0,1] \longrightarrow \mathbb{R}$ with $\tilde{F}(s,0)\equiv 0$ s.t. $p\circ\tilde{F} = F$.

(2) if F is a path homotopy ($F(s,1)\equiv x_1$). then $\tilde{F}(s,1) = x$ for some $x \in p^{-1}(x_0)$

suppose $\Phi(n) = \Phi(m)$ i.e. $\gamma_n \overset{p}{\simeq} \gamma_m$

$F \xrightarrow{\text{life}} \tilde{F}$ s.t $p\circ\tilde{F} = F$.

path life $\begin{cases} \tilde{F}(0,t) = \tilde{\gamma}_n(t) \\ \tilde{F}(1,t) = \tilde{\gamma}_m(t) \end{cases} \Longrightarrow \tilde{F}(0,1) = \tilde{F}(0,1) \Rightarrow \tilde{\gamma}_m(1) = \tilde{\gamma}_n(1)$

$\Longrightarrow \tilde{\gamma}_m \simeq_p \tilde{\gamma}_n \Longrightarrow \gamma_m \simeq_p \gamma_n$ #

Last time. 1. $\pi_1(S^n) = \{e\}$ $n\geq 2$. lebesgue number $\rightsquigarrow$ Van kampen theorem

2. $\pi_1(S^1) \simeq \mathbb{Z}$ lifting lemma $\rightsquigarrow$ Covering space

$\Phi: \mathbb{Z} \longrightarrow \pi_1(S^1)$ lifting $\begin{cases}\text{path lifting} & \text{sur} \\ \text{homotopy lifting} & \text{in}\end{cases}$

homomorphism easy

---

lifting lemma for S^1 let $F: P\times I \longrightarrow S^1$ be cts P is a topological space

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{F}}{\nearrow} & \downarrow p \\ P\times I & \xrightarrow{F} & S^1 \end{array}$$

condition
$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\exists \tilde{F}_0}{\nearrow} & \downarrow \\ P\times\{0\} & \xrightarrow{F|_{\{0\}}} & S^1 \end{array} \Longrightarrow$$

conclusion
$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\exists ! \tilde{F}}{\nearrow} & \downarrow \\ P\times I & \xrightarrow{F} & S^1 \end{array}$$
s.t ① $\tilde{F}|_{P\times I} = \tilde{F}_0$ ② $p\circ\tilde{F} = F$

$\rightsquigarrow$ Special case 1. $P = \{pt\}$
$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow & \downarrow \\ 0 & \rightarrow & S^1 \end{array} \quad \begin{array}{c} \tilde{x} \\ \downarrow \\ x_0 \end{array} \Rightarrow \begin{array}{ccc} & & \mathbb{R} \\ & \overset{\exists !}{\nearrow} & \downarrow \\ I & \longrightarrow & S^1 \end{array}$$

Case 2. $P =$ Interval $I \Longrightarrow$ 1st part of homotopy lifting

suppose F is path homotopy

$p\circ\tilde{F}(s,1) = F(s,1) = \cancel{x_0}\ x_0$. $\underbrace{\boxed{\underset{\text{connected}}{\tilde{F}(s,1)}}}_{\text{is a point.}} \in \underline{p^{-1}(x_0)}$ totally disconnected

Idea: Solve equation $\underline{p} \circ \tilde{F} = \underline{F}$ when p is invertible, we can do sth.

proof: Denote $\hat{U}_1 = S^1 \setminus \{1\}$ $\hat{U}_2 = S^1 \setminus \{-1\}$

$V_j^1 = (j, j+1)$ $V_j^2 = (j-\frac{1}{2}, j+\frac{1}{2})$ $P_j^i = p|_{V_j^i}$

Then ① $P_j^i : V_j^i \to \hat{U}_i$ is a homeomorphism

② $p^{-1}(\hat{U}_i) = \bigcup_{j \in \mathbb{Z}} V_j^i$ disjoint union.

Step 1. existense. near $s_0 \in P$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ V_{x_0} \times I \arrow[ur, "\exists \tilde{F}_{x_0}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

✓

By lebesgue lemma $\exists\ 0 = t_0 < t_1 < \cdots < t_{n+1} = 1$

s.t $F(\{x_0\} \times [t_i, t_{i+1}]) \subseteq U_i = \begin{cases} \hat{U}_1 \\ \hat{U}_2 \end{cases}$

relabel $p_j^i : V_j^i \to U_i$

$0 \le i \le n$

$j \in \mathbb{Z}$

Tube lemma $\Longrightarrow$ ~~s.t. $\{x_0\} \times [t_i, t_{i+1}]$~~

$\exists V_0$, $F(V_0 \times [t_i, t_{i+1}]) \subseteq U_i$

finite intersection

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ V_0 \times \{0\} \arrow[ur, "\tilde{F}_0"] \arrow[r, "F_0"'] & S^1 \end{tikzcd} \Rightarrow \exists \begin{tikzcd} & \mathbb{R} \arrow[d] \\ V_1 \times [0, t_1] \arrow[ur] \arrow[r] & S^1 \end{tikzcd}$$

$V_1 \subseteq V_0$

since $p \circ \tilde{F}_0(s_0, 0) = F_0(s_0, 0) = F(s_0, 0) \in U_0$

$\Longrightarrow \exists j$ s.t $\tilde{F}_0(s_0, 0) \in V_j^0$

$\Longrightarrow \exists V_1 \subseteq V_0$. s.t $\tilde{F}_0(V_1, 0) \subseteq V_j^0$

Define $\tilde{F}_1 = (P_j^0)^{-1} \circ F : \underbrace{V_1 \times [0, t_1]}_{U_0} \longrightarrow V_j^0$ is cts

Then $\tilde{F}_1|_{V_1 \times \{0\}} = (P_j^0)^{-1} \circ F|_{V_1 \times [0, t_1]}$

$p \circ \tilde{F}_1|_{V_1 \times \{0\}} = p \circ \tilde{F}_0|_{V_1 \times \{0\}}$

$\|$ $\|$

$F|_{V_1 \times \{0\}} = F_0|_{V_1 \times \{0\}}$

and $\tilde{F}_1(V_1 \times [0, t_1]) \subseteq V_j^0$

$\tilde{F}_0(\forall \quad) \subseteq V_j^0$

Commutive

$p \circ \tilde{F}_1 = F|_{V_1 \times [0, t_1]}$ is easy

Now

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ V_0 \times \{0\} \arrow[ur, "\tilde{F}_0"] \arrow[r, "F_0"'] & S^1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ V_1 \times [0, t_1] \arrow[ur, "\tilde{F}_1"] \arrow[r, "F_1"'] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ V_1 \times \{t_1\} \arrow[ur, "\tilde{F}_1"] \arrow[r, "F_1"'] & S^1 \end{tikzcd} \longrightarrow \begin{tikzcd} & \mathbb{R} \arrow[d] \\ V_2 \times [t_1, t_2] \arrow[ur, "\tilde{F}_2"] \arrow[r, "F_2"'] & S^1 \end{tikzcd}$$

$\boxed{\text{paste!}}$

Step 2. if $P = \{pt\}$. Existence ✓.

uniqueness: $\tilde{F}_1, \tilde{F}_2$ are both $I \longrightarrow \mathbb{R}$.

Consider ① $S = \{t \mid \tilde{F}_1(t) = \tilde{F}_2(t)\}$ $0 \in S \Rightarrow S \neq \phi$

② S is closed $\{\tilde{F}_1 - \tilde{F}_2 = 0\}$

suppose $\tilde{F}_1(t_0) = \tilde{F}_2(t_0)$ wlog $\tilde{F}_1(t_0) \in V_j^i$

$\Rightarrow \exists\, t_0 \in T_0$ s.t $\tilde{F}_1(T_0), \tilde{F}_2(T_0) \subseteq V_j^i$

$\Rightarrow P_j^i \circ \tilde{F}_1 (T_0) = P_j^i \tilde{F}_2 \Rightarrow T_0 \subseteq S$

$\Rightarrow$ ③ S is open $\rightsquigarrow \tilde{F}_1 = \tilde{F}_2$

Final Step. general P.

By Step 1. for x. $\exists\, \tilde{F}_x : V_x \longrightarrow \mathbb{R}$.

If $x_0 \in V_{x_1} \cap V_{x_2}$. use Step 2. then $F_{x_1}(\{x_0\} \times I) = F_{x_2}(\{x\} \times I)$.

Some applications

1. $\mathbb{R}^2 \not\cong \mathbb{R}^n$ $n > 2$.

2. $\pi_1(S^1 \times S^2) \not\cong \pi_1(S^3)$

3. $\pi_1(\text{mobius}) \cong \mathbb{Z}$

def (retract). $\iota: A \hookrightarrow X$ inclusion

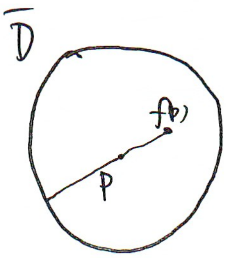
$r: X \longrightarrow A$. $r \circ \iota = Id_A$ $\qquad$ r is a retraction

A is a retract of X

$\Rightarrow \underset{\substack{\Downarrow \\ \text{sur}}}{r_*} \circ \underset{\substack{\Downarrow \\ \text{in}}}{\iota_*} = (Id_A)_* = \underline{\underline{Id_\pi}}$

Cor No retract from $\bar{D}^2$ to S^1

$n=2$. Brouwer



if $f(x)\neq x$

$\overline{f(p)p}$ is cts $\Longrightarrow$ $\exists$ retraction ↯

#

FTA. $p(z)\neq 0 \quad \forall z.$ $\quad a_0\neq 0$

$f: S^1\to S^1 \quad z\mapsto \frac{p(z)}{|p(z)|}$

① $f\sim$ Const $\qquad$ ② $f\sim z^n$

$F(t,z)=\frac{p(tz)}{|p(tz)|} \qquad G(t,z)=\frac{z^n+t\,a_{n-1}z^{n-1}+\cdots+t^n}{|\qquad\qquad|}\left(=\frac{p(\frac{z}{t})}{|p(\frac{z}{t})|}\right)$

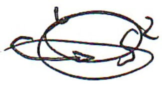
① $\Longrightarrow$ $f_*(m)=0 \quad \forall m \qquad$ ② $f_*(1)=n$.

#

BU. $f: S^n\to\mathbb{R}^n \quad \exists x_0 \ f(x_0)=f(-x_0)$

$n=1$ trivial

$n=2 \qquad g(x)=\frac{f(x)-f(-x)}{|f(x)-f(-x)|} \qquad S^2\to S^1$



$\boxed{g(-x)=-g(x)} \qquad g\circ\iota: S^1\to S^1 \qquad h(-x)=h(x)$

Fact $h_*(m)=0$

$\boxed{S^1\hookrightarrow S^2\to S^1}$

$\exists$ lifting $\tilde{h}: S^1\to\mathbb{R}$. $\Longrightarrow$ $p\circ\tilde{h}(-x) = -p\circ\tilde{h}(x)$

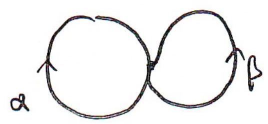
$\Longrightarrow$ $\hat{h}(-x) - \hat{h}(x)=\boxed{m}+\frac{1}{2}$

fix it!

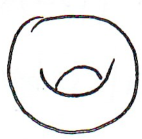
Last week $\pi_1(S^n) = \begin{cases} \mathbb{Z} & n=1 \leftarrow \text{Covering} \\ \{e\} & n \geq 2 \leftarrow \end{cases}$

Compare two sets and their fundamental groups

$S^1 \vee S^1$ $\qquad\qquad$ $S^1 \times S^1$



Not Abel



Abel

trivial

$S^1 \hookrightarrow S^1 \times S^1$

$[\alpha]_p^m [\alpha_p]^{-n} = [\alpha]_p^{m-n}$ $\xrightarrow{\text{should be}}$ $[\alpha]^{m_1}[\beta]^{n_1} \cdots [\alpha]^{m_k}[\beta]^{n_k}$

Group Theory - free group

Any set $S \rightsquigarrow$ Free Group $\langle S \rangle$

- elements: words $s_1 \cdots s_m$ $\quad s_i \in S \cup S^{-1}$
- operation: connect words ~~with trivial reduce~~
  - cancelation $tss^{-1}t = tt$
  - identity $s^{-1}s = \phi$
  - inverse $(s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1}$

$\pi_1(S^1 \vee S^1, p) = \langle [\alpha]_p, [\beta]_p \rangle$.
(we haven't prove it)

More generally. $X = U \cup V$, $U, V, U \cap V$ are path-connected $p = U \cap V$. then what's the relation between $\pi_1(X,p)$ and $\pi_1(U,p)$ $\pi_1(V,p)$

$G * H$ denote the "free product" of Groups.

G, H are groups.

- elements $s_1 \cdots s_m$ $\quad s_i \in G$ or H
- operations "connect words" operations in G & H"

Group Homomorphism $\Phi: \pi_1(U,p) * \pi_1(V,p) \longrightarrow \pi_1(X,p)$ | by the definition of Fundamental Group

use lebesgue property $\Rightarrow \Phi$ is surjective

$\Rightarrow \pi_1(X,p) \simeq \pi_1(U,p) * \pi_1(V,p) \big/ \ker \Phi$

Note. a path can lie in $U \cap V$ totally, but its preimage in the free product should be two elements without any relation ...

Notation

$\iota_1 : U \hookrightarrow X$ $\quad r \in \Omega(U\cap V, p)$

$\iota_2 : V \hookrightarrow X$

$\iota_{12} : U\cap V \hookrightarrow U$ $\quad (\iota_1)_*(\iota_{12})_*[r]_p = (\iota_2)_*(\iota_{21})_*[r]_p$

$\iota_{21} : U\cap V \hookrightarrow V.$

$\Rightarrow$ ① $(\iota_{12})_*[r]_p * ((\iota_{21})_*[r]_p)^{-1} \in \ker \Phi.$

② ~~for~~ Φ is normal subgroub

In general G, H, F groups $\begin{cases} \varphi : F \to G \\ \psi : F \to H \end{cases}$ group homomorphism.

N = the smallest normal subgroup of $G*H$ that contains all elements $\varphi(s)\psi(s)^{-1}$ $s \in F$

$$G *_F H = G*H/N$$

presentation of group. $G \rightsquigarrow \langle G \rangle$

$\varphi : \langle G \rangle \longrightarrow G$ is a sur homo

$s_1 \cdots s_m \mapsto s_1 \cdots s_m$

$\Rightarrow \langle G \rangle / \ker\varphi \simeq G$

Let S is a generation set of G $\Rightarrow \langle G \rangle = \langle S \rangle$

$G = \langle S_1 | R_1 \rangle$ $\quad H = \langle S_2 | R_2 \rangle \rightsquigarrow G*H = \langle S_1, S_2 | R_1, R_2 \rangle.$

Now $G *_F H = \langle S_1, S_2 | R_1, R_2, \text{"}\underset{\varphi(s)\psi(s)^{-1}}{F}\text{"} \rangle$

or $F = \langle S_3 | R_3 \rangle$ $\quad G *_F H = \langle S_1, S_2 | R_1, R_2, \tilde{S}_3 \rangle$

$\tilde{S}_3 = \{\varphi(s)\psi(s)^{-1} \mid s \in S_3\}.$

e.g. $\mathbb{Z}\times\mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

$\mathbb{Z}*\mathbb{Z} = \langle a, b \rangle.$

$G *_F \{e\} = G/N$

Van Kampen Theorem

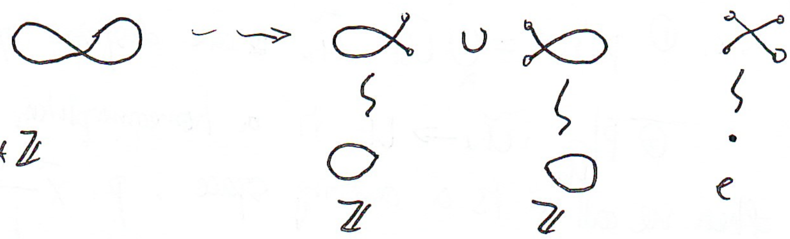
suppose $X = U \cup V$, $U, V, U\cap V$ open, path connected $p \in U\cap V$

Then $\pi_1(X, p) \simeq \pi_1(U, p) *_{\pi_1(U\cap V)} \pi_1(V, p) = \pi_1(U, p) * \pi_1(V, p)/N$

i.e. $\ker\Phi$ = smallest normal subgroup generated by $((\iota_{12})_*[r]_p) * ((\iota_{21})_*[r]_p)^{-1}$

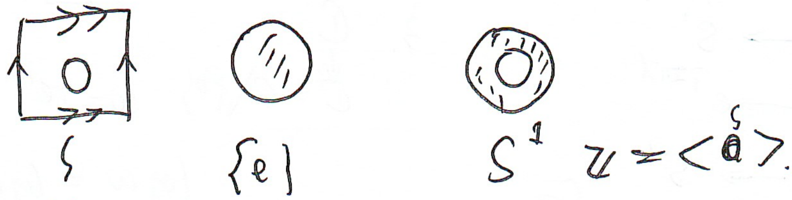
Application of Van Kampen Theorem

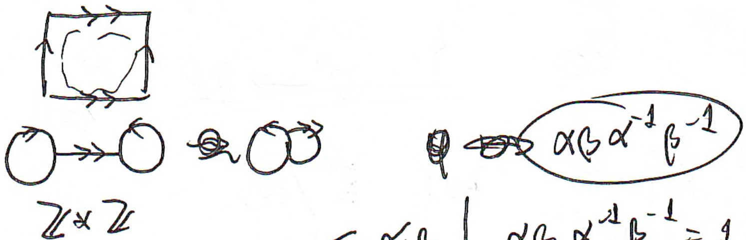
① $X = S^1 \vee S^1$



$\Rightarrow \pi_1(X, p) = \mathbb{Z} * \mathbb{Z}$

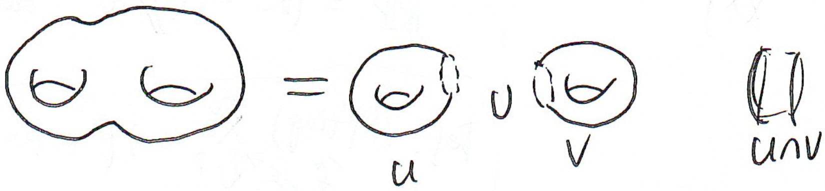
② $X = \mathbb{T}^2 = S^1 \times S^1$





$\langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle$

③ $X =$



$\mathbb{Z} * \mathbb{Z}$ $\quad$ $\mathbb{Z} * \mathbb{Z}$ $\quad$ $\langle s \rangle$.

$\langle \alpha_1, \beta_1 \rangle$ $\quad$ $\langle \alpha_2, \beta_2 \rangle$

$\pi_1(U \cap V) \longrightarrow \pi_1(U)$ $\qquad$ $\pi_1(U \cap V) \longrightarrow \pi_1(V)$

$s \mapsto \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} = s \mapsto \beta_2\alpha_2\beta_2^{-1}\alpha_2^{-1}$

$\Rightarrow \pi_1(\Sigma) = \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} = e \rangle$

Def. let $X, \tilde{X}$ be p.c. topological space $p: \tilde{X} \to X$ cts. suppose $x \in X$, $\exists U \ni x$.
s.t. ① $p^{-1}(U) = \bigcup_\alpha \tilde{U}_\alpha$, $\tilde{U}_\alpha$ are disjoint open sets in $\tilde{X}$
② $p|_{\tilde{U}_\alpha}: \tilde{U}_\alpha \to U$ is a homeomorphism
then we call $\tilde{X}$ is a covering space, $p: \tilde{X} \to X$ is a covering space
$p^{-1}(X_0)$ is the fiber at x_0.

Rmk. 1. if X is not path-connected, let X_1 is a path connected component of X
then $p: p^{-1}(X_1) \to X_1$ is a covering map
2. if X p.c. but $\tilde{X}$, take $\tilde{X}_1 = $ p.c.c of $\tilde{X}$ $\Rightarrow p|_{\tilde{X}_1}$ is covering map

e.g. 1. $\mathbb{R} \to S^1$
$x \mapsto e^{i2\pi x}$

2. $S^1 \to S^1$
$n \neq 0$. $e^{i\theta} \mapsto e^{in\theta}$
$|n|$ sheet covering

3. $\mathbb{C} \xrightarrow{z \mapsto e^z} \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ $\omega = re^{i\theta} = e^a e^{ib}$

$\log \omega = \log r + (2\pi k + \theta) i$

4. $S^n \to \mathbb{RP}^n = S^n / \{\pm 1\}$ double ~~sheet~~ covering

BUT $\mathbb{R}^{n+1} / \{0\}$
$\downarrow p$
$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\}) / \sim$
is not a covering
considering dimension

5. $\tilde{X} \xrightarrow{p_1} X$, $\tilde{Y} \xrightarrow{p_2} Y$ $\Rightarrow$ $\tilde{X} \times \tilde{Y} \xrightarrow{p_1 \times p_2} X \times Y$

6. $\Sigma_{k(g-1)+1} \to \Sigma_g$

Def. (lifting)
\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}
if: $p: \tilde{X} \to X$ is a covering map. $f: Y \to X$ cts
we say $\tilde{f}: Y \to \tilde{X}$ cts is lifting of f,
if the diagram can commute ($p \circ \tilde{f} = f$)

lifting with basepoint
\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}
use connection argument

prop. let Y is connected, then "lifting with basepoint" is unique

Rmk. at most one, maybe not exist ...

proof. if we have two liftings. $\tilde{f}_1, \tilde{f}_2$. s.t.
let $Y_0 = \{y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$. $\Rightarrow y_0 \in Y$
② if $y \notin Y_0 \Rightarrow \tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \exists \alpha \neq \beta$. $\tilde{f}_1(y) \in \tilde{U}_\alpha$, $\tilde{f}_2(y) \in \tilde{U}_\beta$
$(\tilde{f}_1(y), \tilde{f}_2(y) \in p^{-1}(f(y)))$ $\subseteq U$
since $\tilde{f}_i$ is cts
$\overset{\exists V}{\Rightarrow} y \in V \notin Y_0 \Rightarrow Y_0$ is closed
③ if $y \in Y_0 \Rightarrow \tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{f}_1(y), \tilde{f}_2(y) \in \tilde{U}_\alpha$
$\overset{\exists V}{\Rightarrow} \tilde{f}_1(V), \tilde{f}_2(V) \subseteq \tilde{U}_\alpha \Rightarrow p \circ \tilde{f}_1 = p \circ \tilde{f}_2$
but $p|_{\tilde{U}_\alpha}: U_\alpha \to U$ is homeo
$\Rightarrow \tilde{f}_1 = \tilde{f}_2$ on V #

last time we use the same conditions to prove the lifting lemma for $\mathbb{R} \to S^1$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ P\times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\, \tilde{F}"] & X \end{tikzcd} \quad \leftarrow \text{We want} \quad \Longleftarrow \quad \text{suppose } \exists\, \tilde{F}_0 \quad \begin{tikzcd} & \tilde{X} \arrow[d] \\ P\times\{0\} \arrow[r, "F_0 = F|_{P\times\{0\}}"'] \arrow[ur] & X \end{tikzcd}$$

s.t. $\tilde{F}|_{P\times\{0\}} = \tilde{F}_0$

Cor path lifting $P = \{0\}$

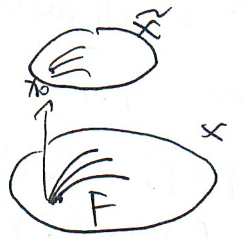
$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (I, 0) \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists!\, \tilde{\gamma}"] & (X, x_0) \end{tikzcd}$$

Cor homotopy lifting. $P =$ Interval I.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (I\times I, I\times\{0\}) \arrow[r, "F"'] \arrow[ur, "\exists!\, \tilde{F}"] & (X, x_0) \end{tikzcd}$$

Cor path homotopy lifting

$\boxed{\text{use path lifting}}$.



prop. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective

proof Suppose $p_*([\tilde{\gamma}]_p) = e$ in $\pi_1(X, x_0)$

$$[p\circ\tilde{\gamma}]_p \Rightarrow p\circ\tilde{\gamma} \underset{p}{\overset{F}{\simeq}} \gamma_{x_0}$$

$$\Rightarrow \tilde{\gamma} \overset{\tilde{F}}{\simeq} \gamma_{\tilde{x}_0}$$

$$\Rightarrow [\tilde{\gamma}]_p = e \in \pi_1(\tilde{X}, \tilde{x}_0) \quad \square$$

In general, the lifting $\tilde{\gamma}$ of ~~a loop~~ $\gamma \in \Omega(X, x_0)$ is a path starting at $\tilde{x}_0$, need not to be a loop   Q: loop lifting?

prop. The lifting $\tilde{\gamma}$ of $\gamma \in \Omega(X, x_0)$ (with starting point $\tilde{x}_0$). is in $\Omega(\tilde{X}, \tilde{x}_0)$

$\Longleftrightarrow [\gamma]_p \in p_*(\pi_1(\tilde{X}, x_0))$

proof $(\Leftarrow)$ suppose $[\gamma]_p \in p_*(\pi_1(\tilde{X}. x_0))$ $\left(\gamma \underset{p}{\overset{F}{\simeq}} p\circ\tilde{\gamma}_1\right)$

i.e. $[p\circ\tilde{\gamma}_1]_p = p_*([\tilde{\gamma}_1]_p)$

$\Rightarrow \tilde{F}$ is path homotopy : $\tilde{\gamma} \underset{p}{\overset{\tilde{F}}{\simeq}} \tilde{\gamma}_1 \Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}_1(1) = \tilde{x}_0$

$\Rightarrow \tilde{\gamma}_0 \in \Omega(\tilde{X}, \tilde{x}_0)$

$(\Rightarrow)$. $[\tilde{\gamma}]_p \in \pi_1(\tilde{X}, \tilde{x}_0)$ $[p\circ\tilde{\gamma}]_p \overset{p_*}{\in} \pi_1(\tilde{X}, x_0)$ $\square$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad [\gamma]_p$

Q

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\exists ? \tilde{f}"] & (X, x_0) \end{tikzcd}$$

$\forall \gamma \in \Omega(Y, y_0)$

$f \circ \gamma \in \Omega(X, x_0)$

$\cancel{p}\, \tilde{f} \circ \gamma \in \Omega(\tilde{X}, \tilde{x}_0)$

by condition $p \circ \tilde{f} \circ \gamma = f \circ \gamma$

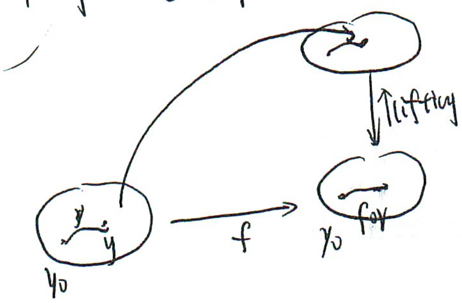
$\Rightarrow \underset{\text{in } Y}{f_*([\gamma]_p)} = \underset{\text{in } \tilde{X}}{p_*([\tilde{f} \circ \gamma]_p)}$

for γ is arbitrary

$\Rightarrow \boxed{f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi(\tilde{X}, \tilde{x}_0))}$

Thm. Y is p.c. locally p.c. then $\tilde{f}$ exists $\Leftrightarrow f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

proof. ① Define $\tilde{f}$ $\boxed{\text{let } \tilde{f}(y) = \tilde{f \circ \gamma}(1)}$



② $\tilde{f}$ is well defined.

suppose $\gamma, \gamma' \in \Omega(Y, y_0, y)$

$\Rightarrow \gamma * \bar{\gamma}' \in \Omega(Y, y_0)$ condition

$\Rightarrow (f \circ \gamma) * (f * \bar{\gamma}') \in \Omega(X, x_0)$

$[(f \circ \gamma) * (f * \bar{\gamma}')]_p = f_*([\gamma * \bar{\gamma}']_p) \subseteq p_* \cdots$

$\tilde{f \circ \gamma}(1) = \tilde{f * \gamma'}(1)$

③ $\tilde{f}$ is cts. V p.c. $V \subseteq f^{-1}(U)$ $p_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} U$

$\tilde{f} = p_\alpha^{-1} \circ f$ on V

$\Rightarrow \tilde{f}$ cts on V $\square$

Cor $n \geq 2$. $f : S^n \to S^1$ is null homotopic.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \underset{\text{s.c}}{S^n} \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & S^1 \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R} \arrow[d] \\ \underset{\mathbb{Z}_2}{\mathbb{RP}^2} \arrow[r] \arrow[ur] & \underset{\mathbb{Z}}{S^1} \end{tikzcd}$$

Recall. $p: \mathbb{R} \to S^1 \rightsquigarrow \pi_1(S^1) = \mathbb{Z}$

In general, consider covering map with base point.

$$p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$$

We can define a map $\alpha: \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$ (α depend on the choice of $\tilde{x}_0$)

$[\gamma]_p \longmapsto \tilde{\gamma}(1)$ path lifting uniqueness

prop. (1) α is well-defined

(2) $\tilde{X}$ path connected, α is a surjective

(3) $\tilde{X}$ is simply connected, α is a bijective.

proof: (1) suppose $\gamma \underset{p}{\simeq} \gamma'$, since homotopy lifting lemma, and the uniqueness of path lifting

$$\tilde{\gamma} \underset{p}{\simeq} \tilde{\gamma}' \Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}'(1)$$

(2) $\forall \tilde{x}_1 \in p^{-1}(x_0)$ take $\tilde{\lambda}: \tilde{x}_0 \rightsquigarrow \tilde{x}_1$

then $\lambda = p \circ \tilde{\lambda}$ is a loop, with the uniqueness.

$\tilde{\lambda}$ is λ's lifting $\Rightarrow \alpha([\lambda]_p) = \tilde{\lambda}(1) = \tilde{x}_1$

(3) suppose $\alpha([\gamma_1]_p) = \alpha([\gamma_2]_p) \Rightarrow \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$

$\Rightarrow \tilde{\gamma}_1 * \bar{\tilde{\gamma}}_2 \in \Omega(\tilde{X}, \tilde{x}_0) \Rightarrow \tilde{\gamma}_1 * \bar{\tilde{\gamma}}_2 \underset{p}{\simeq} \gamma_{\tilde{x}_0}$

$\Rightarrow (p \circ \tilde{\gamma}_1) * p \circ (\bar{\tilde{\gamma}}_2) \underset{p}{\simeq} \gamma_{x_0}$

i.e. $\gamma_1 * \bar{\gamma}_2 \underset{p}{\simeq} \gamma_{x_0} \Rightarrow [\gamma_1]_p = [\gamma_2]_p$. $\square$

e.g. $n \geq 2$.

$S^n \xrightarrow{\text{two-sheet}} \mathbb{RP}^n = S^n / \pm 1$

$\Rightarrow |\pi_1(\mathbb{RP}^n)| = 2$

$\Rightarrow \pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

Def. A covering $p: \tilde{X} \to X$ is called a universal covering if $\pi_1(\tilde{X}) = \{e\}$.

Notation: $p: \hat{X} \to X$

Example ① $\mathbb{R} \to S^1$

② $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = T^2$

③ $S^n \to \mathbb{RP}^n$

④ $SU(2) \to SO(3)$ "Dirac belt trick"

⑤ $S^1 \vee S^1$