special covering spaces via group action

group $G \curvearrowright \tilde{X}$ (group action). $X = \tilde{X}/G$

spaces of orbits

Assume the group action is properly discontinuous

i.e. $\forall \tilde{x} \in \tilde{X}\ \exists \tilde{U} \ni \tilde{x}$, s.t $g \cdot \tilde{U} \cap \tilde{U} = \phi \quad \forall \tilde{x} \in \tilde{X}$

$\Rightarrow$ the quotient map. $p: \tilde{X} \longrightarrow \tilde{X}/G = X$ is a covering map

$\alpha: \underbrace{\pi_1(X, x_0) \longrightarrow \underline{\underline{p^{-1}(x_0)}}_{\text{set}}}_{\text{well-defined After choosing a basepoint.}} \longleftrightarrow G$ ~~fix~~ $\tilde{x}_0$ for Any $\tilde{x}_1 \in \boxed{p^{-1}(x_0)}$ ~~in the~~ orbit of $\tilde{x}_0$

$\exists g \cdot \tilde{x}_0 = \tilde{x}_1$

p.d $\rightsquigarrow$ uniqueness

$\beta: \pi_1(X, x_0) \longrightarrow G$. (with a choice of $\tilde{x} \in p^{-1}(x_0)$)

prop. ~~β is a group homomorphism~~ suppose $G \curvearrowright \tilde{X}$ p.d., then for any $\tilde{x}_0 \in p^{-1}(x_0)$ then β is a group homomorphism

pf: suppose $\beta([\gamma_i]_p) = g_i$, $i = 1, 2$. i.e. $\boxed{g_i \cdot \tilde{x}_0 = \tilde{\gamma}_i(1)}$

Then $g_1 \tilde{\gamma}_2$ is a path from $g_1 \cdot \tilde{\gamma}_2(0) = g_1 \cdot \tilde{x}_0 = \tilde{\gamma}_1(1)$

to $g_1 \cdot \tilde{\gamma}_2(1) = g_1 \cdot g_2 \tilde{x}_0$

$\Rightarrow \tilde{\gamma}_1 * (g_1 \cdot \tilde{\gamma}_2) : \tilde{x}_0 \rightsquigarrow \underline{\underline{g_1 \cdot g_2 \tilde{x}}}$

$\widetilde{\gamma_1 * \gamma_2} = \tilde{\gamma}_1 * g \tilde{\gamma}_2$

so $\beta([\gamma_1]_p [\gamma_2]_p) = g_1 \cdot g_2$.

cor. If $\tilde{X}$ s.c. $G \curvearrowright \tilde{X}$ p.d then $\pi_1(X) \simeq G$ (β is the isom)

example. $g = e^{i 2k\pi/p}$ $\quad \mathbb{Z}_p \curvearrowright S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C} \times \mathbb{C}$

lens space

When do we have universal covering

$p: \hat{X} \longrightarrow X$ suppose $\exists$ $P_\alpha : \widetilde{U_\alpha} \xrightarrow{\sim} U$

$x_0 \in U$.

$(\hat{X}, \hat{x}_0)$

$\downarrow p$ $\Rightarrow$ ~~take any $\gamma \in \Omega(X, x_0)$~~ and $\boxed{\gamma \in \Omega(U, x_0)}$

(X, x_0)

$\rightsquigarrow$ lifting $\hat{\gamma} \in \Omega(\widetilde{U_\alpha}, \widetilde{x_0})$ (using homeomorphism)

$\subseteq \Omega(\hat{X}, \widetilde{x_0})$

$\Rightarrow [\hat{\gamma}]_p = \{e\}$ in $\pi_1(\hat{X}, \widetilde{x_0})$

$\overset{p}{\Rightarrow} [\gamma_p] = \{e\}$ in $\pi_1(X, x_0)$

In fact, What we do is that: $\iota: U \hookrightarrow X$

$\iota_*([\gamma]_p^U) = \{e\}$. i.e. $\iota_*(\pi_1(U, x_0)) = \{e\}$

Def. We say X is semi-locally simply connected.

if. $\forall x_0 \in U$. s.t. $\iota_*(\pi_1(U, x_0)) = \{e\}$.

Rmk. "locally ~~path~~ simply connected": Any $V \ni x_0$. $\exists$ open U. $x_0 \in U \subseteq V$

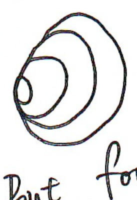
$\pi_1(U, x_0) = \{e\}$

But "semi-..." U can be bad. but the loops in the big space X

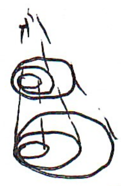
must be contractible.

e.g. Hawaiian earring

is Not SLSC.



But for its cone space



is. SLSC But Not LSC.

Thm: suppose X is path connected, locally path connected. then

X is semi locally simply connected

$\exists \hat{X} \Longleftarrow X$ is semi locally simply connected

How to construct? Idea: $\mathbb{R} \to S^1$

Fix. x_0 $\hat{X} = \{[\gamma]_p \mid \gamma$ is a path with $\gamma(0) = x_0\}$

use condition to construct topo on $\hat{X}$

then ① p is covering map

② $\pi(\hat{X}) = \{e\}$.

Note For Any $H \subset \pi_1(X, x_0)$

$\exists\, p : \tilde{X}_H \longrightarrow X_0$ s.t. $\pi_1(\tilde{X}_H, x_0) = H$

$\tilde{X}_H = \tilde{X}/\tilde{H}$

Today: Brouwer fixpoint theorem

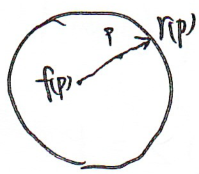
Thm. Any continuous map $f: \overline{B_n} \to \overline{B_n}$ has a fixed point.

Rmk. Any $f \in \mathcal{C}(X,X)$ has a fixedpoint ~ "FPP"

- S^n No FPP $\boxed{SPC}$ $\boxed{cpt}$
- $[0,1)$ ~~not~~ No FPP
- FPP $\not\Rightarrow$ cpt
- cpt + contractible $\not\Rightarrow$ FPP
- X, Y FPP $\not\Rightarrow$ $X \times Y$ FPP.

Though it's a Topological property it's kind of different from the previous ~~topological~~ property.

Recall $n=2$.
① $\pi_1(S^1) \simeq \mathbb{Z}$
② $\pi_1(S^1) \neq \{e\} \Rightarrow \nexists$ retract $r: \overline{D} \to S^1$
③ No retraction $\Longrightarrow$ Any $f: \overline{D} \to \overline{D}$ has fixed point.



$r(p) = p + \lambda(p)(p - f(p))$ $\quad \|r(p)\| = 1$

$\lambda(p) = \cdots$ cts

Observation: the same argument in ③ still holds for $n \geq 2$.

$\Rightarrow$ prop 1 || $\nexists$ Any C^1 retraction $r: \overline{B^n} \to S^{n-1} \Rightarrow$ Any C^1 map $f: \overline{B^n} \to \overline{B^n}$ has a fixed point.

prop 2 || Any C^1 map $f: \overline{B^n} \to \overline{B^n}$ has a fixed point $\Rightarrow$ Any cts $f: \overline{B^n} \to \overline{B^n}$ has a fixed point

prop 3 || $\nexists$ C^1 retraction $r: \overline{B^n} \to S^{n-1}$

proof of prop 2. suppose $f: \overline{B^n} \to \overline{B^n}$ is cts. By SWT. $\exists$ C^1 (polynomial)

$f_\ell: \overline{B^n} \to \mathbb{R}^n$ s.t. $|f(x) - f_\ell(x)| < \frac{1}{\ell}$ $\forall x \in \overline{B}$

let $g_\ell = \frac{\ell}{\ell+1} f_\ell$ then $g_\ell \rightrightarrows f$, $g_\ell: \overline{B} \to \overline{B}$ has FP.

$\exists x_\ell \in \overline{B}$. $g_\ell(x_\ell) = x_\ell$. $\exists x_{\ell_i} \to x_0 \in \overline{B}$

$f(x_0) = \lim g_{\ell_i}(x_{\ell_i}) = \lim x_{\ell_i} = x_0$ □

Why study smooth map? ~ tool: differential

d is a functor

Inverse function thm. If $f: U \to V$ C^1, $(df)_x$ is invertible

then $\exists U_1 \ni x$, s.t. $f: U_1 \to f(U_1)$ is a diffeo.

"local diffeo $\mathbb{R}$"

Cor 1. if $f: U \to V$. $(df)_x$ is invertible for $\forall x \Rightarrow f$ is open

cor 2. if $f: U \to V$ ... and f is bi $\Rightarrow f$ is global diffeo

proof of prop 3. if $\exists$ C^1 retraction $f: \overline{B^n} \to S^{n-1}$

$f_t(x) = (1-t)x + t f(x)$ $\quad$ $Id: \overline{B^n} \to \overline{B^n}$

$= x + t\underbrace{(f(x) - x)}_{g(x)})$

let $F(t) = \int_{\overline{B^n}} \det(df_t)_x \, dx$ $\qquad$ NOTE if f is a diffeo, then $F(t) = Vol(f_t(\overline{B^n}))$

Then Claim A. $F(t)$ is a polynomial

B. $F(1) = 0$

C. $\exists t_0 > 0$ $\forall t \in [0, t_0]$, $F(t) = Vol(\overline{B^n})$

A. $f_t(x) = x + t g(x)$ $\quad (df_t)_x = Id + t\, dg_x$ $\quad \int \det(df_t) =$ polynomial

B. Need "$\det(df_1)_x = 0$ $\quad f_1 = f(x)$ $\quad \langle f_1(x+tv), f_1(x+tv) \rangle = 1$

$\Rightarrow 2\langle (df_1)_x(v), f_1(x) \rangle \quad \frac{d}{dt}\Big|_{t=0} \langle \dots \rangle = 0.$

C. Only need to prove a family of maps are diffeos.

① $\exists t_1$, f_t is injective $t \in [0, t_1]$

local diffeo --

② t_2

③ t_3 surjective

proof of ① Suppose $x_1 \neq x_2$, $f_t(x_1) = f_t(x_2)$ $\quad$ g is lipschitz

$\Downarrow$

$\|x_1 - x_2\| = t\|g(x_1) - g(x_2)\| \le Lt\|x_1 - x_2\|$

$\Rightarrow t \geq \frac{1}{L}$ ↯

② $(df_t)_x = I + t(dg)_x$ $\quad \boxed{t_2 > 0.}$

③ $t \le t_2$ $\quad f: \overline{B^n} \to \mathbb{R}^n$ is open

$f(\overline{B^n})$ open in $\mathbb{R}^n$

$\shortparallel$

G_t

suppose $G_t \neq B^n$

take $y_0 \in \partial G_t \cap B^n$

Take $x_t \in B^n$ s.t. $f_t(x_t) \to y_0$

$\Rightarrow x_{t_i} \to x_0 \in \overline{B^n}$

$\Rightarrow f_t(x_{t_i}) \to f(x_0) \Rightarrow f(x_0) = y_0 \in G_t$

$\overset{\text{injective}}{\Rightarrow} x_0 \in S^{n-1} \Rightarrow f_t(x_0) = x_0 \in S^{n-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\quad y_0 \in B^n$ #

Rmk. suppose $K \subseteq \mathbb{R}^n$ is convex, compact

Then $K \cong \overline{B^m}$ for some $m \leq n$

Brouwer FPT (Version 2) $\emptyset \neq K \subseteq \mathbb{R}^n$ cpt convex ~~then~~ Any cts $f: K \to K$ has a f.p.

Rmk. infinite demension.

For $\ell^2 = \{(a_1, \cdots a_n, \cdots)\}$ $d((a_i),(b_i)) = \sqrt{\sum (a_i - b_i)^2}$

$f((a_i)) \to \left(\sqrt{1 - \|a_i\|^2}, a_1, \cdots a_n, \cdots\right)$

① f is cts ② f has no fixed point.

However. the version 2 can be extend to infinite demension

schauder. $\emptyset \neq K$ cpt. convex in a normed vector space

$f: K \to K$ has F.P.

NOTE. ball is not cpt in ℓ^2.

Some knowledge and detail for before

Contractible space ‖ $(X, \mathcal{T})$, $Id_X \in C(X,X)$ is null homotopic, then we call X is contractible.

Example. - star-shaped Area in $\mathbb{R}^n$ is contractible.

- $C(X)$ (i.e. retract to its top point?)

X is contractible $\Longleftrightarrow$ X is homotopy to {pt}.

Covering space v.s. Group Action

—NOTE: A topological space with G Action ~~is~~ imply the space is compatible with multiplication of G

properly discontinuous ‖ $G \curvearrowright \tilde{X}$, ~~is~~ $\forall x \in \tilde{X}$, $\exists\, \tilde{U} \ni x$. s.t. $\forall g \neq e$, $g \cdot \tilde{U} \cap \tilde{U} = \phi$.

In fact it gives the "basic open sets" lying different sheets.

$$\begin{array}{c} \tilde{X} \\ \downarrow p \\ X = \tilde{X}/G \end{array}$$

the quotient map $p: \tilde{X} \to \tilde{X}/G = X$ is a covering map.

proof. $x \in X$, $\tilde{x} \in p^{-1}(x) \xrightarrow{\text{d.p.}} \tilde{U} \ni \tilde{x}$.

denote $p(\tilde{U}) = U$, $p^{-1}(U) = p^{-1}(p(\tilde{U})) = \cup\, g \cdot \tilde{U}$

$\Rightarrow U$ is open

consider $p|_{\tilde{U}}: \tilde{U} \to U$.

surjective ✓

injective p.d. ✓

p is open $\Rightarrow$ homeo ✓

$\Rightarrow p_g: g \cdot \tilde{U} \to U$ $\square$

So can we regard $g \in G$ as a transform ~~from~~ in different sheets?

universal covering ‖ $p: \tilde{X} \to X$ is a covering map, when $\pi_1(\tilde{X}) = \{e\}$, then $\tilde{X}$ is called universal covering, rewrite as $\hat{X}$

with some caculations, we see

when X has a "universal covering"

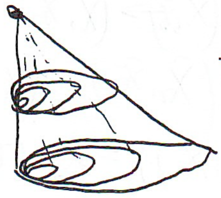
X should be a "semi-locall simply connected space"

$\Downarrow$

$\forall x \in X$. $\exists\, U \ni x$, $i: U \hookrightarrow X$, $i_*(\pi_1(U, x)) = \{e\}$.

Some examples: Hawaiian earring  $\overset{\text{def}}{=} H$

Any U of o contains some circles $\Rightarrow H$ is not "SLSC."

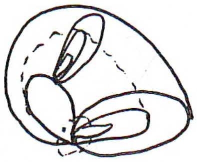
$C(H)$:  ~~the~~ "SLSC" But NOT "locally simply connected"

$\Downarrow$

$\forall V \in \mathcal{N}_x, \exists\, x \in U \subseteq V$

U is simply connected

so the "L_x" is necessary

$C(H)/\sim$ 

The Galois Theory for covering space

$$\begin{tikzcd} \tilde{X}_H \arrow[r, "\pi_1"] \arrow[d] & H \\ (X, x_0) \arrow[r, "\pi_1"] & \pi(X) \arrow[u, phantom, "\subseteq"] \end{tikzcd}$$

"the bigger covering space, the smaller ~~the~~ $\pi_1(\tilde{X})$ to do with?"

Though it satisfy our goal to the theory of "covering space".

Thm. || X is p.c. l.p.c. S.L.S.C. then for any subgroup H, there exists a covering space of X, $p: \tilde{X}_H \to X$ and basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

$$\begin{tikzcd} (\tilde{X}_H, \tilde{x}_0) \arrow[r, squiggly] \arrow[d, "p"] & H \\ (X, x_0) \arrow[r, squiggly] & \pi_1(X, x_0) \arrow[u, phantom, "\subseteq"] \end{tikzcd}$$

the ~~facts~~ method is "to construct a equivalent relation $\underset{H}{\sim}$ on $\tilde{X}$, s.t. $\tilde{X}/_{\underset{H}{\sim}} = \tilde{X}_H$

"$[\gamma]_p \underset{H}{\sim} [\gamma']_p \Leftrightarrow \gamma(1) = \gamma'(1)$ & $[\gamma * \bar{\gamma}']_p \in H$

def || "Isom of Covering space"

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\cong"] \arrow[dr, "p_1"'] & \circlearrowright & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

{ In fact, we want to classify all the covering space of X.

~~the~~ deck transform group of the covering space — $\mathrm{Aut}(p) = \{h: \tilde{X} \to \tilde{X} \mid \text{homeo}\}$

~~$\forall x_0 \in X$.~~ h is bijective from $p_1^{-1}(x_0)$ to $p_2^{-1}(x_0)$. If choose $\tilde{x}_1 \in p^{-1}(x_0)$, $\tilde{x}_2 = h(\tilde{x}_1)$. then ~~be~~ by def.

h is iso. of $\tilde{X}_1, \tilde{X}_2$. $\forall x_0 \in X$, h

$$\begin{tikzcd} (\tilde{X}_1, \tilde{x}_1) \arrow[rr, "\cong"] \arrow[dr, "p_1"'] & \circlearrowright & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

$\Rightarrow p_2 \circ h = p_1 \quad p_2 = p_1 \circ h^{-1}$

$\Rightarrow p_{2*} h_* = p_{1*}, \quad p_{2*} = p_{1*} h_*^{-1}$

$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_2(\pi_2(\tilde{X}_2, \tilde{x}_2))$

So we have " two covering space are isomorphism, then they have the $\overset{\text{same}}{\frown}$ subgroup under p_*

the ~~relation~~ reflection is also sufficial

X is p.c. l.p.c. then two p.c. covering space: $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$, $p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$

have ~~isomorphi~~ homeomorphism with the basepoint $\Longleftrightarrow$ $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

(X, x_0)
$\downarrow p$ if Y is l.p.c.
(Y, y_0) so is X.

$\forall\, x_1 \in X$, $p(x_1) =: y_1 \in Y$

by def. $x_1 \in U_\alpha$.

$p|_{U_\alpha} : U_\alpha \to U \subseteq Y$ open

~~$\forall\, \in U$~~ $\forall\, x_1 \in V \subseteq X$

$\Rightarrow x_1 \in U_\alpha \cap V$

& $p|_{V \cap U_\alpha}$ is also a homeo.

$p(V \cap U_\alpha)$ is open since $V \cap U_\alpha$ is open $\subseteq U_\alpha$.

$$\begin{tikzcd} & \tilde{X}_1 \arrow[d, "p_1"] \\ \tilde{X}_2 \arrow[ur, "\tilde{p}_2"] \arrow[r, "p_2"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ \tilde{X}_1 \arrow[ur, "\tilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd}$$

Now, it's not hard to see the order reserse

Brouwer fixed point Theorem

$n=2$. ① $\pi_1(S^1) \cong \mathbb{Z}$

② $\nexists$ retraction

③ "$\nexists \ldots$" $\Rightarrow$ "fixed point"

$\Downarrow$ $r(p) = p + \lambda(p)(p - f(p))$

$\|r(p)\| = 1 \Rightarrow \lambda^2(p)\|p - f(p)\|^2 + 2p\cdot(p - f(p))\lambda(p) + \|p\|^2 - 1 = 0$

$$\lambda(p) = \frac{-p\cdot(p - f(p)) + \sqrt{(p\cdot(p - f(p)))^2 + \|p - f(p)\|^2\cdot(1 - \|p\|^2)}}{\|p - f(p)\|^2}$$ is smooth enough and has nothing to do with dimension.

$\boxed{f(p) \neq p \quad \forall p}$

p $f(p)$

prove by three steps. $\cancel{①}$: $\nexists\, r: \overline{B^n} \to S^{n-1}$

Step 1. " $\nexists$ C^1 retraction $r: \overline{B^n}$ $\Rightarrow$ Any C^1 f has a fixed point " ✓.

(like ③).

Step. 2. "If Any C^1 f has a fixed point $\Rightarrow$ Any cts f has a fixed point

SWT $f_n \rightrightarrows f$, $\|f - f_\ell\| < \frac{1}{\ell}$ in $\overline{B^n}$

$$g_\ell = \frac{\ell}{\ell+1} f_\ell \qquad g_\ell \rightrightarrows f \quad \text{and} \quad g: \overline{B^n} \to \overline{B^n}$$

$$g_\ell(x_\ell) = x_\ell \qquad \{x_\ell\} \rightsquigarrow \{x_{\ell_i}\} \quad x_{\ell_i} \to x_0.$$

$$f(x_0) = \lim_{\ell\to+\infty} g_\ell(x_0) = \lim_{\ell_i\to+\infty} g_{\ell_i}(x_0) = \lim_{\ell_i\to+\infty} \lim_{k\to+\infty} g_{\ell_i}(x_{\ell_i})$$

$$= \lim \lim x_{\ell_i} \cancel{\ominus} = x_0.$$

Step 3. There is no C^1 retraction. $r: \overline{B^n} \to S^{n-1}$

$\overline{B^n}$ is contractable. $f_t(x) = (1-t)x + t\, r(x).$ $\quad \left(r \sim Id. \right)$

$$= x + t(r(x) - x))$$

$$= x + t\, g(x)$$

$\cancel{(df_t)_x = I_t}$

$$(df_t)_x = I + t(dg)_x.$$

1° $F(t) = \int_{\overline{B^n}} \det (df_t)_x \, dx = \int_{\overline{B^n}} \det\left[I + t(dg)_x \right] dx$

$\cancel{\int_{\overline{B^n}} (df_t)_x}$

it's trivial to see $F(t)$ is a polynomial

and when t is small enough, $\det(I + t(dg)_x) \neq 0.\ \forall x.$

$t \in [0, t_1]$

when $t=1$ suppose $x + tv \in \overline{B^n}$ $\quad f_1(x) = r(x) \in S^{n-1}$

$$\langle f_1(x+tv), f_1(x+tv) \rangle = 1 \qquad \|f_1(x+tv)\|^2 = 1$$

$\cancel{\frac{d}{dt}\Big|\sum f_1^{i\,2}(x+tv)\Big| = \sum 2 f_1^i(x+tv) \nabla f_1^i(x+tv)\cdot\vec{v} \quad 2 f_1(x+tv) (df_1)_{x+tv} \vec{v}}$

$\Rightarrow f_1 = \begin{pmatrix} f_1^i \end{pmatrix} \qquad df_1 = \begin{pmatrix} \dfrac{\partial f_1^i}{\partial x^j} \end{pmatrix}$ $\quad$ it's better to use estein sum

$$0 = \frac{d}{dt}\Big|_{t=0} 1 = \frac{d}{dt}\Big|_{t=0} \sum_i f_1^{i\,2}(x+tv) = \sum 2 f^i(x) \sum \frac{\partial f_1^i}{\partial x^j} v^j \Rightarrow$$

$$= \cancel{f_1^T df_1 v} \; 2 \langle f, df_1 \vec{v} \rangle \qquad \det(df_1)_x = 0$$

An Application of Brouwer fixed point theorem.

Topological invariant of dimension

$U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$, U, V are open if $m \neq n$ then $U \not\cong V$

Brouwer's Topological Invariant of Domain

if $U \subseteq \mathbb{R}^n$ is open & f is continuous injective $\Rightarrow$ f is open.

Rmk. fail in infinite dimension: $f: \ell^2 \to \ell^2$ $(a_1, \cdots) \mapsto (0, a_1, \cdots)$

Domain $\Rightarrow$ dimension

proof: WLOG. $n > m$. and $f: U \to V$ is a homeomorphism.

suppose $i: \mathbb{R}^m \to \mathbb{R}^n$, $(x_1, \dots x_m) \mapsto (0, \cdots 0, x_1, \cdots x_m)$

$\Rightarrow i \circ f$ is a continuous injective.

global version But $i \circ f$ is not open $\square$

local version ‖ suppose $f: \overline{B^n} \to \mathbb{R}^n$ is continuous injective, $f(0) \in \mathrm{Int}\, f(\overline{B_n})$

(global $\Rightarrow$ local is trivial) $f(0) \in f(B^n(0))$ is enough

local $\Rightarrow$ global

$f(U)$ is open

$x \in U$. $B(x, 2\varepsilon) \in U$.

$f|_{\overline{B(x,2\varepsilon)}}: \overline{B(x,2\varepsilon)} \to \mathbb{R}^n$ is c.i.

$\Rightarrow f(x) \in \mathrm{int}\, f(\overline{B(x,\varepsilon)}) \in f(U)$

two ideas:

1° $n=1$ $f: [-1,1] \to \mathbb{R}$ is cts inj

$f([-1,1])$ is connected cpt set $\leadsto$ closed interval $[a,b]$

$\Downarrow$ inj $\Rightarrow a < b$

interval

if $f(0) = a$ $f(\frac{1}{2}) = y_1 > a$ $f(-\frac{1}{2}) = y_2 > a$

and $y_1 \neq y_2$ use connectness ...

n=2. suppose $f: D \to \mathbb{R}^2$ is cts inj

By contradiction

$f(D)$ since f is inj $\Rightarrow f(0) \notin f(S^1)$

$\Rightarrow \exists\, \varepsilon > 0.\ B(f(0), \varepsilon) \subseteq \mathbb{R}^2 \setminus f(S^1)$

$\exists\, c \in B(f(0), \varepsilon)$ But $c \notin f(D)$

$g: S^1 \to S^1 .\ s \mapsto \dfrac{f(s) - c}{\|f(s) - c\|}$

on one hand $g \underset{\uparrow}{\sim} f_0 = \dfrac{f(0) - c}{\|f(0) - c\|} \Rightarrow g$ is null homotopic

$\dfrac{f(ts) - c}{\|f(ts) - c\|}$

$g \sim h_0 \quad s \mapsto \dfrac{f(s) - f(0)}{\|f(s) - f(0)\|}$

$\dfrac{f(s) - \lambda f(t)}{\|f(s) - \lambda f(t)\|}$

$h_0 \sim h \quad \dfrac{f(s) - f(-st)}{\|f(s) - f(-st)\|}$

$h = \dfrac{f(s) - f(-s)}{\|f(s) - f(-s)\|}$ perserve antipody point.

$\boxed{\text{Borsuk-Ulam}} \rightsquigarrow$ Contradiction.

Note. if we have n-dim Borsuk-Ulam Theorem. We success...

2° (using ~~Bo~~ Brouwer Fixed point theorem)

Idea. construct $h(\bar{B}^n) \to \mathbb{R}^n$ s.t. 1. $\|h(f(x)) - x\| \le 1$ 2. $h \circ f(x) \neq 0$. $\to Id - h\circ f: \bar{B}^n \to \bar{B}^n$ But do not have F.P.

proof.

$f(0) \notin \operatorname{Int} f(\bar B) \Rightarrow \exists \underbrace{c}_{\notin f(\bar B)}, \ \|c - f(0)\| < \underset{=}{\varepsilon}$ determine later

denote $\Sigma_1 = f(\bar B)\setminus B(c,\varepsilon)$

$\Sigma_2 = \partial B(c,\varepsilon) \qquad \Sigma = \Sigma_1 \cup \Sigma_2$

$f: \underset{\text{cpt}}{\bar B} \longrightarrow \underset{T_2}{f(\bar B)}$ cts & bi $\Rightarrow$ f is a homeomorphism

$\Rightarrow f^{-1} : \underbrace{f(\bar B)}_{\text{closed in } \mathbb{R}^n} \to \bar B$ is cts

Tietze extension $\leadsto \exists\, g : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $g|_{f(\bar B)} = f^{-1}$

g is non-zero on $\Sigma_1 \Rightarrow \exists\, \delta$, st. $\|g(y)\| \geq \delta$, $y \in \Sigma_1$

$\Rightarrow$ Stone Weierstrass $\Rightarrow$ p is a polynomial $\|p(y) - g(y)\| < \delta/2 \quad \forall y \in \Sigma$.

Fact 1. $\underline{\exists\, a_0 \in B(0, \delta/2)}$, $a_0 \in p(\Sigma_2)$ (measure review).

We need extension to deal the domain f^{-1} does not contact.

consider $\tilde p = p - a_0$ $\quad |p| > \delta/2$ on $\Sigma_1 \longrightarrow |\tilde p| > 0$ on Σ_1

$|p| > 0$ on Σ_2

$\boxed{\tilde p \neq 0 \text{ on } \Sigma}$

Consider $\Phi : f(\bar B) \to \Sigma$ $\quad \Sigma_1$

$$y \mapsto \begin{cases} y & \\ c + \varepsilon \dfrac{y-c}{\|y-c\|} & y \notin \Sigma_1 \end{cases}$$

let $h: \tilde p \circ \Phi : f(\bar B) \to \mathbb{R}^n$

$$\|\tilde p \circ \Phi \circ f(x) - x\| \cdot \begin{cases} \|\tilde p \circ f(x) - g \circ f(x)\| \leq \delta \leq \varepsilon_1 & \Sigma_1 \\ \| \underbrace{\tilde p \circ \Phi \circ f(x) - g \circ \Phi \circ f(x)}_{\delta} + \underbrace{g \circ \Phi \circ f(x)}_{\varepsilon} - \underset{\sim}{x} \| \leq \varepsilon_1 & \end{cases}$$

The Invariance of dimension $\rightsquigarrow$ the dim of manifolds is well-defined

- Topological manifold : A2. T2. locally euclidean
  $\forall x,\ \exists x \in U_x \subseteq X$ $\quad V_x \subseteq \mathbb{R}^n$
  and homeomorphism $\varphi_x : U_x \to V_x$

$\rightsquigarrow X$ is ~ of dim n

manifold with boundary [E with boundary

$\forall x \in M.\ \exists U \ni x.\quad U \simeq \mathbb{R}^n$ / or $U \simeq \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$

$\partial M := \{x \in M \mid \nexists\, U \text{ s.t. } U \simeq \mathbb{R}^n\}$.

$\boxed{\text{manifold of dim } 1}$

Def. If $X \simeq [0,1]$, then we call it Jordan arc (a simple arc)
If $X \simeq S^1$ then we call it Jordan curve (a simple closed curve)

prop. let $\gamma \subseteq \mathbb{R}^2$ be a Jordan arc or Jordan curve

$\boxed{\text{Rmk : No parameterization.}}$

① $\mathbb{R}^2 \setminus \gamma$ has exactly 1 unbounded connected component
② Any connected component of $\mathbb{R}^2 \setminus \gamma$ is path-connected
③ For Any path-component A of $\mathbb{R}^2 \setminus \gamma$. $\underset{\overline{A} \cap \overline{A^c}}{\underline{\underline{\partial A}}} \subseteq \gamma$

proof. (1) $\gamma \simeq [0,1]$ or $S^1 \rightsquigarrow \gamma$ cpt $\rightsquigarrow \gamma$ is bounded
$\rightsquigarrow \gamma \subseteq B(x_0, r)\quad \overline{B(x_0,r)}^c$ is unbounded

(2) $\gamma \xrightarrow{\text{cpt}}$ closed $\quad \mathbb{R}^2 \setminus \gamma$ open $\Rightarrow$ locally path connected

(3) suppose $x_0 \in \gamma \Rightarrow B(x_0, r) \subseteq \mathbb{R}^2 \setminus \gamma \Rightarrow x_0$ is interior in $\square$

Thm || let C be ~~the~~ a Jordan arc. in $\mathbb{R}^2$
then $\mathbb{R}^2 \setminus C$ is connected

Rmk. if $C \simeq (0,1)$, the thm may fail in general

proof : prove by contradiction. $\mathbb{R}^2 \setminus C$ is not connected

suppose $C \subseteq B(x_0, r) \overset{x_0 \in A}{\Longrightarrow} \exists$ a bounded connected component A

$\Rightarrow A \subseteq B(x_0, r)$ $\boxed{\text{Claim : } \exists \text{ a retraction from } \overline{B(x_0,r)} \text{ to } C}$.

$f : [0,1] \xrightarrow{\sim} C \Rightarrow f^{-1} : \underset{\overline{B(x_0,r)}}{\underset{\cap}{C}} \to [0,1]$ cts

tietze
$\exists g : \overline{B(x_0,r)} \to [0,1]$, then $f \circ g : \overline{B(x_0,r)} \to C$ is cts retraction

Consider $h : \overline{B(x_0,r)} \longrightarrow \overline{B(x_0,r)}$

$$x \longmapsto h(x) = \begin{cases} f\circ g(x) & x \in \bar{A} \\ x & x \in A^c \cap \overline{B(x_0,r)} \end{cases}$$

Note $\bar{A} \cap \left(A^c \cap \overline{B(x_0,r)}\right) = \partial A \cap B(x_0,r) \subseteq C$

from paste lemma (closed) h is cts.

$x_0 \notin \operatorname{Im}(h)$ since $\begin{cases} x_0 \notin C \\ x_0 \notin A^c \end{cases}$

Consider $h_0 : \overline{B(x_0,r)} \longrightarrow \partial B(x_0,r)$

$$h_0 \quad x \mapsto x_0 + r \frac{\cancel{h(x)}\; x - x_0}{\cancel{h(x) - h(x_0)}\; \|x - x_0\|}$$

$h_0 h$ is a retraction ~~and~~ contradiction! □

Rmk. for $n \geq 2$, let $K \subseteq \mathbb{R}^n$ is cpt retract~~ion~~, then $\mathbb{R}^n \setminus K$ is connected ; it's necessary

con. let γ be a $\boxed{\text{Jordan arc or Jordan curve}}$ in $\mathbb{R}^2$

let for Any connected component A of $\mathbb{R}^2 \setminus \gamma$.

then we have $\partial A = \gamma$

proof. if $\mathbb{R}^2 \setminus \gamma$ is connected $\Rightarrow A = \mathbb{R}^2 \setminus \gamma \Rightarrow \mathring{\gamma} = \emptyset \Rightarrow \bar{A} = \mathbb{R}$

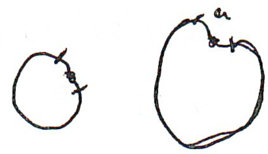
$\Rightarrow \partial A = \gamma$

if ~~ϕ~~ $\mathbb{R}^2 \setminus \gamma$ is ... $\Rightarrow (\gamma \simeq S^1)$ $\gamma = J$ is a J C.

$\Downarrow$

$\exists$ Another component $B \neq A$

We have $\partial A \subseteq \gamma = J \simeq S^1$ (



if $\exists\, a \in J \setminus \partial A \Rightarrow \partial A \subseteq C$

for $x \in A$ $y \in B$, since $\mathbb{R}^2 \setminus C$ is p.c.

$\Rightarrow \exists$ path p in $\mathbb{R}^2 \setminus C$ from x to y

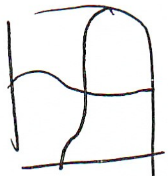
$p \cap C = \emptyset$

$t_0 = \inf\{p(t) \in A^c\}$ $\Rightarrow p(t_0) \in \bar{A} \cap A^c = \partial A \subseteq C$. ↯ □

Thm. Jordan Curve Theorem : let $J \subseteq \mathbb{R}^2$ be a Jordan curve

① $\mathbb{R}^2 \setminus J$ has exactly 2 components.

② each components has J as its boundary key lemma

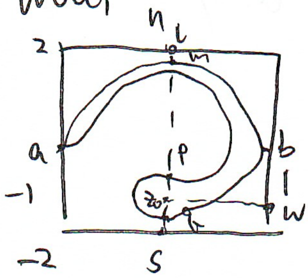


using poincaré miranda

proof of Jordan curve Theorem

① J cpt. $\exists\ a, b \in J$. $d(a,b) = \operatorname{diam} J$

WLOG $a = (-1, 0)$, $b = (1, 0)$



$\exists\, l \in \overline{ns} \cap J$ attains max y value.

$J_n \subset J$: $a \xrightarrow{l} b$

~~m ... J_m~~

a, b leads ~~$J = J_n \cup J_m$~~ $J = J_n \cup J_s$

$m \in \overline{ns} \cap J_n$ min y

$\Rightarrow \overline{ms} \cap J_s \neq \phi$

if not $(\overline{nl} + \widehat{lm} + \overline{ms}) \cap J_s = \phi$ ↯

② Now suppose p, q in $\overline{ms} \cap J_s$ are the points with max y and min y.

take $z_0 = \frac{m+p}{2}$. U ⊂ $\mathbb{R}^2 \setminus J$ contains z_0.
is the component

We hope it's bounded

if it's unbounded. since unbounded component exists uniquely

suppose $*$ lie in R^c, there is a path from z_0 to $*$

denote the first intersection point of α and J is w

α_w is the path

if $w \in \partial R^-$ $(\overline{nl} + \widehat{lm} + \overline{mz_0} + \alpha_w + \overline{ws}) \cap J_s = \phi$

if $w \in \partial R^+$ $\overline{nw} + \alpha_w + \overline{z_0 s} \cap J_n = \phi$

③ uniqueness $z_0 \in U$ is bounded. $\widetilde{U}$ is another

$\beta = \overline{nl} + \widehat{lm} + \overline{mp} + \widehat{pq} + \overline{qs}$ $\beta \cap \widetilde{U} = \phi$

~~find~~ $a, b \in \partial \widetilde{U} \subseteq \overline{\widetilde{U}}$ (since $\mathbb{R}^n$ is locally connected, Any component is open)

$\overline{aa_1} + \widehat{a_1 b_1} + \overline{b_1 b} \subseteq \widetilde{U}$ ↯

Classification of Curves

1 dim mani L.E. $\rightsquigarrow$ $\forall x \in M$. $\exists U_x$, $U_x \overset{\varphi}{\cong} \mathbb{R}$

denote such U_x chart (φ, u)

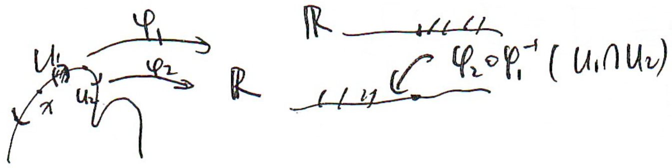
There are only two $\boxed{\text{connected}}$ 1 dim mani & 1 dim mani with boundary
in homeomorphism sense

$$\underbrace{\text{1 dim mani}} \downarrow \quad \mathbb{R}^1 \;\&\; \underset{\text{cpt}}{S^1} \qquad\qquad \underbrace{\text{1 dim mani with boundary}} \downarrow \quad \underset{\text{cpt}}{[0,1]} \;\&\; [0,1)$$

connected ? compact ? with Boundary ?

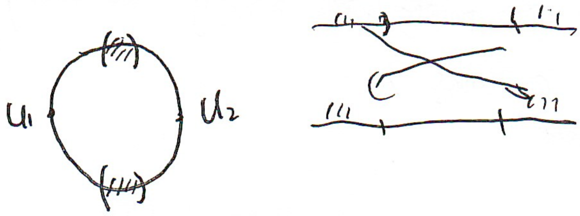
Today The First Part.

Idea.
①



paste φ $U_1 \cup U_2 \to \mathbb{R}$

②



$\rightsquigarrow S^1$

lemma (φ_1, U_1) (φ_2, U_2) $U_1 \not\subseteq U_2$. $U_2 \not\subseteq U_1$ W is a component of $U_1 \cap U_2$

$\underbrace{\varphi_1(W)}_{\substack{\text{connected } \mathbb{R} \\ \text{open set in}}} = (a, b)$ $\varphi_2(W) = (c, d)$ $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$

$\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(W) \to \varphi_2(W)$

Suppose φ_{12} is monotone $\boxed{\text{increasing}}$ Then, either $\begin{matrix} a = -\infty, & b \in \mathbb{R} \\ c \in \mathbb{R} & d = +\infty \end{matrix}$ or $\begin{matrix} a \in \mathbb{R} & b = +\infty \\ c = -\infty & d \in \mathbb{R} \end{matrix}$

proof. $a = -\infty$, $b = +\infty$ ✗
$c = -\infty$, $d = +\infty$ ✗.

$\boxed{a, c \in \mathbb{R}}$ (similar $b, d \in \mathbb{R}$).

if so, Claim $\varphi_1^{-1}(a) = \varphi_2^{-1}(c)$

$\varphi_1^{-1}(a) \in U_a$, $\varphi_2^{-1}(c) \in U_c$

$a \in \varphi_1(U_a)$ $c \in \varphi_2(U_c)$ $\varphi_{12}(a,b) \longrightarrow (c,d)$ $\nearrow$ & homeomorphism

$\Rightarrow \varphi_{12}\left(\varphi_1(U_1) \cap (a,b)\right) \cap \varphi_{12}(U_c) \neq \phi$

~~$\Rightarrow \varphi_2(U_1 \cap \varphi(a,b))$~~ T_2 $\Rightarrow \varphi_1^{-1}(a) = \varphi_2^{-1}(c)$,

$\varphi_1^{-1}(a)$ $\in \mathrm{Int}\, U_1$ $\Rightarrow \varphi_1^{-1}(a) \in \mathrm{Int}(U_1 \cap U_2)$

$= \varphi^{-1}(c) \in \mathrm{Int}\, U_2$

But $\varphi^{-1}(a) \in \partial \varphi^{-1}(a,b)$ ↯ $\square$

lemma Any continuous injective map $f: (a,b) \longrightarrow \mathbb{R}$ is monotone.

lemma Paste Method 1

lemma Paste Method 2

---

prop. Let M 1 dim mani (φ_1, U_1) (φ_2, U_2) Then $U_1 \cap U_2$ has at most two component, and

① $U_1 \cap U_2$ connected $\Rightarrow \exists (\varphi, U)$. $U = U_1 \cap U_2$

② two components $U_1 \cup U_2 \cong S^1$

the proof of Classification

Lindeloff countable covering using chart

let $\tilde{U}_1 = U_1$ $\tilde{U}_{n+1} = \tilde{U}_n \cup U_{k(n)}$ $k(n) = \inf\{k \mid U_k \cap \tilde{U}_n \neq \phi\}$

$U_k \notin \tilde{U}_n$

Claim $\bigcup_N \tilde{U}_n = M$

Chain union $\Rightarrow$ Connected

suppose $x \in M$, $x \notin \bigcup \tilde{U}_n$ $\Rightarrow \exists U_m \ni x$ m is smallest

$\Rightarrow U_m \cap (\bigcup \tilde{U}_n) = \phi$ (if not, After finite time. U_m will be chosen)

$\Rightarrow \bigcup \tilde{U}_n$ is Closed. $\Rightarrow \bigcup \tilde{U}_n = M$.

Case 1. $\exists n$ $\tilde{U}_n \cap U_{k(n)}$ has 2 components.

$\tilde{U}_{n+1} \cong S^1$ is cpt $\Rightarrow$ $\underline{\tilde{U}_{n+1}}_{\text{open}}$ is closed $\Rightarrow \tilde{U}_{n+1} = M$.

Case 2.

$\tilde{U}_1 \subseteq \tilde{U}_2 \subseteq \cdots$

2.1 stop after finite steps ~~stop~~ trivial

2.2. $(\psi_n, U_n, (a_n, b_n))$

$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n - 1, b_n) \\ (a_n, b_n + 1) \end{cases}$$ ~~how~~ depend on how to paste.

$$\psi_{n+1}\big|_{\cancel{(a_n, b_n)}\ \tilde{U}_n} = \psi_n$$

$\psi = \lim \psi_n$ $\quad \psi : M \to (a, b).$

Classification of compact surface

compact surface of boundary ⇝ its boundary is compact 1-dim curve without boundary

(A cpt surface take away a point is not cpt)

examples manifest that connected surface can have many pieces of S^1 as its boundary. union of S^1's

theorem. (Classification of compact surface).

idea. holes: boundary components. oriented

Connected sum Non-trivial: independent of position
- - - boundary orient.

Conclusion: connected sum is well-defined.

"zero element" S^2 ~~commutative~~ commutative associated.

~~the~~ use polygonal presentation.

corollary $\mathbb{RP}^2 \# \mathbb{RP}^2 \simeq$ the klein bottle

Dyck surface $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \simeq \mathbb{T}^2 \# \mathbb{RP}^2$

NO cancellation law

def. $\{x_0, \cdots x_m\} \subseteq \mathbb{R}^n$ $x_1 - x_0, \cdots x_m - x_0$ is in general position if they are linear independent.