A continuous but nowhere differentiable function

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Before everything, I need to difine my notations about Fourier Analysis.
1. $D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}$ is the N-th Dirichlet kernel.
2. $F_N(x) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=-k}^{k} e^{2\pi ijx}$ is the N-th Fejer kernel.
3. $S_N(f) = (D_N * f)(x)$.
4. $F_N(f) = (F_N * f)(x)$.
5. $\mathcal{S}$ is the Schwartz function space.
6. for $f \in \mathcal{S}$, Poisson summation formula is

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

Today, I'm going to introduce a continuous but nowhere differentiable function by Fourier Analysis. First of all, I would like to introduce some solutions to this problem. An interesting method to this problem is to prove that

$$E_n = \{f \in C[0,1] : \exists x_0,\ s.t.\ |f(x) - f(x_0)| \leq n|x - x_0|\ for\ \forall x \in [0,1]\}$$

is a nowhere dense set. Then one can see the target result by Baire Catagory Theorem. However, it's not a constructive proof, and we have learnt a constructive proof in Mathematical Analysis A3, which is related to the concept of fractal naturally. Now, I'm going to prove an example which in fact give a family of continuous but nowhere differentiable functions. My main theorem is the one below.

Theorem 1. $f = \sum\limits_{n=1}^{\infty} 2^{-n\alpha} e^{2\pi i 2^n x}$ *is continuous but nowhere differentiable for* $\alpha \in (0,1]$.

Why do we consider such a function? My opinion is that, we must make leading term have high frequency and oscillate.(It's the key point! Latter discussion about decay means is based on the point!)

I will prove the theorem by dividing it into two cases that one is $\alpha \in (0,1)$ and the other is $\alpha = 1$. Indeed, the proof of latter one is harder the former one and based on the former one. Hence, I will focus on the case $\alpha \in (0,1)$ now.

Let's think a natural question, why the function can't be differentiable, and what will happen if it can be differentiated term by term.

$$f' \sim 2\pi i \sum_{n=1}^{\infty} 2^{n(1-\alpha)} e^{2\pi i 2^n x}$$

It seems that, the partial sum of derivation can't converge, that is to say, we can estimate $S_n(f)'$ concretely. But in Fourier Analysis, we have known that, Dirichlet kernel is not a good kernel, so we shall not deal with it casually. Fortunately, f is a funciton with lacunary property, which means that term gaps have exponential growth. Hence we can change it into the difference of two cesaro sums.

Here I think we need to explain more about the decay means in the book. As what I said, Dirichlet kernel is not a good kernel, so we should avoid dealing with it since we need more precious caculation. And as for Fejer kernel, it is a good kernel, but it actually changes almost every term in our original funciton. And for our need, we shall preserve more term of high frequency as we can, since they are what we need to lead bad differentiability.

$$S_n(f) = \sum_{k=1}^{n} 2^{-k\alpha} e^{2\pi i 2^k x} = 2\sigma_{2n'}(f) - \sigma_{n'}(f) := \Delta'_n(f)$$

where n' has the form 2^k, and $n' \leq n < 2n'$.

Now, we can estimate $F_n(f)'(x_0)$ if f is differentiable at $x = x_0$. Fejer kernel is a triangle polynomial with form

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=-k}^{k} e^{2\pi i j x}$$

Then we have natural estiamtion that

$$|F'_n| \leq C_1 n^2$$

$$|F_n|(x) = \frac{\sin^2(n\pi x)}{n \sin^2(\pi x)} \Rightarrow |F'_n|(x) \leq \frac{C_2}{|x|^2}$$

Hence, we calcuate $\sigma_n(f)'(x_0)$ below.

$$\begin{aligned}
|\sigma_n(f)'(x_0)| &= \left| \int_{-1/2}^{1/2} f(x_0 - t) F'_n(t) dt \right| = \left| \int_{-1/2}^{1/2} (f(x_0 - t) - f(x_0)) F'_n(t) dt \right| \\
&\leq \left| \int_{|t| \leq 1/n} + \int_{1/n \leq |t| \leq 1/2} (f(x_0 - t) - f(x_0)) F'_n(t) dt \right| \\
&\leq \left| \int_{|t| \leq 1/n} (f(x_0 - t) - f(x_0)) F'_n(t) dt \right| \\
&+ \left| \int_{1/n \leq |t| \leq 1/2} (f(x_0 - t) - f(x_0)) F'_n(t) dt \right| \\
&= O(logn) + O(1) = O(logn)
\end{aligned}$$

Hence, we shall see $\Delta_n(f)'(x_0) = O(logn)$. However, we also see that, if $n = 2^{k-1}$, we have $\Delta_{2n}(f) - \Delta_n(f) = S_{2n}(f) - S_n(f) = 2^{-k\alpha} e^{2\pi i 2^k x}$. Then, we actually have

$$|\Delta_{2n}(f)'(x_0) - \Delta_n(f)'(x_0)| = 2^{k(1-\alpha)} = n^{1-\alpha} \gg O(logn)$$

Therefore, f is nowhere differentiable as a complex-valued function. However, we actually aim to find a nowhere differentiable real-valued function, so we naturally consider its real part or imaginary part.

Take its real part and imaginary part, we obtain

$$Re(f) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x) \quad Im(f) = \sum_{n=1}^{\infty} 2^{-n\alpha} \sin(2^n x)$$

Let g denote its real part. If g is differentiable at $x = x_0$, by our estiamtion above, we have

$$\Delta_n(g)' = O(logn)$$

whenever $|h| \leq \frac{c}{n}$.

Note that $\Delta_{2n}(g) - \Delta_n(g) = 2^{-k\alpha} \cos(2^k x)$, here $n = 2^{k-1}$. Then we have

$$|2^{-k\alpha} \sin(2^k(x_0 + h))| = O(logn)$$

Now, we only need to let δ denote the distance between $2^k x$ and the nearest number of the form $(k + \frac{1}{2}\pi)$, and we take $h = \delta/2^k$. Done.

Before considering the case for $\alpha = 1$, we conclude what we have done in the case $\alpha \in (0, 1)$. We initially have a Fourier series, and we use a good kernel, modified Fejer kernel, to cut it, which preserves the information of the former n term, and has good decay property at the boundary so that it can converge to f. Finally, we estimate the order of its one term derivation, and it leads our target.

By differentiating term by term, we shall see the intrinsic difference to the case $\alpha \in (0, 1)$. In this case, we shall focus on only one term of high frequency. So, to improve its decay speed, we consider another kernel. Using bump function, we can obtain the function below.

$$\phi(\xi) = \begin{cases} 1 & |\xi| \leq 1, \\ 0 & |\xi| \geq 2, \\ C^{\infty} connection & otherwise. \end{cases}$$

Then $\tilde{\Delta}_N(x) = \sum_{n=-\infty}^{\infty} \Phi(n/N) e^{2\pi i n x}$ is a new kernel, but this form is not suitable for later integral estiamtion, so we shall change its form. Using inversion formula, we obtain a new function ϕ. Since Φ is compact supported, let $\phi(x)$ denote $\check{\Phi}(x)$, then $\phi \in \mathcal{S}$. Hence we can rewrite the kernel by Poisson sum formula

$$\tilde{\Delta}_N(x) = \sum_{n=-\infty}^{\infty} \phi_N(x + n)$$

where $\phi_N(x) = N\phi(Nx)$ so that $\widehat{\phi_N}(\xi) = \Phi(\xi/N)$.

Noting the construction, we can immediately see $\Delta_{N'}(f) = S_N(f)$, where N' is the largest integer of the form 2^k with $N' \leq N$.

For convenience, we can prove a strong result to complete its proof.

Theorem 2. *If $f'(x_0)$ exists, then*

$$(f * \tilde{\Delta}'_N)(x_0 + h_N) \to f'(x_0) \quad as\ N \to \infty$$

whenever $|h_N| \leq \frac{c}{N}$.

To prove the result, we need the following lemma.

Theorem 3. *Set* $\tilde{\Delta}_N(x) = \phi_N(x) + E_N(x)$*, then*
1. $\sup_{|x|\leq 1/2} |E'_N(x)| \to 0\ as\ N \to \infty$
2. $|\tilde{\Delta}'_N(x) \leq cN^2$
3. $|\tilde{\Delta}'_N(x)| \leq \frac{c}{N|x|^3}$
4. $\int_{|x|\leq 1/2} \tilde{\Delta}'_N(x)dx = 0$
5. $\int_{|x|\leq 1/2} x\tilde{\Delta}'_N(x)dx \to -1\ as\ N \to \infty.$

Proof. 1. $E_N = \sum_{n\neq 0} N\phi(N(x+n))$, then $E'_N = \sum_{n\neq 0} N^2\phi(N(x+n))$. Since $\phi \in \mathcal{S}$, we have $\phi' \in \mathcal{S}$ and exists $C > 0$ s.t. $|\phi'_N(x) \leq \frac{C}{|N(x+n)|^k}$ for k>2.
2. $|\tilde{\Delta}'_N(x)| \leq C|\sum_{0<|n|\leq 2N} n = CN^2$.
3. in 1., take k=3.
4. it's periodic.
5. integrating by part, we have

$$\int_{|x|\leq 1/2} x\tilde{\Delta}_N(x)dx = \tilde{\Delta}_N(1/2) - 1 \to -1\ as\ N \to \infty$$

□

Now, we start proving the strong result.

Proof. Let $\delta(t) = f(x_0 + h_N - t) - f(x_0 + h_N) + tf'(x_0)$, so $\delta(t) \to 0\ as\ t \to 0$. Then we have the following integral caculation

$$\left|(f * \tilde{\Delta}'_N)(x + h_N) + \int_{-1/2}^{1/2} t\tilde{\Delta}'_N(t)f'(x_0)dt\right| \leq \left|\delta(t)|t||\tilde{\Delta}'_N(t)dt\right|$$
$$\leq \int_{|x|\leq k/N} + \int_{1/2\leq|x|>k/N} * = I_1 + I_2$$

where k is an arbitrary positive integer. We shall estimate the two parts by previous estiamtions.

$$I_1 \leq \sup_{|t|\leq k/N} |\delta(t)| \int_{|x|\leq k/N} |t||\tilde{\Delta}'_N(t)|dt \leq \sup_{|t|\leq k/N} |\delta(t)| \int_{|x|\leq k/N} |t|cN^2 dt$$
$$\leq \sup_{|t|\leq k/N} |\delta(t)| \frac{k}{N}\frac{2k}{N} cN^2 \to 0\ as\ N \to \infty$$

$$I_2 \leq C \int_{k/N\leq|t|\leq 1/2} |t|\frac{c}{N|t|^3}dt = C\int_{k/N\leq|t|\leq 1/2} \frac{1}{N|t|^2} \leq \frac{C}{k}$$

Then, we finally have

$$|(f * \tilde{\Delta}'_N)(x + h_N) - f(x_0)| \le c/k$$

as N goes to infinity. And since k is arbitrary and c is a constant which is independent with k, so we have done. □

Then we shall see, $|(f * \tilde{\Delta}_{2N})'(x_0) - (f * \tilde{\Delta}_N)'(x_0)| \to 0$, but $|(f * \tilde{\Delta}_{2N})'(x_0) - (f * \tilde{\Delta}_N)'(x_0)| = 2\pi$, so it leads contradiction. Since we have left h_N's room, we can repeat what we have done in case $\alpha \in (0, 1)$, and then we approach our target!