测度论的解题思路

Littlewood 三原理	战略	战术
测度正则性	可测集	开集
Lusin定理	可测函数	连续函数
Egorov定理	收敛函数列	一致收敛函数列

- 可测集=开矩体$+\lim+$*Caratheodory* $=G_\delta \setminus Z = F_\sigma \cup Z$
- 可测函数$=\chi_A+(+,\cdot,\lim)=$连续$+a.e.\lim$

例题

- $f:\mathbb{R}\longrightarrow\mathbb{R}$可加+局部有界$\Longrightarrow$ 线性.

- $f:\mathbb{R}\longrightarrow\mathbb{R}$可加+可测$\Longrightarrow$ 线性.

例题

- 若$f:\mathbb{R}\longrightarrow\mathbb{R}$在一个正测集上有界(例如局部有界), 具有可加性, 则它具有线性性.

证明: 我们要证$f(x)=f(1)x$.

$$f\text{在正测集}E\text{有界}\xRightarrow[[-\delta,\delta]\subset E-E]{\text{Steinhaus定理}} f\text{在原点附近}[-\delta,\delta]\text{有界}.$$

由可加性

$$nf(\frac{m}{n})=f(m)=mf(1)\Longrightarrow f(\frac{m}{n})=\frac{m}{n}f(1).$$

$$\begin{aligned}|f(x)-f(1)x| &\xlongequal[\text{取 } r\in\mathbb{Q}:\ |x-r|<\delta/n]{\forall\ \text{固定}x,n} \left|\frac{1}{n}f(n(x-r))+(r-x)f(1)\right| \\ &\leqslant \frac{M}{n}+\frac{\delta}{n}|f(1)|\to 0\end{aligned}$$

例题

证明: f可测 $\xRightarrow[f|_{[-1,1]}\text{可测}]{Lusin}$ $\exists$ 紧$K \subset [-1,1]$: $m(K) > 0$, $f|_K$ 连续

$\xRightarrow{Tietze}$ $f|_K$ 连续可连续扩张到$C_c(\mathbb{R})$, 一致连续.

$\xRightarrow{Steinhaus}$ $\exists[-\delta_0, \delta_0] \subset K - K$:

$$|f(z)| \overset{z=x-y}{\underset{x,y\in K}{=\!=\!=}} |f(x) - f(y)| \leqslant M.$$

$\xRightarrow{\text{局部有界}}$ f连续.

概念: 《a.e. 连续》, 《与连续函数a.e.相等的函数》

- a.e. 连续： 不连续的点是零测集.
- 与连续函数a.e.相等: 不相等的点是零测集。

(1) $g \overset{a.e.}{=\!=} f \in C(\mathbb{R}) \not\Rightarrow$ g在$\mathbb{R}$上a.e.连续

(例如: $g = \chi_{\mathbb{Q}}, \quad f = 0$)

(2) g在$\mathbb{R}$上a.e.连续 $\not\Rightarrow$ $g \overset{a.e.}{=\!=} f \in C(\mathbb{R})$

反证法： $\chi_{[0,+\infty)} = g \overset{a.e.}{=\!=} f \in C(\mathbb{R})$

$$\text{range}\, f \supset \{0,1\} \xRightarrow{f\text{连续}} \text{range}\, f \supset [0,1] \Longrightarrow f^{-1}(0,1)\text{非空开集}$$

例题

- 设$\{\varphi_k\}_{k=0}^{\infty}$是全体有理多项式, $\varphi_0=0$, $f_k=\varphi_k-\varphi_{k-1}$. 则 $\forall\ f\in\mathscr{L}([a,b],\mathbb{R})$, 存在加括号的方法使得

$$f \xlongequal{a.e.} (f_1+\cdots f_{n_1})+(f_{n_1+1}+\cdots+f_{n_2})+\cdots .$$

证明: 取多项式列$p_k\longrightarrow f\ a.e.$ in $[a,b]$.

取$\{\varphi_k\}_{k=0}^{\infty}$子列$\varphi_{n_k}$: $|p_k-\varphi_{n_k}|<\frac{1}{k}$ in $[a,b]$, 添加$\varphi_{n_0}=\varphi_0=0$.

$$\Longrightarrow f \xlongequal{a.e.} \lim_{k\to\infty}\varphi_{n_k}=\sum_{k=0}^{\infty}(\varphi_{n_{k+1}}-\varphi_{n_k})=(f_1+\cdots f_{n_1})+(f_{n_1+1}+\cdots+f_{n_2})+\cdots$$

例题

- 设$E \in \mathscr{L}(\mathbb{R}^n)$, $m(E) < \infty$ $f_k, f \in \mathscr{L}(E, \mathbb{R})$, $f_k \xrightarrow{a.e.} f$.

$$\Longrightarrow \exists \text{可测集} E_k \uparrow, \exists \text{零测集} Z : E = \lim_{k\to\infty} E_k \sqcup Z, \qquad f_k \rightrightarrows f \quad \text{in} \quad E_k$$

证明:

$$f_k \xrightarrow{a.e.} f \quad \overset{Egorov}{\Longrightarrow} \quad \exists \text{ 可测集} A_k, \quad m(A_k^c) < \frac{1}{k}, \qquad f_k \rightrightarrows f \quad \text{in} \quad A_k$$

$$\Longrightarrow \quad E_k := \bigcup_{j=1}^{k} A_j \uparrow, \ Z = (\lim_{k\to\infty} E_k)^c = \bigcap_{k=1}^{\infty} E_k^c \subset \bigcap_{k=1}^{\infty} A_k^c$$

$$m(Z) = 0, \qquad f_k \rightrightarrows f \quad \text{in} \quad E_k$$

例题

- 例题

$$f_n : [0,1] \longrightarrow \mathbb{R}\text{可测} \xRightarrow{\quad \exists\ C_n>0 \quad} \frac{f_n(x)}{C_n} \xrightarrow{a.e.} 0.$$

例题

证明: $$\lim_{k\to\infty} m[|f_n| > k] = m[f_n = \infty] = 0, \qquad \forall\ n \in \mathbb{N}$$

$$\xRightarrow{\exists k(n)\in\mathbb{N}} m[|f_n| > k(n)] < \frac{1}{2^n}$$

$$\xRightarrow{Borel-C} m\left(\overline{\lim}\left[\left|\frac{f_n(x)}{C_n}\right| > \frac{1}{n}\right]\right) = 0 \quad (C_n := nk(n))$$

$$\Longrightarrow \forall\ \epsilon > 0, \quad m\left(\overline{\lim}\left[\left|\frac{f_n(x)}{C_n}\right| > \epsilon\right]\right) = 0$$

$$\Longrightarrow \frac{f_n(x)}{C_n} \xrightarrow{a.e.} 0.$$

本节主要内容

- Ch3.回顾Littlewood 三原理

- Ch4. Lebesgue 积分.

§1. 积分定义

Littlewood 三原理

- $[a, b]$每一个可测集差不多是有限个区间的并.
- 每一个可测函数差不多是连续函数.
- 每一个逐点收敛的可测函数列差不多是一致收敛函数列.

测度正则性

Theorem 1

$\forall\ E \subset \mathbb{R}^n, m(E) < \infty$. 则 $\forall\ \epsilon > 0$, $\exists$ 有限个开矩体$I_1, \ldots, I_s$:

$$m(E \setminus \bigcup_{k=1}^{s} I_k) < \epsilon, \qquad m(\bigcup_{k=1}^{s} I_k \setminus E) < \epsilon.$$

证明: 不妨设$E = G$是开集, 由测度正则性存在紧集$K \subset G$

$$m(G \setminus K) < \epsilon.$$

$$G = \bigcup \text{开矩体}, \qquad K \subset \bigcup_{\text{有限}} \text{开矩体} \subset G.$$

Lusin定理: 可测是连续的弱化

Theorem 2 (Lusin定理)

设$f : E \longrightarrow \mathbb{R}$可测, 则

$$\forall\, \epsilon > 0,\ \ \exists\, g \in C(\mathbb{R}^n),\ \ \exists\, 闭集 F \subset E:$$

$$m(E \setminus F) < \epsilon, \qquad f|_F = g|_F\,.$$

f定义域有界	$g \in C_c(\mathbb{R}^n)$	
f值域有界	$g \in C_b(\mathbb{R}^n)$	且L^∞范数不增加

Egorov定理

Theorem 3 (Egorov定理)

$$f_k \xrightarrow{a.e.} f$$

$$\xLeftrightarrow{m(E)<+\infty} \quad f_k \xrightarrow{a.un} f$$

$$\Longleftrightarrow \quad \forall\ \epsilon > 0, \exists E_\epsilon \subset E : \quad m(E_\epsilon) < \epsilon, \qquad f_n \overset{onE\backslash E_\epsilon}{\rightrightarrows} f;$$

回顾：记号

- $E \subset \mathbb{R}^n$可测.

S^+	$S^+(E)$	非负简单可测	$\mathbb{R}$值
$\mathscr{L}^+$	$\mathscr{L}^+(E)$	非负可测	$\overline{\mathbb{R}}$值
$\mathscr{L}^1$	$\mathscr{L}^1(E)$	可积	$\overline{\mathbb{R}}$值 (本质$\mathbb{R}$ 值)

积分定义

- 标准四步:　　从小框架(直观)扩充到大框架(洞察力)

$$\chi_A \xrightarrow[\text{代数结构、极限结构、序结构}]{S^+,\quad \mathscr{L}^+} \mathscr{L}^1$$

- Lebesgue 积分记号:

$$\int_E f(x)dx = \int_E f(x)dm(x) = \int_E f$$

- 三个层次

$$\mathbb{R} \Longrightarrow \mathbb{R}^n \Longrightarrow (X, \Gamma, d\mu)$$

积分定义四部曲

(1): 面积=长×宽的推广:

$$\int_E \chi_A := m(A)$$

(2): 线性性的要求:

$$\int_E \sum_{j=1}^{k} c_j \chi_{E_j} \xlongequal[\text{标准表示}]{f=\sum_{j=1}^{k} c_j\chi_{E_j}\in S^+} \sum_{j=1}^{k} c_j m(E_j).$$

注记

注记1：　约定:

$$0 \cdot \infty = 0.$$

$$(\text{宽} = 0 \Longrightarrow \text{面积} = 0)$$

注记2：　Lebesgue 积分剖分值域$\Longrightarrow$ 第一步、第二步成立.

注记3：　Lebesgue 积分几何意义$\xrightarrow[\text{高}=c_j,\text{宽}=m(E_j)]{\text{f下方图形面积}}$ 第二步良定.

续

(3) 单调收敛定理嵌入到积分定义:

$$\int_E f :\xlongequal{S^+ \ni \varphi_k \uparrow f \in \mathscr{L}^+} \lim_{k\to\infty} \int_E \varphi_k$$

续

注记: 该定义自然:

利用可测函数的四部曲, 通过线性性和单调收敛定义积分

$$\chi_A \xrightarrow{+,\cdot,\lim} \mathscr{L}^1$$

$$\Longrightarrow \int_E f = \int_E \chi_A + (+,\cdot,\lim)$$

续

注记:　需要证明良定性:

$$\int_E f \xlongequal{\text{待证}} \sup_{S^+\ni\varphi\leqslant f}\int_E \varphi.$$

续

(4)　　一般情形:

$$\int_E f \;:\xlongequal[\text{f可测}]{f=f^+-f^-} \int_E f^+ - \int_E f^-.$$

注记:　陷阱:

$$+\infty - (+\infty)$$

续

$$f \in \mathscr{L}^1(E) \overset{\text{def}}{\Longleftrightarrow} f^{\pm} \in \mathscr{L}^{+}(E) \cap \mathscr{L}^1(E) \Longleftrightarrow |f| \in \mathscr{L}^1(E) \Longleftrightarrow \int_E |f| < +\infty$$

$$f\text{在E上积分存在(有定义)} \quad \overset{\text{def}}{\Longleftrightarrow} f^{\pm}\text{之一} \in \mathscr{L}^1(E) \Longleftrightarrow \int_E f \in \overline{\mathbb{R}}$$

注记

- 不是所有集合都能定义测度:

  只对可测集.

- 不是所有函数都能定义积分:

  只对可测特征函数的代数和极限运算得到的函数

- 不是所有可测函数都能定义积分:

  需要避开$(+\infty)-(+\infty)$陷阱.

- 不是所有可定义积分的函数都可积

两种积分的差别

Riemann积分	Lebesgue积分
非绝对收敛的积分理论	绝对收敛的积分理论
不适用于高维	适用于高维
交换次序条件苛刻	交换次序条件较弱
非完备	完备
局限性	释放了积分的手脚, 统一了离散和连续

积分良定性: 不依赖于φ_k的选取

S^+性质: 本节总是假设$f, g, f_k, g_k \in S^+$.

(1) 线性: 积分作为映照是线性映照

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

(只要对特征函数验证: 归结于积分的定义和测度论)

积分良定性: 不依赖于φ_k的选取

(2) 单调性: 积分作为映照保序

$$f \leqslant g \Longrightarrow \int_E f \leqslant \int_E g.$$

(只要对特征函数验证: 归结于测度的单调性)

积分良定性: 不依赖于φ_k的选取

(3) 可积性判别: $f \in \mathscr{L}^1 \overset{f\in S^+}{\Longleftrightarrow} m[f > 0] < +\infty.$

证明: 假设$f \in S^+$取的非零值:

$$0 < a_1 < a_2 < \cdots < a_k.$$

$$\Longrightarrow \quad a_1\chi_{[f>0]} \leqslant f|_{[f>0]} \leqslant a_k\chi_{[f>0]}.$$

积分良定性(续)

(4) 单调收敛： (将测度命题由$A=\chi_A$推广到S^+)

(4a) $f_n \downarrow f, \quad f_1 \in \mathscr{L}^1 \xRightarrow{S^+\text{框架内}} \int_E f_n \downarrow \int_E f.$

(4b) $f_n \uparrow f \xRightarrow{S^+\text{框架内}} \int_E f_n \uparrow \int_E f.$

(4c) $f_n \uparrow, g_n \uparrow, \quad \lim f_n \leqslant \lim g_n \xRightarrow{S^+\text{框架内}} \lim \int_E f_n \leqslant \lim \int_E g_n.$

积分良定性(续)

证明: (4a) $S^+ \cap \mathscr{L}^1 \ni f_n \downarrow 0$.

$$\forall\, \epsilon > 0, \qquad f_n \leqslant M\chi_{[f_n \geqslant \epsilon]} + \epsilon\chi_{[0<f_n\leqslant\epsilon]}$$

$$M = \max f_1, \qquad [0 < f_n \leqslant \epsilon] \subset [f_1 > 0]$$

$$0 \leqslant \overline{\lim} \int_E f_n \leqslant \epsilon m[f_1 > 0].$$

积分良定性(续2)

证明：<u>(4b) 情形1</u> $f \in \mathscr{L}^1$:

$$f - f_n \downarrow 0 \xRightarrow{(4a)} \int_E f - f_n \downarrow 0.$$

<u>情形2</u> $f \notin \mathscr{L}^1$:

$$f \notin \mathscr{L}^1 \xRightarrow[f \in S^+]{(3)} \exists a > 0,\ \ m[f \geqslant a] = +\infty$$

$$f_n \geqslant \frac{a}{2}\chi_{[f_n \geqslant \frac{a}{2}]} \xRightarrow{(2)} \int_E f_n \geqslant \frac{a}{2} m[f_n \geqslant \frac{a}{2}] = +\infty.$$

积分良定性(续3)

$$\text{(4c)}\quad f_n\uparrow, g_n\uparrow,\quad \lim f_n \leqslant \lim g_n \xRightarrow{S^+\text{框架内}} \lim \int_E f_n \leqslant \lim \int_E g_n.$$

(注意此时不要求$\lim f_n \in S^+$， $\lim g_n \in S^+$)

$\underline{\text{(4c)}}$ min 阀门: 处理双指标技巧

$\forall$固定m, $\min\{g_n, f_m\}\uparrow f_m$.

$$\lim_{n\to\infty}\int_E g_n \geqslant \lim_{n\to\infty}\int_E \min\{g_n, f_m\} \overset{(4b)}{=\!=\!=} \int_E f_m.$$

积分良定性

注记:　积分的良定性来自于(4c).

积分的等价定义

- $$f \in \mathscr{L}^+(E) \Longrightarrow \int_E f = \sup_{S^+ \ni \varphi \leqslant f} \int_E \varphi$$

积分的等价定义

证明:
$$\begin{aligned}
\int_E f &\xlongequal[\text{def}]{S^+\ni\varphi_k\uparrow f\in\mathscr{L}^+} \lim_{k\to\infty}\int_E \varphi_k \\
&\leqslant \sup_{S^+\ni\varphi\leqslant f}\int_E \varphi \\
&\xlongequal{\exists S^+\ni\psi_k\leqslant f} \lim_{k\to\infty}\int_E \psi_k \\
&\overset{\text{取定}S^+\ni f_k\uparrow f}{\leqslant} \lim_{k\to\infty}\int_E \underbrace{\max\{\psi_1,\psi_2,\cdots,\psi_k,f_k\}}_{\in[f_k,f]}\uparrow \\
&\xlongequal{def} \int_E f
\end{aligned}$$

d. There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

10. Prove Proposition 2.11.

11. Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \leq x \leq a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}.$$

Then f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function on $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

2.2 INTEGRATION OF NONNEGATIVE FUNCTIONS

In this section we fix a measure space $(X, \mathcal{M}, \mu)$, and we define

$$L^+ = \text{the space of all measurable functions from } X \text{ to } [0, \infty].$$

If ϕ is a simple function in L^+ with standard representation $\phi = \sum_1^n a_j \chi_{E_j}$, we define the **integral** of ϕ with respect to μ by

$$\int \phi \, d\mu = \sum_1^n a_j \mu(E_j)$$

(with the convention, as always, that $0 \cdot \infty = 0$). We note that $\int \phi \, d\mu$ may equal ∞. When there is no danger of confusion, we shall also write $\int \phi$ for $\int \phi \, d\mu$. Also, it is sometimes convenient to display the argument of ϕ explicitly, especially when $\phi(x)$ is given by a formula in terms of x or when there are other variables involved; in this case we shall use the notation $\int \phi(x) \, d\mu(x)$. (Some authors prefer to write $\int \phi(x) \, \mu(dx)$ instead.) Finally, if $A \in \mathcal{M}$, then $\phi \chi_A$ is also simple (viz., $\phi \chi_A = \sum a_j \chi_{A \cap E_j}$), and we define $\int_A \phi \, d\mu$ (or $\int_A \phi$ or $\int_A \phi(x) \, d\mu(x)$) to be $\int \phi \chi_A \, d\mu$. The same notational conventions will also apply to the inegrals of more general functions to be defined below. To summarize:

$$\int_A \phi \, d\mu = \int_A \phi = \int_A \phi(x) \, d\mu(x) = \int \phi \chi_A \, d\mu, \qquad \int = \int_X .$$

2.13 Proposition. *Let ϕ and ψ be simple functions in L^+.*

a. If $c \geq 0$, $\int c\phi = c \int \phi$.

b. $\int (\phi + \psi) = \int \phi + \int \psi$.

c. If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.

d. The map $A \mapsto \int_A d\mu$ is a measure on $\mathcal{M}$.

Proof. (a) is trivial. For (b), let $\sum_1^n a_j \chi_{E_j}$ and $\sum_1^m b_k \chi_{F_k}$ be the standard representations of ϕ and ψ. Then $E_j = \bigcup_{k=1}^m (E_j \cap F_k)$ and $F_k = \bigcup_1^n (E_j \cap F_k)$ since $\bigcup_1^n E_j = \bigcup_1^n F_k = X$, and these unions are disjoint. Hence the finite additivity of μ implies that

$$\int \phi + \int \psi = \sum_{j,k} (a_j + b_k) \mu(E_j \cap F_k),$$

and the same reasoning show that the sum on the right equals $\int (\phi + \psi)$. Moreover, if $\phi \leq \psi$, then $a_j \leq b_k$ whenever $E_j \cap F_k \neq \varnothing$, so

$$\int \phi = \sum_{j,k} a_j \mu(E_j \cap F_k) \leq \sum_{j,k} b_k \mu(E_j \cap F_k) = \int \psi,$$

which proves (c). Finally, if $\{A_k\}$ is a disjoint sequence in $\mathcal{M}$ and $A = \bigcup_1^\infty A_k$,

$$\int_A \phi = \sum_j a_j \mu(A \cap E_j) = \sum_{j,k} a_j \mu(A_k \cap E_j) = \sum_k \int_{A_k} \phi,$$

which establishes (d). ∎

We now extend the integral to all functions $f \in L^+$ by defining

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \ \phi \text{ simple} \right\}.$$

By Proposition 2.13c, the two definitions of $\int f$ agree when f is simple, as the family of simple functions over which the supremum is taken includes f itself. Moreover, it is obvious from the definition that

$$\int f \leq \int g \text{ whenever } f \leq g, \text{ and } \int cf = c \int f \text{ for all } c \in [0, \infty).$$

The next step is to establish one of the fundamental convergence theorems.

2.14 The Monotone Convergence Theorem. *If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n \to \infty} f_n$ $(= \sup_n f_n)$, then $\int f = \lim_{n \to \infty} \int f_n$.*

Proof. $\{\int f_n\}$ is an increasing sequence of numbers, so its limit exists (possibly equal to ∞). Moreover, $\int f_n \leq \int f$ for all n, so $\lim \int f_n \leq \int f$. To establish the reverse inequality, fix $\alpha \in (0, 1)$, let ϕ be a simple function with $0 \leq \phi \leq f$, and let $E_n = \{x : f_n(x) \geq \alpha \phi(x)\}$. Then $\{E_n\}$ is an increasing sequence of measurable sets whose union is X, and we have $\int f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \phi$. By Proposition 2.13d and Theorem 1.8c, $\lim \int_{E_n} \phi = \int \phi$, and hence $\lim \int f_n \geq \alpha \int \phi$. Since this is true for all $\alpha < 1$, it remains true for $\alpha = 1$, and taking the supremum over all simple $\phi \leq f$, we obtain $\lim \int f_n \geq \int f$. ∎

The monotone convergence theorem is an essential tool in many situations, but its immediate significance for us is as follows. The definition of $\int f$ involves the supremum over a huge (usually uncountable) family of simple functions, so it may be difficult to evaluate $\int f$ directly from the definition. The monotone convergence theorem, however, assures us that to compute $\int f$ it is enough to compute $\lim \int \phi_n$ where $\{\phi_n\}$ is any sequence of simple functions that increase to f, and Theorem 2.10 guarantees that such sequences exist. As a first application, we establish the additivity of the integral.

2.15 Theorem. *If $\{f_n\}$ is a finite or infinite sequence in L^+ and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$.*

Proof. First consider two functions f_1 and f_2. By Theorem 2.10 we can find sequences $\{\phi_j\}$ and $\{\psi_j\}$ of nonnegative simple functions that increase to f_1 and f_2. Then $\{\phi_j + \psi_j\}$ increases to $f_1 + f_2$, so by the monotone convergence theorem and Theorem 2.13b,

$$\int (f_1 + f_2) = \lim \int (\phi_j + \psi_j) = \lim \int \phi_j + \lim \int \psi_j = \int f_1 + \int f_2.$$

Hence, by induction, $\int \sum_1^N f_n = \sum_1^N \int f_n$ for any finite N. Letting $N \to \infty$ and applying the monotone convergence theorem again, we obtain $\int \sum_1^\infty f_n = \sum_1^\infty \int f_n$. ∎

2.16 Proposition. *If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.*

Proof. This is obvious if f is simple: if $f = \sum_1^n a_j \chi_{E_j}$ with $a_j \geq 0$, then $\int f = 0$ iff for each j either $a_j = 0$ or $\mu(E_j) = 0$. In general, if $f = 0$ a.e. and ϕ is simple with $0 \leq \phi \leq f$, then $\phi = 0$ a.e., hence $\int f = \sup_{\phi \leq f} \int \phi = 0$. On the other hand, $\{x : f(x) > 0\} = \bigcup_1^\infty E_n$ where $E_n = \{x : f(x) > n^{-1}\}$, so if it is false that $f = 0$ a.e., we must have $\mu(E_n) > 0$ for some n. But then $f > n^{-1}\chi_{E_n}$, so $\int f \geq n^{-1}\mu(E_n) > 0$. ∎

2.17 Corollary. *If $\{f_n\} \subset L^+$, $f \in L^+$, and $f_n(x)$ increases to $f(x)$ for a.e. x, then $\int f = \lim \int f_n$.*

Proof. If $f_n(x)$ increases to $f(x)$ for $x \in E$ where $\mu(E^c) = 0$, then $f - f\chi_E = 0$ a.e. and $f_n - f_n\chi_E = 0$ a.e., so by the monotone convergence theorem, $\int f = \int f\chi_E = \lim \int f_n\chi_E = \lim \int f_n$. ∎

The hypothesis that the sequence $\{f_n\}$ be increasing, at least a.e., is essential for the monotone convergence theorem. For example, if X is $\mathbb{R}$ and μ is Lebesgue measure, we have $\chi_{(n,n+1)} \to 0$ and $n\chi_{(0,1/n)} \to 0$ pointwise, but $\int \chi_{(n,n+1)} = \int n\chi_{(0,1/n)} = 1$ for all n. As one sees by sketching the graphs, the trouble in these examples is that the area under the graph "escapes to infinity" as $n \to \infty$, so the area in the limit is less than one would expect. This is typical of the cases when the integral of the limit is not the limit of the integrals, but in this situation there is still an inequality that remains valid. We deduce it from the following general result.

2.18 Fatou's Lemma. *If* $\{f_n\}$ *is any sequence in* L^+, *then*

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

Proof. For each $k \geq 1$ we have $\inf_{n \geq k} f_n \leq f_j$ for $j \geq k$, hence $\int \inf_{n \geq k} f_n \leq \int f_j$ for $j \geq k$, hence $\int \inf_{n \geq k} f_n \leq \inf_{j \geq k} \int f_j$. Now let $k \to \infty$ and apply the monotone convergence theorem:

$$\int (\liminf f_n) = \lim_{k \to \infty} \int \big(\inf_{n \geq k} f_n\big) \leq \liminf \int f_n.$$

∎

2.19 Corollary. *If* $\{f_n\} \subset L^+$, $f \in L^+$, *and* $f_n \to f$ *a.e., then* $\int f \leq \liminf \int f_n$.

Proof. If $f_n \to f$ everywhere, the result is immediate from Fatou's lemma, and this can be achieved by modifying f_n and f on a null set without affecting the integrals, by Proposition 2.16. ∎

2.20 Proposition. *If* $f \in L^+$ *and* $\int f < \infty$, *then* $\{x : f(x) = \infty\}$ *is a null set and* $\{x : f(x) > 0\}$ *is* σ*-finite.*

The proof is left to the reader (Exercise 12).

Exercises

12. Prove Proposition 2.20. (See Proposition 0.20, where a special case is proved.)

13. Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. However, this need not be true if $\int f = \lim \int f_n = \infty$.

14. If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$. (First suppose that g is simple.)

15. If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

16. If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

17. Assume Fatou's lemma and deduce the monotone convergence theorem from it.

2.3 INTEGRATION OF COMPLEX FUNCTIONS

We continue to work on a fixed measure space $(X, \mathcal{M}, \mu)$. The integral defined in the previous section can be extended to real-valued measurable functions f in an obvious