本讲内容

- Fubini定理的证明

- Tonelli 定理 (重积分化为累次积分):

$$f \in L^+(\mathbb{R}^{n+m}) \Longrightarrow \int_{\mathbb{R}^{n+m}} f(x,y)dxdy = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^m} f(x,y)dy\Big)dx.$$

注记

- 注记1:

$$f \in L^+(\mathbb{R}^{n+m}) \Longrightarrow \int_{\mathbb{R}^m} f(x,y)dy \in L^+(\mathbb{R}^n)$$

$$\Longrightarrow f(x,\cdot) \in L^+(\mathbb{R}^m) \qquad a.e.\ x \in \mathbb{R}^n.$$

注记续

- Tonelli定理对于下列函数成立:
  - $f|_{\mathbb{R}^{n+m}\setminus(\{0\}\times\mathbb{R}^m)} = 0$
  - $f(0,\cdot)$ 在$\mathbb{R}^m$上不可测
- Tonelli定理成立的函数, 其在垂直截面上的性质可能很差.

构造集合

$$\Gamma = \left\{ f \in L(\mathbb{R}^{n+m}) : f\text{满足三条性质} \right\}$$

性质1. $$\int_{\mathbb{R}^{n+m}} f(x,y)dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)dy \right) dx.$$

性质2. $$\int_{\mathbb{R}^m} f(x,y)dy \in L(\mathbb{R}^n)$$

性质3. $f(x,\cdot) \in L(\mathbb{R}^m) \qquad a.e.\ x \in \mathbb{R}^n.$

下列性质成立:

$$\Gamma\cap L^+\text{是锥}\qquad(\text{对加法和关于非负数的乘法封闭})$$

$$\Gamma\cap L^1\text{是向量空间}\qquad(\Gamma\text{是向量空间})$$

$$\Gamma\ni f_n\nearrow f\quad\xRightarrow{f_n,f\in L^+}\quad f\in\Gamma$$

$$\Gamma\ni f_n\searrow f\quad\xRightarrow{f_n,f\in L^1}\quad f\in\Gamma$$

预备引理图示

$\Gamma \cap L^+$	锥	关于$\nearrow$ lim封闭(在L^+的框架内)
$\Gamma \cap L^1$	向量空间	关于$\searrow$ lim封闭(在L^1的框架内)

预备引理续

证明:

$$\Gamma \text{ 非空:} \quad 0 \in \Gamma.$$

Γ 关于代数运算的封闭性显然.

预备引理续

Γ 关于极限运算的封闭性:

(1) 来自单调收敛定理.

(2) 意味着

$$\Gamma \cap L^1 \ni f_n \searrow f \in L^1 \Longrightarrow f \in \Gamma \cap L^1.$$

转化为情形一: $f_1 - f_n \nearrow f_1 - f$

利用标准四步骤:

$$\chi_E \xrightarrow[+,\cdot,\lim]{S^+} L^+$$

$$\chi_E \in \Gamma \quad \xRightarrow[(\forall E \subset \mathbb{R}^{n+m}\text{可测})]{\text{预备引理}} L^+(\mathbb{R}^{n+m}) \subset \Gamma$$

- 矩体:

$$E = I \times J, \quad I \subset \mathbb{R}^n, \quad J \subset \mathbb{R}^m, \quad I, J\text{是矩体}.$$

$$\text{性质(1)中的积分相等} = |I|\,|J|.$$

五种情形续

- 开集:

$$E \xlongequal{\text{二进制剖分}} \bigsqcup_{\text{可列}} I_k, \qquad I_k \text{ 是矩体}.$$

$$\chi_E = \sum_{\text{可列}} \chi_{I_k} \in \Gamma \cap L^+ \subset \Gamma.$$

- 紧集:

$$B(0,R) = E \bigsqcup \overbrace{(E^c \cap B(0,R))}^{\text{开集}}, \qquad R \gg 1$$

$$\xrightarrow{\text{预备引理}} \chi_E = \chi_{B(0,R)} - \chi_{E^c \cap B(0,R)} \in \Gamma \cap L^1 \subset \Gamma.$$

五种情形(续)

- 零测集E:

$$\exists 开集 G_k \supset E: \qquad \lim_{k\to\infty} m(G_k) = m(E) = 0$$

$$\Longrightarrow \quad 可设 G_k \downarrow, \qquad m(G_1) < +\infty$$

$$\Longrightarrow \quad H = \bigcap_{k=1}^{\infty} G_k 是 E 的等测包$$

$$\xrightarrow[G_k \in \Gamma \cap L^1]{预备引理} \quad \chi_H \in \Gamma \cap L^1 \subset \Gamma.$$

五种情形(续)

$$0 = m(E) = m(H) \xlongequal{\chi_H \in \Gamma} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} \chi_H(x,y)dy \Big) dx$$

$$\Longrightarrow \qquad \int_{\mathbb{R}^m} \chi_H(x,y)dy \xlongequal{a.e.} 0$$

$$\chi_H \xlongequal{a.e.} 0$$

五种情形(续)

$$\chi_H \in \Gamma, \qquad \chi_H \xlongequal{a.e.} 0 \Longrightarrow \qquad \chi_E \in \Gamma$$

五种情形(续)

- 可测集E:

$$E = \bigcup_{k=1}^{\infty} F_k \bigsqcup Z, \quad \text{紧集} F_k \uparrow \subset \mathbb{R}^{n+m}, \ m(Z) = 0.$$

$$\Longrightarrow \chi_E = \chi_Z + \lim_{k\to\infty} \chi_{F_k} \subset \Gamma \cap L^+ \subset \Gamma.$$

Fubini-Tonelli定理

Theorem 1 (Fubini-Tonelli 定理)

设$f \in L^+(\mathbb{R}^{n+m}) \cup L^1(\mathbb{R}^{n+m})$, 则

$$\int_{\mathbb{R}^{n+m}} f(x,y)dxdy = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} f(x,y)dy \Big) dx.$$

Fubini-Tonelli定理续

- 注记

$$f\in L^1(\mathbb{R}^{n+m}) \Longrightarrow f(x,\cdot)\in L^1(\mathbb{R}^m) \qquad a.e.\ x\in\mathbb{R}^n$$

$$\Longrightarrow f(x,\cdot)\ \ a.e.\text{可测,}\ \ a.e.\text{有限.}$$

$$f\in L^+(\mathbb{R}^{n+m}) \Longrightarrow f(x,\cdot)\in L^+(\mathbb{R}^m) \qquad a.e.\ x\in\mathbb{R}^n$$

$$\Longrightarrow f(x,\cdot)\ \ a.e.\text{可测.}$$

Fubini-Tonelli定理续

证明: $f \in L^1(\mathbb{R}^{n+m}) \xRightarrow{\textit{Tonelli}} f = f^+ - f^-, \ \ f^\pm \in \Gamma \cap L^1$

$$\xRightarrow{\text{预备引理}} f \in \Gamma \cap L^1 \subset \Gamma.$$

关于Fubini-Tonelli定理应用的注记

Fubini-Tonelli定理应用的两部曲:

- Tonelli定理验证f可积性.

$$\int_{\mathbb{R}^{n+m}} |f(x,y)|dxdy = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} |f(x,y)|dy \Big) dx < +\infty \Longrightarrow f \in L^1.$$

- Fubini定理计算f积分值.

$$f \in L^1 \Longrightarrow \int_{\mathbb{R}^{n+m}} f(x,y)dxdy = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} f(x,y)dy \Big) dx.$$

例题

Fubini定理对于下列函数不成立:

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}, \qquad (x,y) \in (0,1] \times (0,1].$$

证明:

$$\int_0^1 \frac{1}{x^2+a^2}dx = \frac{x}{x^2+a^2}\Big|_0^1 + \int_0^1 \frac{2x^2}{(x^2+a^2)^2}dx$$

$$\xRightarrow{\text{移项通分}} \int_0^1 \frac{x^2-a^2}{(x^2+a^2)^2}dx = -\frac{1}{1+a^2}$$

例题续

$$\int_0^1 \Big(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \Big) dy = \int_0^1 -\frac{1}{1 + y^2} dy = -\frac{\pi}{4},$$

$$\int_0^1 \Big(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \Big) dx = \int_0^1 \frac{1}{1 + x^2} dx = \frac{\pi}{4}.$$

例题续

$$\int_0^1 \Big(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \Big) dy \neq \int_0^1 \Big(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \Big) dx$$

$$\int_0^1 \int_0^1 |f(x, y)| dxdx = +\infty.$$

交换积分次序的五个等价定理

测度的σ可加性等价于:

Levi	L^+	$f_n \uparrow f \xRightarrow{f_n \in L^+} \int f_n \uparrow \int f$
Fatou	L^+	$\int \underline{\lim}\, f_n \overset{f_n \in L^+}{\leqslant} \underline{\lim} \int f_n$
Lebesgue	L^1	$f_n \xrightarrow{a.e.} f \xRightarrow[f_n\text{可测}]{\lvert f_n\rvert \leqslant g \in L^1} f_n \xrightarrow{L^1} f, \int f_n \longrightarrow \int f$
Fubini	$L^+ \cup L^1$	$\int_{\mathbb{R}^{n+m}} f \overset{f \in L^+ \cup L^1}{=\!=\!=} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} f(x,y) dm(y) \Big) dm(x).$

例题

设$f \in L^1(\mathbb{R})$, 且

$$\int_{\mathbb{R}} f(x)\varphi(x)dx = 0, \qquad \forall\, \varphi(x) \in L^\infty(\mathbb{R}).$$

证明: $f \xlongequal{a.e.} 0$

证明

$$\int_E f(x)dx = 0, \qquad \forall\ m(E) < \infty.$$

$$取E = [-R, R] \cap [f \geqslant 0] \Longrightarrow f|_{[-R,R]\cap[f\geqslant 0]} \xlongequal{a.e.} 0$$

$$取E = [-R, R] \cap [f < 0] \Longrightarrow f|_{[-R,R]\cap[f<0]} \xlongequal{a.e.} 0$$