本节主要内容

- 抽象积分理论中的Fubini定理
- Vitali覆盖定理

Theorem 1 (Fubini 定理)

设(X, Γ_X, μ), (X, Γ_Y, ν)是σ有限正测度空间,

$$f \in L^+(X \times Y) \cup L^1(X \times Y, \mu \times \nu).$$

则

$$\begin{aligned}\int_{X\times Y} f \, d\mu \times \nu &= \int_X \Big(\int_Y f(x,y) d\nu(y) \Big) d\mu(x) \\ &= \int_Y \Big(\int_X f(x,y) d\mu(x) \Big) d\nu(y).\end{aligned}$$

- 注记

$$f \in L^1(X \times Y, \mu \times \nu)$$

$$\Longrightarrow \quad f(x,\cdot) \in L^1(Y) \qquad (\mu - a.e.\ \ x \in X)$$

$$\Longrightarrow \quad f(x,\cdot)\ \text{可测} \qquad (\mu - a.e.\ \ x \in X)$$

$$\Longrightarrow \quad f(x,y)\ \text{有限} \qquad (\mu - a.e.\ \ x \in X, \quad \nu - a.e.\ \ y \in Y)$$

Fubini-Tonelli定理续

- 注记

$$f \in L^{+}(X \times Y, \mu \times \nu) \implies f(x, \cdot) \in L^{+}(Y) \qquad \mu - a.e.\ x \in X$$

- 乘积测度空间

$$(X \times Y, \quad \Gamma_{X\times Y}, \quad \mu \times \nu)$$

乘积测度空间σ代数

- $\Gamma_{X\times Y}$ 是$\Gamma_X\times\Gamma_Y$生成的最小σ代数.

乘积测度空间的乘积测度

- 乘积测度是由Caratheodory构造出的测度, 它的出发点是

$$\mu \times \nu(A \times B) = \mu(A) \times \nu(B),$$

- 它的定义域$\hat{\Gamma}_{X\times Y}$:

$$\hat{\Gamma}_{X\times Y} \supset \Gamma_{X\times Y}, \qquad \text{加入所有的零测集的子集}$$

- 乘积测度空间$(X\times Y,\hat{\Gamma}_{X\times Y},\mu\times\nu)$是完备测度空间
- $(X\times Y,\Gamma_{X\times Y},\mu\times\nu)$可能非完备.
- 若(X,Γ_X,μ), (X,Γ_Y,ν)是σ有限测度空间,

  则$(X\times Y,\Gamma_{X\times Y},\mu\times\nu)$是$\sigma$有限测度空间.

逐项积分定理: 计数测度

- 取计数测度ν:

Fubini 定理 $\Longleftarrow\!\Longrightarrow$ 逐项积分定理 $\Longleftarrow\!\Longrightarrow$ 测度σ可加性

$\Longleftarrow\!\Longrightarrow$ Levi单调收敛定理

$\Longleftarrow\!\Longrightarrow$ Lebesgue 控制收敛定理

逐项积分定理的条件

(X, Γ_X, μ)是σ有限测度空间.

$$f(x,n) := f_n(x)$$

- $f(x,n) \in L^+(X \times \mathbb{N}) \Longleftrightarrow f_n \in L^+(E).$
- $f(x,n) \in L^1(X \times \mathbb{N}, \mu \times \nu) \Longleftrightarrow \sum_{n=1}^{\infty} \int_X |f_n(x)| d\mu(x) < +\infty$

注记

数列 $\Longleftrightarrow$ 函数 (维数升高一维)

级数(极限) $\Longleftrightarrow$ 积分 (离散测度)

$$Levi = Lebesgue = Fubini = \text{交换积分次序}$$

Theorem 2 (逐项积分定理)

(X, Γ_X, μ)是σ有限测度空间.

$$f_n(x) =: f(x,n) \in L^+ \cup L^1(X \times \mathbb{N}, \mu \times \nu))$$

$$\Longrightarrow \int_X \sum_{n=1}^{\infty} f_n(x) d\mu(x) = \sum_{n=1}^{\infty} \int_X f_n(x) d\mu(x).$$

测度的σ可加性等价于:

Levi	L^+	$f_n \uparrow f \xRightarrow{f_n \in L^+} \int f_n \uparrow \int f$
Fatou	L^+	$\int \underline{\lim} f_n \overset{f_n \in L^+}{\leqslant} \underline{\lim} \int f_n$
Lebesgue	L^1	$f_n \xrightarrow{a.e.} f \xRightarrow[f_n\text{可测}]{\lvert f_n\rvert \leqslant g \in L^1} f_n \xrightarrow{L^1} f, \int f_n \uparrow \int f$
Fubini	$L^+ \cup L^1$	$\int_{X\times Y} f \xlongequal{f \in L^+ \cup L^1} \int_X \Big(\int_Y f(x,y) d\nu(y) \Big) d\mu(x).$

- Radon-Nikodym 定理

  设ν, μ是可测空间(X, Γ)测度, ν有限, μ是σ有限, 则

$$\nu \ll \mu \quad \Longleftrightarrow \quad d\nu = f d\mu \quad (\exists f \in L^{+} \cap L^{1}(X, \mu))$$

$$\text{即}\nu(A) = \int_A d\nu = \int_A f d\mu, \ \ \forall A \in \Gamma.$$

- Radon-Nikodym 定理刻画: 哪些测度本质上就是函数?

Theorem 3

$$m_{n+1}(U(f)) = \int_E f(x)dm(x) = \int_0^{\infty} m\Big\{x \in E : t < f(x)\Big\}dt$$

- $f \in \mathscr{L}^+(E)$, f的图形的下方

$$U(f) = \Big\{(x,t) \in E \times [0,+\infty) : 0 \leqslant t < f(x)\Big\}$$

- 积分是曲边梯形的面积.

$$\text{微元面积} = \text{长} \times \text{宽} = \overbrace{m\Big\{x \in E : t < f(x)\Big\}}^{\text{图形下方长度}} dt$$

证明： 图形下方是可测集(f非负有界):

$$U(f)=\lim_{k\to\infty}\bigsqcup_{j=1}^{2^{2k}}\left(f^{-1}[\frac{j-1}{2^k},\frac{j}{2^k})\times[0,\frac{j}{2^k})\right).$$

蛋糕分层

积分是高维的测度续

$$
\begin{aligned}
m_{n+1}(U(f)) &= \int_{E\times(0,+\infty)} \chi_{U(f)}(x,t)dm_n(x)dt \\
&= \int_E \left(\int_0^{+\infty} \chi_{U(f)}(x,t)dt \right) dm_n(x) \\
&= \int_E \int_0^{f(x)} dtdm_n(x) \\
&= \int_E f(x)dm_n(x) \\
&= \int_0^{\infty} \left(\int_E \chi_{U(f)}(x,t)dm_n(x) \right) dt \\
&= \int_0^{\infty} m\left\{x \in E : t < f(x)\right\} dt
\end{aligned}
$$

积分是高维的测度

Theorem 4

$$(\mu \times m)(U(f)) = \int_X f(x)d\mu(x) = \int_0^\infty \mu\left\{x \in X : t < f(x)\right\}dm(t)$$

- 测度空间(X, Γ, μ), *Borel*测度空间$(\mathbb{R}, \mathscr{B}(\mathbb{R}), m)$
- $f \in \mathscr{L}^+(X)$, f的图形的下方

$$U(f) = \left\{(x,t) \in X \times [0, +\infty) : t < f(x)\right\}$$

- 积分是曲边梯形的面积.

$$\text{微元面积= 长} \times \text{ 宽} = \overbrace{\mu\left\{x \in X : t < f(x)\right\}}^{\text{图形下方水平集长度}}\ dt$$

Theorem 5

假设$f \in C^1(\mathbb{R}^n, \mathbb{R})$, $g \in C(\mathbb{R}^n, \mathbb{R})$. 则

$$\int_{\mathbb{R}^n} g(x)|\nabla f(x)|dx = \int_{\mathbb{R}} \int_{f^{-1}(r)} g(x)d\sigma(x)dr.$$

余面积公式

- 水平集$f(x)=r$:

  切空间是$n-1$维的(对于a.e.r, Sard定理)，法向量为$\frac{f(x)}{|f(x)|}$.

- 假设x_0在水平集$f(x)=r$上，沿着法向量移动距离dh, 则

$$x-x_0=\frac{\nabla f(x_0)}{|\nabla f(x_0)|}dh$$

- 相应的函数的改变量

$$\Delta r=f(x)-f(x_0)=\nabla f(x_0)(x-x_0)+\cdots=|\nabla f(x_0)|dh+O(\Delta h^2)$$

$$dt=|\nabla f(x_0)|dh$$

- 两种坐标: $dx,(dh,ds)$. 切空间面积微元 $ds=d\sigma(s)$, $\mathbb{R}^n$ 上的体积元

$$dx=dhds=|\nabla f(x_0)|^{-1}drds$$

$$|\nabla f(x_0)|dx=drds$$

极坐标积分公式

Theorem 6

假设$g \in C(\mathbb{R}^n, \mathbb{R})$. 则

$$\int_{\mathbb{R}^n} g(x)dx = \int_0^\infty \int_{|x|=t} g(s)d\sigma(s)dt.$$

$$f(x) = |x| \Longrightarrow \nabla f(x) = \frac{x}{|x|} \Longrightarrow |\nabla f(x)| = 1.$$

极坐标：　$x = r\theta, \qquad \theta \in S^{n-1}.$

体积元：　$dx = r^{n-1}drd\theta = r^{n-1}drd\sigma, \qquad d\sigma$是面积元

$$\int_{\mathbb{R}^n} g(x)dx = \int_0^\infty \int_{S^{n-1}} g(r\theta)d\theta dr.$$

Vitali覆盖

- Vitali覆盖是分析的手术刀:

$$\text{无穷} \xRightarrow{\text{转化为}} \text{可列、有限}$$

- 应用:

$$\text{导数} \xRightarrow{\text{转化为}} \text{差商}$$

Vitali覆盖的定义

$$E \subset \mathbb{R}, \qquad \Gamma\text{是}E\text{的Vitali覆盖}$$

- Γ是E 的覆盖
- $\Gamma =$ 非退化的闭区间族
- 每点处无穷小层次覆盖:

$$\forall\ x \in E, \quad \inf\left\{|I| : \quad x \in I \in \Gamma\right\} = 0$$

Theorem 7 (Vitali覆盖定理)

$E \subset \mathbb{R}$, $m^*(E) < \infty$, Γ 是E的Vitali覆盖.

$$\Longrightarrow E \subset \bigsqcup_{j=1}^{\infty} I_j \sqcup Z \qquad (\exists I_j \in \Gamma, m(\bigsqcup_{j=1}^{\infty} I_j) < \infty,\ m(Z) = 0)$$

$$\overset{\text{逐次收割}}{\Longleftrightarrow} \forall \epsilon > 0, \quad E \subset \bigsqcup_{j=1}^{n(\epsilon)} I_j \sqcup E_\epsilon \quad (\exists I_j \in \Gamma, \quad m(E_\epsilon) < \epsilon.)$$

- Vitali覆盖定理在$\mathbb{R}^n$中成立:

$$\text{闭区间} \quad \longrightarrow \quad \text{闭球}$$

$$|I| \quad \longrightarrow \quad \text{闭球直径}.$$

Vitali覆盖定理的证明

- 宇宙取为测度有限开集. 不妨设

$$E \subset G \subset \mathbb{R}, \qquad m(G) < +\infty, \qquad I \subset G(\mathit{Vitali}) \qquad (\forall\ I \in \Gamma).$$

续

- 归纳选取:

$$\text{设已选取} I_1, \ldots, I_n \in \Gamma \text{互不相交}$$

$$\Longrightarrow \qquad \text{选取} I_1, \ldots, I_{n+1} \in \Gamma \text{互不相交}$$

续

- 贪婪算法(每步择优):

$$可设 E \setminus \bigsqcup_{j=1}^{n} I_j \neq \emptyset$$

$$\delta_n = \sup \left\{ |I| : \ I \in \Gamma, \ I与 I_1, \ldots, I_n互不相交 \right\}$$

$$\xrightarrow{|I| \leqslant m(G) < \infty} \quad \delta_n \in (0, +\infty)$$

$$\Longrightarrow \quad 选取 I_{n+1} : \quad |I_{n+1}| > \frac{\delta_n}{2}.$$

Vitali覆盖定理的证明续

- 假设上述过程一直继续下去:

$$\text{存在互不相交可列个} I_j \in \Gamma, \text{满足}$$

$$\sum_{k=1}^{\infty} |I_k| \leqslant m(G) < +\infty \Longrightarrow |I_k| \to 0$$

Vitali覆盖定理的证明续

- 断言:

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}:$$

$$E \setminus \bigsqcup_{k=1}^{N} I_k \subset \bigcup_{k=N+1}^{\infty} 5I_k \qquad \sum_{k=N+1}^{\infty} |I_k| < \frac{\epsilon}{5}.$$

- 记号: $5I$与I同心, $|5I| = 5|I|$

Vitali覆盖定理的证明续

- 利用断言证明定理:

$$m(E\setminus\bigsqcup_{k=1}^{\infty}I_k)\leqslant m(E\setminus\bigsqcup_{k=1}^{N}I_k)<\epsilon.$$

续

断言的证明:

$$\text{只要证明:}\quad E\setminus\bigsqcup_{k=1}^{N} I_k \subset \bigsqcup_{k=N+1}^{\infty} 5I_k, \qquad \forall N\in\mathbb{N}.$$

$$\forall\, x\in E\setminus\bigsqcup_{k=1}^{N} I_k \quad \stackrel{\exists I}{\Longrightarrow} \quad x\in I\in\Gamma,$$

$$I\cap\bigsqcup_{k=1}^{N} I_k = \emptyset$$

$$I\cap\bigsqcup_{k=1}^{\infty} I_k \neq \emptyset$$

否则$|I|\leqslant \delta_k \leqslant 2|I_{k+1}| \longrightarrow 0.$

续

- I与$I_1, \dots I_N, \dots I_{n_0-1}$不相交，I与$I_{n_0}$相交
- $n_0 > N, \quad |I| \leqslant \delta_{n_0-1} < 2|I_{n_0}|.$
- $x \in \Gamma \Longrightarrow x \in 5I_{n_0}$，断言成立.