本讲主要内容

- 全变差计算方法
- L^∞版本的微积分基本定理
- L^1版本的逐项求导

- 内容一：　全变差计算方法

全变差计算方法

Theorem 1

$$f \in AC[a,b] \quad \overset{f\in W^{1,1}[a,b]}{\Longleftrightarrow} \quad V_a^x f = \int_a^x |f'|$$

特例:　全变差是积分

$$f \in C^1[a,b] \Longrightarrow V_a^x f = \int_a^x |f'|$$

充分性:

$$f \in W^{1,1}[a,b] \Longrightarrow |f'| \in L^1[a,b] \Longrightarrow \int_a^x |f'| \in AC[a,b]$$

$$\xrightarrow{V_a^x f = \int_a^x |f'|} V_a^x f \in AC[a,b]$$

$$\Longrightarrow f \in AC[a,b]$$

$$(\forall\, y > x \Longrightarrow |f(y) - f(x)| \leqslant V_a^y - V_a^x).$$

全变差计算方法

必要性：$\left(\forall\, y > x \Longrightarrow |f(y) - f(x)| \leqslant V_a^y - V_a^x\right)$

$$\xrightarrow{f\in AC[a,b]} \quad |f'(x)| \overset{a.e.}{\leqslant} \frac{d}{dx} V_a^x f$$

$$\Longrightarrow \quad \int_a^x |f'| \leqslant V_a^x f \xlongequal{f(x)=f(a)+\int_a^x f'} V_a^x \int_a^x f'$$

$$\leqslant V_a^x \int_a^x (f')^+ \uparrow + V_a^x \int_a^x (f')^- \uparrow$$

$$= \int_a^x (f')^+ + \int_a^x (f')^- = \int_a^x |f'|$$

- 内容二：　L^∞版本的微积分基本定理

记号

- $L^\infty[a,b]=[a,b]$上几乎处处有界可测函数全体.

- Lipschitz函数$f:[a,b]\longrightarrow\mathbb{R}$:

$$f\in Lip[a,b] \Longleftrightarrow \quad \exists M>0,\ \ \forall\ x,y\in[a,b]:$$

$$|f(x)-f(y)|\leqslant M|x-y|$$

$L^\infty[a,b]$ 框架下微积分基本定理

- $L^\infty[a,b]$框架下微积分基本定理:

$$L^\infty[a,b] \underset{\frac{d}{dx}}{\overset{\int_a^x}{\rightleftharpoons}} Lip[a,b] \qquad \text{（双射）}$$

(a.e. 相等视为恒等) (相差常数视为恒等)

$\Updownarrow$ $\Updownarrow$

$\int_a^x$ 单射 $\frac{d}{dx}$ 单射

续

证明: (1) $f \in L^\infty[a,b] \subset L^1[a,b]$

$$\Longrightarrow \int_a^x f \in AC[a,b].$$

$$\Longrightarrow \left|\int_a^y f - \int_a^x f\right| \leqslant \int_x^y |f| \leqslant M|y-x| \qquad (\forall\, x < y)$$

$$\Longrightarrow \int_a^x f \in Lip[a,b].$$

续

(2). $f \in Lip[a,b] \subset AC[a,b] \subset W^{1,1}[a,b]$.

$$\Longrightarrow \quad f' \text{几乎处处存在}, \quad |f(y) - f(x)| \leqslant M|y - x|$$

$$\Longrightarrow \quad |f'| \overset{a.e.}{\leqslant} M$$

$$\Longrightarrow \quad f' \in L^{\infty}[a,b].$$

具体实例

$$f(x) = x^{\alpha}\sin\frac{1}{x^{\beta}}, \quad x\in[0,1], \quad \alpha,\beta>0.$$

$W^{1,1}[0,1]$	$\alpha>\beta$
$BV[0,1]$	$\alpha>\beta$
$AC[0,1]$	$\alpha>\beta$
$Lip[0,1]$	$\alpha\geqslant\beta+1$
$C^1[0,1]$	$\alpha>\beta+1$

例题1

$$\alpha > \beta > 0 \Longrightarrow f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}} \in AC[0,1]$$

续

假设$\alpha > \beta > 0$, 则

$$f'(x) \xlongequal{a.e.} \alpha x^{\alpha-1} \sin \frac{1}{x^\beta} - \beta x^{\alpha-\beta-1} \cos \frac{1}{x^\beta}$$

$$\Longrightarrow \quad |f'(x)| \overset{a.e.}{\leqslant} \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} \in L^1[0,1]$$

$$\Longrightarrow \quad f(x) - f(\epsilon) = \int_\epsilon^x f' \quad \text{(Riemann框架下基本定理)}$$

$$\xrightarrow[\text{控制收敛}]{\epsilon\to 0^+} \quad f(x) = f(0) + \int_0^x f'(x)dx \in AC[a,b]$$

例题2

$$0 < \alpha \leqslant \beta \Longrightarrow f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}} \notin W^{1,1}[0,1].$$

续

假设$0 < \alpha \leqslant \beta$, 则

$$f'(x) \xlongequal{a.e.} \alpha x^{\alpha-1} \sin \frac{1}{x^\beta} - \beta x^{\alpha-\beta-1} \cos \frac{1}{x^\beta} \notin L^1[0,1]$$

$$\text{右边第一项} \in L^1[0,1], \qquad \text{右边第二项} \notin L^1[0,1],$$

续

$$0 < \alpha \leqslant \beta \Longrightarrow x^{\alpha-\beta-1}\left|\cos\frac{1}{x^\beta}\right| \notin L^1[0,1]$$

证明: 做变换$t = x^{-\beta}$, 利用$|\cos t| \geqslant |\cos t|^2 = \frac{1}{2}(\cos(2t)+1)$

$$\begin{aligned}\int_\epsilon^1 x^{\alpha-\beta-1}\left|\cos\frac{1}{x^\beta}\right|dx &= \frac{1}{\beta}\int_1^{\epsilon^{-\beta}} t^{-\frac{\alpha}{\beta}}|\cos t|dt \\ &\geqslant \frac{1}{2\beta}\Big(\int_1^{\epsilon^{-\beta}} t^{-\frac{\alpha}{\beta}}\cos(2t)dt + \int_1^{\epsilon^{-\beta}} t^{-\frac{\alpha}{\beta}}dt\Big) \\ &\to \frac{1}{2\beta}\Big(\int_1^{\infty} t^{-\frac{\alpha}{\beta}}\cos(2t)dt + \int_1^{\infty} t^{-\frac{\alpha}{\beta}}dt\Big) = \infty.\end{aligned}$$

$$Dirichlet判别法: t^{-\frac{\alpha}{\beta}} \downarrow 0$$

$$\int_0^1 x^{\alpha-\beta-1}\left|\cos\frac{1}{x^\beta}\right|dx \geqslant \int_\epsilon^1 x^{\alpha-\beta-1}\left|\cos\frac{1}{x^\beta}\right|dx \to +\infty$$

例题2'

$$0 < \alpha \leqslant \beta \xRightarrow{\text{直接证明}} f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}} \notin \mathrm{BV}[0, 1].$$

证明: 取区间剖分

$$\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$$

$$\left(\frac{1}{x_i}\right)^{\beta} = (n-1-i)\pi + \frac{\pi}{2}, \qquad i = 1, 2, \cdots, n-1.$$

$$f(x_{i-1})f(x_i) < 0, \qquad |f(x)| \leqslant 1.$$

$$\begin{aligned}\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| &\geqslant \sum_{i=2}^{n-1}|f(x_i)| \\ &\geqslant \sum_{i=2}^{n-1}|f(x_i)|^{\frac{\beta}{\alpha}} \\ &= \sum_{i=2}^{n-1}\frac{1}{(n-i)\pi+\frac{\pi}{2}} \\ &= \sum_{k=1}^{n-2}\frac{1}{k\pi+\frac{\pi}{2}} \to \infty\end{aligned}$$

$$\Longrightarrow V_0^1 f = \infty.$$

例题3

$$f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}} \in \mathrm{Lip}[0,1] \overset{\alpha,\beta>0}{\Longleftrightarrow} \alpha \geqslant \beta + 1.$$

证明:

$$f \in Lip[0,1] \xLeftrightarrow{\text{微积分基本定理}} f' \in L^{\infty}[0,\infty]$$

$$f'(x) \overset{a.e.}{=\!=} \alpha x^{\alpha-1} \sin \frac{1}{x^{\beta}} - \beta x^{\alpha-\beta-1} \cos \frac{1}{x^{\beta}}$$

$$\xRightarrow{\alpha,\beta>0} \quad f'(x) \in L^{\infty}[0,1] \text{当且仅当} \alpha \geqslant \beta+1>1$$

例题4

$$f \in AC[-1,1], \quad g \in C^1[-1,1] \subset AC[-1,1], \qquad f\text{和}g\text{可复合}.$$

$$\not\Rightarrow \qquad f \circ g \in AC[-1,1].$$

续

证明: 取

$$
\begin{aligned}
f(y) &= y^{\frac{1}{3}} \in AC[-1,1] \setminus C^1[-1,1] \\
g(x) &= \left(x \sin \frac{1}{x}\right)^3 \in C^1[-1,1] \\
f \circ g(x) &= x \sin \frac{1}{x} \in C[-1,1] \setminus AC[-1,1]
\end{aligned}
$$

注意: 可复合的条件要求在$[-1,1]$而不是$[0,1]$上考虑.

续

证明: 取

$$f(y) = \int_0^y f' \xrightarrow{f' \in L^1[-1,1]} f \in AC[-1,1].$$

$$g'(x) = 3x^2 \sin^3 \frac{1}{x} - 3x \sin^2 \frac{1}{x} \cos \frac{1}{x} \in C[-1,1] \xrightarrow{g'(0)=0} g \in C^1[-1,1]$$

$$f \circ g(x) = x \sin \frac{1}{x} \notin AC[-1,1]$$

例题5

$$f \in \mathrm{Lip}[0,1], \quad g \in AC[0,1], \qquad f\text{和}g\text{可复合}.$$

$$\Longrightarrow \qquad f \circ g \in AC[0,1].$$

续

证明: $$|f(x) - f(y)| \leqslant M|x - y|, \qquad x, y \in [0, 1].$$

$$g \in AC[a, b]$$

$$\Longleftrightarrow \quad \forall\, \epsilon > 0, \exists \delta > 0 : m(\bigsqcup_{k=1}^{n}(a_k, b_k)) < \delta \Longrightarrow \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \epsilon$$

$$\Longrightarrow \quad \sum_{k=1}^{n} |f \circ g(b_k) - f \circ g(a_k)| \leqslant M \sum_{k=1}^{n} |g(b_k) - g(a_k)| \leqslant M\epsilon$$

$$\Longrightarrow \quad f \circ g \in AC[0, 1]$$

- 内容三: L^1框架下逐项微分

逐项微分Fubini定理

$$(1) \quad f_k \in AC[a,b]$$

$$(2) \quad \sum_{k=1}^{\infty} f_k(c)\text{收敛}, \quad \exists c \in [a,b]$$

$$(3) \quad \sum_{k=1}^{\infty} \int_a^b |f_k'(x)|dx < \infty$$

$$\Longrightarrow \quad \sum_{k=1}^{\infty} f_k \in AC[a,b] \qquad \left(\sum_{k=1}^{\infty} f_k\right)' \overset{a.e.}{=\!=\!=} \sum_{k=1}^{\infty} f_k'$$

注记: 在L^1框架下的逐项微分Fubini定理

- L^1框架: 逐项微分后的级数的L^1可积性.
- 逐项微分的C^1条件弱化为弱C^1条件, 即绝对连续条件.
- 级数收敛的要求可简化为在一点处收敛
- 逐项微分的等式是几乎处处成立的等式

注记

- 逐项微分后的级数的L^1可积性.

$$f_k'(x) =: f'(x,k) \in L^1([a,b]\times\mathbb{N}) \iff \sum_{k=1}^{\infty}\int_a^b |f_k'(x)|dx < \infty$$

$$\iff \sum_{k=1}^{\infty}|f_k'(x)| \in L^1[a,b]$$

证明

$$\xrightarrow[L^1\text{框架}]{\text{条件(3)}} \quad \sum_{k=1}^{\infty}\int_a^x f_k'(t)dt \xlongequal{Fubini} \int_a^x \sum_{k=1}^{\infty} f_k'(t)dt < \infty$$

$$\xrightarrow[f_k\in AC[a,b]]{\text{条件(2)}} \quad \int_c^x f_k'(t)dt = f_k(x) - f_k(c)$$

续

$$\Longrightarrow \quad \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k(c) + \int_c^x \underbrace{\sum_{k=1}^{\infty} f_k'(t)}_{\in L^1[a,b]} dt \in AC[a,b]$$

$$\Longrightarrow \quad \Big(\sum_{k=1}^{\infty} f_k\Big)' \xlongequal{a.e.} \sum_{k=1}^{\infty} f_k'$$