本讲主要内容

- 习题课

例题1

$$\mathbb{R} \ni \lambda_n \to +\infty \Longrightarrow m\left\{x \in \mathbb{R} : \lim_{n\to\infty} \sin \lambda_n x \exists\right\} = 0$$

证明:　令

$$A := \left\{ x \in \mathbb{R} : \lim_{n\to\infty} \sin \lambda_n x \exists \right\}$$

A可测: Cauchy 序列ϵ语言+可测函数关于一个区间的原像可测。

$$A = \bigcap_{m\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{i>j>N} f_{i,j}^{-1}(-\frac{1}{m}, \frac{1}{m}).$$

其中

$$f_{i,j}(x) :== |f_i(x) - f_j(x)| \quad 可测.$$

自然地考虑函数

$$f(x) := \chi_A(x) \lim_{n\to\infty} \sin \lambda_n x \qquad 零扩充$$

利用积分理论研究该函数:

$$\int_E f(x)dx \xlongequal[\forall\text{有界可测}E]{\text{控制收敛}} \lim \int_E \chi_A(x)\sin\lambda_n x dx \xlongequal{Lebesgue} 0$$

$$\xrightarrow{\text{Lebesgue点定理}} f(x) \xlongequal{a.e.} \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f = 0$$

$$0=\int_E f^2(x)dx \xlongequal[\forall\text{有界可测}E]{\text{控制收敛}} \lim\int_{E\cap A}\sin^2\lambda_n x dx=\frac{1}{2}m(E\cap A)$$

$$\Longrightarrow m(A)=0.$$

注记

利用积分论处理测度论技巧:

$$\text{零测集} \xRightarrow{\text{自然构造}} \text{零函数} \xRightarrow{\text{Lebesgue点理论}} \text{积分理论}.$$

例题2

假设

(1) $f_k, f \in L^1(E)$, $E \subset \mathbb{R}^n$可测.

(2) $f_k \xrightarrow{a.e.} f$

则

$$f_k \xrightarrow{L^1} f \quad \Longleftrightarrow \quad \|f_k\|_{L^1(E)} \longrightarrow \|f\|_{L^1(E)}$$

证明

$$f \in L^1(E) \Longrightarrow \quad \forall \epsilon > 0, \quad \exists\, A \subset E, \quad \exists\, \delta > 0:$$

$$m(A) < \infty, \quad \int_{A^c} |f| < \frac{\epsilon}{2}, \quad \int_C |f| < \frac{\epsilon}{2} \quad \text{if} \quad m(C) < \delta$$

$$\xrightarrow{\text{Egorov}} \quad \exists B_0 \subset A: \quad m(A \setminus B_0) < \delta, \qquad f_n \rightrightarrows f \quad \text{on} \quad B_0$$

$$\Longrightarrow \quad \int_E |f| = \int_{A^c} |f| + \int_{A \setminus B_0} |f| + \int_{B_0} |f| < \epsilon + \int_{B_0} |f|$$

续

$$\int_E |f| < \epsilon + \varliminf_{k\to\infty} \int_{B_0} |f_k| \qquad \text{(Fatou引理)}$$

$$= \epsilon + \varliminf \left(\|f_k\|_{L^1(E)} - \int_{B_0^c} |f_k| \right)$$

$$\Longrightarrow \quad \varlimsup \int_{B_0^c} |f_k| < \epsilon$$

$$\Longrightarrow \quad \|f_k - f\|_{L^1(E)} \leqslant \int_{B_0^c} |f_k| + \int_{B_0^c} |f| + \sup_{B_0} |f_k - f| m(B_0)$$

例题3

设$\forall\ f \in L^1(\mathbb{R}^n), \quad x_0 \in \mathbb{R}^n, \quad a \in \mathbb{R} \setminus \{0\}.$

$$\int_{\mathbb{R}^n} f(x+x_0)dx = \int_{\mathbb{R}^n} f(x)dx;$$

$$\int_{\mathbb{R}^n} f(ax)dx = \frac{1}{|a|^n}\int_{\mathbb{R}^n} f(x)dx;$$

证明: 只要验证$f=\chi_A$情形, 这归结于测度论的性质.

例题4

$$f \in L^1[0,+\infty) \Longrightarrow \lim_{n\to\infty} f(x+n) \xlongequal{a.e.} 0.$$

证明:

$$\sum_{n=0}^{\infty} \int_0^1 |f(x+n)|\, dx = \int_0^{\infty} |f(x)|\, dx < +\infty$$

$$\Longrightarrow \lim_{n\to\infty} f(x+n) \xlongequal{a.e.in[0,1]} 0.$$

例题5

设$a_n > 0$, $\sum\limits_{n=1}^{\infty} \frac{1}{a_n} < +\infty$.

$$f \in L^1(\mathbb{R}) \Longrightarrow \lim_{n\to\infty} f(a_n x) \xlongequal{a.e.} 0.$$

证明:

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f(a_n x)|\, dx = \sum_{n=1}^{\infty} \frac{1}{a_n} \int_{\mathbb{R}} |f(x)|\, dx < +\infty.$$

- 假设$E \subset \mathbb{R}^n$, $m(E) < \infty$, $f_n \in \mathscr{L}^1(E)$, 则

$$f_n \rightrightarrows f \Longrightarrow f \in \mathscr{L}^1(E), \quad \int_E f_n \longrightarrow \int_E f.$$

- $m(E) = \infty$结论不成立.

反例 $$f_n = \frac{1}{2^n}\chi_{(0,2^n]}.$$

例题7: 积分公式

- $E \subset \mathbb{R}$ Lebesgue可测,
- $f \in L^+(E)$,
- $\varphi : [0, +\infty) \to \mathbb{R} \uparrow$, 内闭绝对连续, $\varphi(0) = 0$.

$$\Longrightarrow \int_E \varphi(f(x))dx = \int_0^\infty m[x \in E : f(x) > t]\varphi'(t)dt.$$

证明

$$\varphi(a) = \int_0^a \varphi'(t)dt, \qquad \forall a \in [0,\infty)$$

$$\Longrightarrow \varphi(f(x)) = \int_0^{f(x)} \varphi'(t)dt = \int_{\mathbb{R}^+} \chi_{[0,f(x)]}(t)\varphi'(t)dt.$$

$$\begin{aligned}\Longrightarrow \int_E \varphi(f(x))dx &= \int_{\mathbb{R}} \chi_E(x)\Big(\int_{\mathbb{R}^+} \chi_{[0,f(x)]}(t)\varphi'(t)dt\Big)dx \\ &= \int_{\mathbb{R}^+} \varphi'(t)dt \int_{\mathbb{R}} \chi_E(x)\chi_{[f(x)>t]}(x)dx \\ &= \int_0^\infty m[x \in E : f(x) > t]\varphi'(t)dt.\end{aligned}$$

例题8: 导数是伸缩率

- $E \subset [a, b]$,
- $f : [a, b] \longrightarrow \mathbb{R}$
- f在E上每一点可导, $|f'(x)| \leqslant M, \forall x \in E$.

$$\Longrightarrow m^*(f(E)) \leqslant Mm^*(E).$$

证明

固定$\epsilon > 0$.

$$E_n := \left\{x \in E : |f(y - f(x)| \leqslant (M + \epsilon)|y - x|, \forall y \in [a,b] \cap B(x, \frac{1}{n})\right\}.$$

$$\lim_{n\to\infty} m^*(E_n \uparrow) = m^*(E)$$

$$E_n \setminus \{a,b\} \subset \bigcup_{k=1}^{\infty} I_{n,k}, \qquad \text{开区间} I_{n,k} \subset (a,b):$$

$$\sum_{k=1}^{\infty} m(I_{n,k}) < m^*(E) + \epsilon, \qquad m(I_{n,k}) < \frac{1}{n}, \qquad \forall k \in \mathbb{N}.$$

证明续

$$s, t \in E_n \bigcap I_{n,k} \Longrightarrow |f(s) - f(t)| \leqslant (M+\epsilon)|s-t| \leqslant (M+\epsilon) m(I_{n,k})$$

$$\begin{aligned} m^*(f(E_n)) &\leqslant m^*(f(E_n \cap \bigcup_k I_{n,k})) \\ &\leqslant \sum_k m^*(f(E_n \cap I_{n,k})) \\ &\leqslant \sum_k \operatorname{diam}(f(E_n \cap I_{n,k})) \\ &\leqslant (M+\epsilon) \sum_k m(I_{n,k})) \\ &\leqslant (M+\epsilon)(m^*(E_n) + \epsilon) \end{aligned}$$

令$n \to \infty$, $\epsilon \to 0$即可.

例题9: 映照像的测度

- $f : [a, b] \longrightarrow \mathbb{R}$可测
- $E \subset [a, b]$可测
- f在E上每一点可导

$$\Longrightarrow m^*(f(E)) \leqslant \int_E |f'(x)| dx.$$

证明

固定$\epsilon > 0$.

$$E_n := \left\{x \in E : (n-1)\epsilon \leqslant |f'(x)| < n\epsilon\right\}$$ 可测(差商可测从而导数可测).

$$\begin{aligned} m^*(f(E_n)) &\leqslant n\epsilon m^*(E_n) \\ &\leqslant \int_{E_n} |f'(x)|dx + \epsilon m^*(E_n) \end{aligned}$$

$$f(E) = f(\bigcup_n E_n) = \bigcup_n f(E_n)$$

$$m^*(f(E)) \leqslant \int_E |f'(x)|dx + \epsilon m^*(E)$$

例题10

- $f \in C[a,b] \cap W^{1,1}[a,b]$,
- 除了一个至多可数集外, f'存在而且有限

$$\Longrightarrow f \in AC[a,b]$$

证明

$$A := \left\{x \in [a,b] : f'(x)\text{不存在}\right\}, \quad \text{至多可数}$$

$$\forall (\alpha,\beta) \subset [a,b]$$

$$\begin{aligned} |f(\beta) - f(\alpha)| &\leqslant m(f(\alpha,\beta)) \quad (\text{介值定理}) \\ &= m(f((\alpha,\beta) \cap A^c)) \quad (A\text{至多可数}) \\ &\leqslant \int_{(\alpha,\beta)\cap A^c} |f'(x)| dx \\ &\leqslant \int_{(\alpha,\beta)} |f'(x)| dx \end{aligned}$$

$$\Longrightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)| \leqslant \int_{\bigsqcup_{i=1}^{n}(a_i,b_i)} |f'(x)| dx.$$

例题11: 微积分基本定理

- f在$[a,b]$处处可导, $f' \in L^1[a,b]$

$$\Longrightarrow f(x) = f(a) + \int_a^x f'(t)dt.$$

例题12:分部积分公式

- $f, g \in AC[a,b]$

$$\Longrightarrow \int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

证明

- $f, g \in AC[a,b] \Longrightarrow fg \in AC[a,b]$
- $f, g \in AC[a,b] \Longrightarrow f'g \in L^1[a,b], \quad fg' \in L^1[a,b]$

$$\int_a^b (f(x)g(x))'dx = f(x)g(x)\Big|_a^b$$

$$\int_a^b (f(x)g(x))'dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

例题13:积分第一中值定理

- $f \in C[a,b], \qquad g \in L^1[a,b] \cap L^+[a,b]$

$$\Longrightarrow \int_a^b f(x)g(x)dx \xlongequal{\exists \zeta \in [a,b]} f(\zeta) \int_a^b g(x)dx$$

证明同Riemann积分情形。

例题14:积分换元公式(变量代换公式)

- $g : [a, b] \longrightarrow [c, d]$a.e.可导.
- $f \in L^1[c, d]$
- $F(x) = \int_c^x f(t)dt$

下列等价

(1) $F(g(t)) \in AC[a, b]$

(2) $f(g(t))g'(t) \in L^1[a, b]$, 而且

$$\int_{g(\alpha)}^{g(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(g(t))g'(t)dx \quad \forall[\alpha, \beta] \subset [a, b]$$

证明

- $(2) \Longrightarrow (1)$:

$$F(g(t)) - F(g(a)) \xlongequal{\text{F的定义}} \int_{g(a)}^{g(t)} f(x)dx \xlongequal{\text{假设(2)}} \int_a^t f(g(u))g'(u)du$$

$$\Longrightarrow F \circ g \in AC[a,b].$$

证明

- (1) $\Longrightarrow$ (2):

$$F(x)=\int_c^x f(t)dt \quad\Longrightarrow\quad F\in AC[a,b]$$

$$\xrightarrow{\text{需要证明}} \quad m(F(Z))=0,\quad \forall\text{零测集}Z.$$

$$(F(g(t)))' \quad \underset{\text{需要证明}}{\overset{a.e.}{=\!=\!=}} \quad f(g(t))g'(t)\in L^1[a,b]$$

$$\Longrightarrow \quad \int_{g(\alpha)}^{g(\beta)} f(x)dx = F(g(\beta))-F(g(\alpha))$$

$$=\int_\alpha^\beta (F(g(t)))'dt=\int_\alpha^\beta f(g(t))g'(t)dt$$

补充证明

$$F \in AC[a,b] \quad \Longrightarrow \quad m(F(Z)) = 0, \quad \forall \text{零测集} Z.$$

- 存在开集G:

$$Z \setminus \{a,b\} \subset G \subset (a,b), \quad m(G) < \epsilon.$$

- $G = \bigsqcup_{i=1}^{\infty} (a_i, b_i)$:
- 存在 $c_i, d_i \in [a_i, b_i]$:

$$F([a_i, b_i]) = [F(c_i), F(d_i)]$$

$$
\begin{aligned}
m^*(F(Z)) &= m^*(F(Z \setminus \{a, b\})) \\
&\leqslant m^*(F(\bigsqcup_{i=1}^{\infty}(a_i, b_i))) \\
&\leqslant \sum_i m^*(F([a_i, b_i])) \\
&= \sum_i m([F(c_i), F(d_i)]) \\
&= \sum_i |F(c_i) - F(d_i)| < \epsilon
\end{aligned}
$$

补充证明: 复合函数的导数

假设下列条件满足:

- 函数几乎处处可导

$$\begin{aligned} g : [a,b] &\longrightarrow [c,d] \\ F : [c,d] &\longrightarrow \mathbb{R} \\ F \circ g : [a,b] &\longrightarrow \mathbb{R} \end{aligned}$$

- $F'(x) \xlongequal{a.e.} f(x)$:
- $m(F(Z)) = 0, \quad \forall$零测集$Z \subset [c,d]$

$$\Longrightarrow (F(g(t)))' \xlongequal{a.e.} f(g(t))g'(t)$$

证明:

- 构造集合

$$
\begin{aligned}
Z &:= \{x \in [c,d] : F\text{在}x\text{处不可导}\} \\
A &:= g^{-1}(Z) \\
B &:= [a,b] \setminus A
\end{aligned}
$$

- 固定$t \in B$, g在t点可导 $(g$ a.e.可导)

$$\Longrightarrow g\text{在}t\text{点连续}, \quad g(t) \in Z^c, \quad F'(g(t)) = f(g(t))$$

- 构造

$$\varphi(h) = \begin{cases} \frac{F(g(t+h)-F(g(t)))}{g(t+h)-g(t)}, & g(t+h)-g(t) \neq 0 \\ f(g(t)) & g(t+h)-g(t) = 0 \end{cases}$$

证明续

- $\lim_{h\to 0}\varphi(h) = f(g(t))$
- 在B上成立(差商+极限)

$$(F(g(t)))' \xlongequal{\quad} f(g(t))g'(t)$$

- 在A上成立

$$\begin{aligned} g(A) &\subset Z \\ m(g(A)) &= m(F(g(A))) = 0 \\ g'(t) &\xlongequal{a.e.} 0, \qquad (F(g(t)))' \xlongequal{a.e.} 0 \\ \Longrightarrow & \quad (F(g(t)))' \xlongequal{a.e.} f(g(t))g'(t) \end{aligned}$$

补充证明

假设下列条件成立：

- $f : [a, b] \to \mathbb{R}$
- $E \subset [a, b]$, f在E中每点处可导
- $m(f(E)) = 0$

$$\Longrightarrow \qquad f'(x) \xlongequal{a.e. x \in E} 0$$

$$B=\{x\in E:|f'(x)|>0\}=\bigcup_{n=1}^{\infty}B_n$$

$$B_n=\left\{x\in E:|f(y)-f(x)|\geqslant\frac{1}{n}|y-x|\quad\forall y\in B(x,\frac{1}{n})\right\}.$$

只要证明B_n是零测集.

$\forall I$长度小于$\frac{1}{n}$的区间, $A := I \cap B_n$

$$f(A) \subset f(B_n) \subset f(E)\text{零测集}$$

$$\forall \epsilon > 0, \text{存在开区间列}\{I_k\}: \quad f(A) \subset \bigcup_k I_k, \quad \sum_k m(I_k) < \epsilon.$$

$$A_k := A \cap f^{-1}(I_k)$$

$$A = \bigcup_k A_k, \qquad A_k \subset A = I \cap B_n$$

$$f(A_k) \subset I_k$$

$$m^*(A_k) \leqslant \mathrm{diam}(A_k) \leqslant n\mathrm{diam}(f(A_k))$$

$$\Longrightarrow \qquad m^*(A) \leqslant n \sum_k \mathrm{diam}(I_k) < n\epsilon.$$