例题续

证明：构造单射 $\{x \in (a,b) : f'_-(x) < f'_+(x)\} \longrightarrow \mathbb{Q}^3$

$$x \longmapsto (r_x, s_x, t_x)$$

- $r_x \in (f'_-(x), \quad f'_+(x)), \qquad x \in (s_x, t_x) \subset [s_x, t_x] \subset (a,b)$

$$\xRightarrow[\forall\, v \in (s_x, x), u \in (x, t_x)]{\text{极限保号性}} \frac{f(v)-f(x)}{v-x} < r_x < \frac{f(u)-f(x)}{u-x}.$$

$$\xRightarrow[\text{支撑线}]{} f(z)-f(x) > r_x(z-x), \qquad \forall\, z \in (s_x, t_x) \setminus \{x\}.$$

- 单射 $(r_x, s_x, t_x) = (r_y, s_y, t_y) \Longrightarrow x = y$. 否则

$$\xRightarrow{\text{上式}} f(y)-f(x) > r_x(y-x), \qquad f(x)-f(y) > r_y(x-y), \text{与} r_x = r_y \text{矛盾}$$

(严格凸支撑线不同)

不存在万能的尺子, 测度论由此应运而生

- 不存在$\mu : 2^{\mathbb{R}} \longrightarrow [0, +\infty]$满足
  - $\mu(\phi) = 0.$
  - $\mu(\bigsqcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k), \qquad \forall E_k \subset \mathbb{R}.$
  - $\mu(E) = \mu(x + E), \qquad \forall E \subset \mathbb{R}.$
  - $\mu([0, s]) = s, \qquad \forall s > 0.$

不可测集存在性的证明（反证法）

- 取代表元集合K(选择公理)

$$[0,1]/\mathbb{Q} \xlongequal{\text{双射}} K.$$

- 相似集合的可列并介于两个区间：

$$[0,1] \subset \bigsqcup_{r\in[-1,1]\cap\mathbb{Q}} (K+r) \subset [-1,2].$$

  - 考虑$[-1,1]\cap\mathbb{Q}$的原因:

$$\forall x\in[0,1] \Longrightarrow x = k + r_0 \quad (\exists k\in K,\ \ r_0\in\mathbb{Q})$$
$$r_0 = x - k \in [0,1]-[0,1] = [-1,1].$$

  - $K+[-1,1]\subset[0,1]+[-1,1]=[-1,2]$
  - $k_1+r_1=k_2+r_2 \xRightarrow[\text{代表元唯一性}]{\text{两边在同一等价类}} k_1=k_2,\ \ r_1=r_2.$

- 代表元集合的平移可记为K_j, 它们相似且互不相交

$$[0,1] \subset \bigsqcup_{j=1}^{\infty} K_j \subset [-1,2].$$

- 这与下列事实矛盾

$$1 \leqslant \sum_{j=1}^{\infty} \mu(K_j) \leqslant 3$$

$$\sum_{j=1}^{\infty} \mu(K_j) \xlongequal{\mu(K_j)=\mu(K)} 0\text{或} + \infty.$$

任意正测集包含不可测子集

- 任意正测集包含不可测子集.

$$A=\bigcup_{n=1}^{\infty}(A\cap[-n,n])$$

$$\xRightarrow{\text{不妨设}A\text{有界}}\quad \text{不妨设正测集}A\subset[0,1].$$

取代表元集合W:

$$W\subset A,\qquad A/\mathbb{Q}\ \xlongequal{\text{双射}}\ W.$$

$$A\subset\bigsqcup_{r\in[-1,1]\cap\mathbb{Q}}(r+W)\subset[-1,2]\Longrightarrow W\text{是不可测集合}.$$

本讲内容

- Cantor集

- Cantor 函数

反直观, 逻辑起着关键作用

- Cantor集

Cantor集	有理数集	推广的Cantor集	无理数集
零测集	零测集	正测集	正测集
不可数集	可数集	不可数集	不可数集
第一纲集	第一纲集	第一纲集	第二纲集
奇异	中规中矩	奇异	中规中矩

- Cantor 函数
  - $[0,1] \twoheadrightarrow [0,1] \uparrow$ 连续
  - 将零测集映照为正测集.
  - 导函数几乎处处为零.

- 一尺之锤, 日截其半, 万世不竭.

—-庄子

$$\bigcap_{n=1}^{\infty}[0, \frac{1}{2^n}] = \{0\}.$$

三等分Cantor集

- $C_0 = [0, 1].$
- $C_1 = [0, \frac{1}{3}] \sqcup [\frac{2}{3}, 1].$ 被挖去集合$(\frac{1}{3}, \frac{2}{3})$
- $C_2 = [0, \frac{1}{9}] \sqcup [\frac{2}{9}, \frac{3}{9}] \sqcup [\frac{6}{9}, \frac{7}{9}] \sqcup [\frac{8}{9}, 1].$ 被挖$(\frac{1}{9}, \frac{2}{9}) \sqcup (\frac{7}{9}, \frac{8}{9})$
- $\vdots$
- C_n是2^n个区间的并, 每个区间长度均为$\frac{1}{3^n}$.
  C_n是由C_{n-1}挖去是2^{n-1}个区间得到.

集合的个数如细胞分裂, 指数增长.

Cantor集续

	区间长度	个数	去掉长度	个数
C_0	1	1	0	0
C_1	$\frac{1}{3}$	2	$\frac{1}{3}$	1
C_2	$\frac{1}{3^2}$	2^2	$\frac{1}{3^2}$	2
$\vdots$				
C_n	$\frac{1}{3^n}$	2^n	$\frac{1}{3^n}$	2^{n-1}
$\vdots$				

Cantor集续

- $C_{n-1} \xrightarrow[\text{去掉中间的开区间}]{\text{三等分}} C_n.$

- $C_n = C_{n-1} \setminus \bigsqcup_{2^{n-1}\text{个开区间}} \text{长度为}\frac{1}{3^n}$

- $C_n = \bigsqcup_{2^n\text{个闭区间}} \text{长度为}\frac{1}{3^n}$

Cantor集续2

- Cantor三分集

$$C=\bigcap_{n=1}^{\infty}C_n \xlongequal{C_n\downarrow} \lim_{n\to\infty}C_n.$$

C_n是有限个闭区间的并, C是C_n的极限, 其性质来源于C_n.

Cantor集性质

Cantor集是

- 紧集 $\quad (C=\bigcap_{n=1}^{\infty} C_n, \quad C_n$紧)
- 零测集 $\quad (|C| \leqslant |C_n| \leqslant \frac{2^n}{3^n} \to 0.$ 无内点)
- 连续统的势 $\quad$ (见Cantor函数)
- 完全集(无孤立点的闭集)

$$(\forall x \in C \xrightarrow[x\in C_n]{C_n\text{的端点}\in C} x\text{是}C\text{的极限点})$$

- 第一刚集 $\quad$ (C是无处稠密集合, 即其闭包无内点, 第一刚集是可列个无处稠密集合的并)

小集合

	集合论	测度论	拓扑学
	势	测度	开集
小集合	可数集	零测集	第一纲集

Baire纲定理

- $A \subset \mathbb{R}$无处稠密集合 (A的闭包无内点).
- 第一纲集= $\bigcup_{\text{可列}}$ 无处稠密集合.
- Baire纲定理: 完备度量空间是第二纲集.

p进制表示定理: $p = 2, 3, \cdots$.

- $\forall\ x \in (0,1), \exists\,!\ a_n \in \{0, 1, \cdots, p-1\}$, 有无穷多项$a_n$非零,

$$\xlongequal{\text{唯一表示}} x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = 0.a_1 a_2 \cdots (p),$$

- 当且仅当x 是p进制下的有穷小数, 表示唯二:

$$x \xlongequal{x_n \neq 0} 0.a_1 a_2 \cdots a_n(p) = 0.a_1 \cdots a_{n-1}(a_n-1)(p-1)(p-1)\cdots(p).$$

(唯二表示特点: 各项全为0或全为p-1, 仅有限项例外)

厘米尺子的制作

取1米长度的无刻度的尺子

- 将尺子$[0,1]$十等分, 若x位于第a_1+1个区间，则

$$x = 0.a_1\cdots \quad (10)$$

- 再将第a_1+1个区间十等分, 若x位于第a_2+1个区间，则

$$x = 0.a_1a_2\cdots \quad (10)$$

- 再将第a_2+1个区间十等分, 若x位于第a_3+1个区间，则

$$x = 0.a_1a_2a_3\cdots \quad (10)$$

Cantor集的三进制表示

- $$C = \{x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n = 0, 2\}$$

证: 取三进制唯一表示

$$x = 0.a_1a_2a_3\cdots(3)$$

$$\text{Cantor集的构造过程} \xRightarrow{\text{被去掉点的特征}} \text{三进制表示包含1.}$$

第一次去掉$a_1 = 1$的项,

端点$\frac{1}{3} = 0.02222\cdots(3) \in C$, 端点$\frac{2}{3} = 0.2(3) \in C$

第二次去掉$a_2 = 1$的项,

⋮

- Cantor函数 $f: C \longrightarrow [0,1] \uparrow$ 满射

  $$f(0.(2c_1)(2c_2)\cdots(3)) = 0.c_1c_2\cdots(2), \qquad c_k = 0, 1.$$

  即 $$f\left(\sum_{n=1}^{\infty}\frac{x_n}{3^n}\right) \xlongequal{x_n=0,2} \frac{1}{2}\sum_{n=1}^{\infty}\frac{x_n}{2^n}.$$

- 单调增加: $c_{n+1} = 0, \quad d_{n+1} = 1$

  $$0.(2c_1)\cdots(2c_n)(2c_{n+1})\cdots(3) < 0.(2c_1)\cdots(2c_n)(2d_{n+1})\cdots(3)$$

  $$\Longrightarrow 0.c_1\cdots c_nc_{n+1}\cdots(2) \leqslant 0.c_1\cdots c_nd_{n+1}\cdots(2).$$

Cantor函数的局部常值扩充

- 设(a,b)是Cantor三分集构造过程中去掉的区间. 则

$$\begin{aligned} a &= 0.(2c_1)\cdots(2c_{n-1})100\cdots(3) \\ b &= 0.(2c_1)\cdots(2c_{n-1})200\cdots(3) \end{aligned}$$

- $f(a) = 0.c_1\cdots c_{n-1}0111\cdots(2) = 0.c_1\cdots c_{n-1}1(2) = f(b)$

- 局部常值扩充

$$f|_{(a,b)} = f(a) = f(b)$$

- $f:[0,1] \longrightarrow [0,1]$ ↑满射

- $f \in C[0,1]$ (f不具有跳跃点, 否则利用单调性f非满射)

Cantor函数图形

- $f : [0, 1] \longrightarrow [0, 1]$连续
- $f(0) = 0, \quad f(1) = 1$
- $f|_{[\frac{1}{3}, \frac{2}{3}]} = \frac{1}{2}$.
- $f|_{[\frac{1}{9}, \frac{2}{9}]} = \frac{1}{4}, \qquad f|_{[\frac{7}{9}, \frac{8}{9}]} = \frac{3}{4}$.

$$\vdots$$

Cantor函数性质

- Cantor函数$f : [0,1] \longrightarrow [0,1]$ ↑满射, 连续.
- f将零测集映为正测集
$$f(C) = [0,1].$$
- f 在$(0,1) \setminus C$上局部为常数.
$$f'\big|_{(0,1)\setminus C} = 0, \qquad f' \overset{a.e.}{=\!=} 0.$$
- 微积分基本定理失效: $\quad 0 = \int_0^1 f'(x)dx < f(x)\big|_0^1 = 1.$

由微积分基本定理看Cantor函数

$$L^1[a,b]/\sim \ \underset{\frac{d}{dx}}{\overset{\int_a^x}{\rightleftarrows}} \ AC[a,b]/\mathbb{R} \qquad （双射）$$

- Cantor函数$f \notin AC[0,1]$.

问题：Cantor 函数是否是Lipschitz函数？

Cantor集的性质

- Cantor集合C:

$$\mathring{C} = \emptyset, \qquad \partial C = C' = \overline{C} = C.$$

Cantor函数在Cantor集中的点不可导

- $\forall\ x \in C \Longrightarrow f'(x)$不存在.

$$C \ni x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \qquad 取 \quad C \ni x_n = \begin{cases} x + \frac{2}{3^n}, & \text{if } a_n = 0; \\ x - \frac{2}{3^n}, & \text{if } a_n = 2. \end{cases}$$

$$则 \qquad \lim_{n\to\infty} x_n = x.$$

(1) $a_n = 0$:

$$f(x_n) - f(x) = \frac{1}{2^n}, \qquad x_n - x = \frac{2}{3^n}.$$

(2) $a_n = 2$:

$$f(x_n) - f(x) = -\frac{1}{2^n}, \qquad x_n - x = -\frac{2}{3^n}.$$

综上, 两种情形下都有

$$\lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = +\infty.$$

- $K_0 = [0,1]$, $\quad K_{n-1} \xrightarrow[\text{去掉中间的同心开区间}]{\text{剩余每个区间长度为}\lambda_n} K_n.$
- λ_k满足

$$1 > 2\lambda_1 > 4\lambda_2 > \cdots > 2^n\lambda_n > 2^{n+1}\lambda_{n+1} > \cdots$$

- 任给$\theta \in [0,1)$, $\forall n \in \mathbb{N}$, 取λ_n满足

$$2^n\lambda_n = \frac{n\theta+1}{n+1}\downarrow, \qquad \lim_{n\to\infty}\lambda_n = 0.$$

- 推广的Cantor集

$$K := \lim_{n\to\infty} K_n, \qquad m(K) = \lim_{n\to\infty} m(K_n) = \lim_{n\to\infty} 2^n\lambda_n = \theta.$$

- K是连续统势，测度是$\theta \in [0,1)$, 第一纲集($\lim_{n\to\infty}\lambda_n = 0.$, 紧集$K$的闭包无内点)

推广的Cantor集的大小

	集合论	测度论	拓扑学
推广的Cantor集	大	可大可小	小

Cantor集的维数

- 欧氏空间维数

$$\mathbb{R}^2\text{的维数} \xrightarrow[\text{将标准立方体扩大3倍再分解}]{3E=E_1\sqcup\cdots\sqcup E_{3^2}} \frac{\log 3^2}{\log 3}.$$

- Cantor集的Hausdorff 维数

$$\text{Cantor集的维数} \xrightarrow[\text{将标准Cantor集扩大3倍再分解}]{3C=C_1\sqcup C_2} \frac{\log 2}{\log 3}.$$

小结

- Cantor 集合:
  - Cantor 集是一个矛盾统一体, 在某种意义下是很大的集合, 在某种意义下是小的集合.
  - 它是分数维集合, 在分形理论中起着重要作用.
  - Cantor 集是实分析的夜明珠, 照彻直观思维的死角.
  - 在人类对世界的认识中, 有理数集、Cantor集、推广的Cantor集、无理数集具有同等的重要性.
  - 有理数集和无理数集在人类直观的范围内. Cantor集和推广的Cantor集在人类直观的范围之外. 这也证实了数学是人类的眼睛.
- Cantor 函数:
  - Cantor函数是[0, 1]区间单调增加的连续满函数.
  - 与直观情形不同, 该连续函数将零测集映照为正测集.
  - 与直观情形不同, 该单调满射的导函数几乎处处为零.

1 The Baire category theorem

Although Baire proved his theorem on the real line, his result actually holds in the more general setting of complete metric spaces. For the purpose of the applications we have in mind it is better to have access to this more general formulation right away. Fortunately, the proof of the theorem remains very simple and elegant.

To state the main result, we begin with a list of definitions. Let X be a metric space with metric d, carrying the natural topology induced by d. In other words, a set $\mathcal{O}$ in X is open if for every $x \in \mathcal{O}$ there exists $r > 0$ so that $B_r(x) \subset \mathcal{O}$, where $B_r(x)$ denotes the open ball centered at x and of radius r,

$$B_r(x) = \{y \in X : \ d(x, y) < r\}.$$

By definition, a set is closed if its complement is open.

We define the **interior** E° of a set $E \subset X$ to be the union of all open sets contained in E. Also, the **closure** $\overline{E}$ of E is the intersection of all closed sets containing E. Since one checks easily that the union of any collection of open sets is open, and the intersection of any collection of closed sets is closed, we see that E° is the "largest" open set contained in E, and $\overline{E}$ is the "smallest" closed set containing E.

Suppose E is a subset of X. We say that the set E is **dense** in X if $\overline{E} = X$. Also, the set E is **nowhere dense** if the interior of its closure is empty, $(\overline{E})^\circ = \emptyset$. For instance, any point in $\mathbb{R}^d$ is nowhere dense in $\mathbb{R}^d$. Also, the Cantor set is nowhere dense in $\mathbb{R}$, but the rationals $\mathbb{Q}$ are not since $\overline{\mathbb{Q}} = \mathbb{R}$. We note here that in general E is closed and nowhere dense if and only $\mathcal{O} = E^c$ is open and dense.

We now describe the central notion of category due to Baire, and the dichotomy it introduces.

- A set $E \subset X$ is of the **first category** in X if E is a countable union of nowhere dense sets in X. A set of the first category is sometimes said to be "meager." A set E that is not of the first category in X is referred to as being of the **second category** in X.

- A set $E \subset X$ is defined to be **generic** if its complement is of the first category.

Thus the idea of category is to describe "smallness" in purely topological terms (involving closures, interiors, etc.) It reflects the idea that elements of a set of the first category are to be thought of as "exceptional," while those of a generic set are to be considered "typical." Connected with this is the fact that a countable union of sets of the first category is of the first category, while the countable intersection of generic sets is a generic set. Also we record here the useful fact that any open dense set is generic (this follows from our remark earlier).

In general relying on one's intuition about the category of sets requires a little caution. For instance, there is no link between this notion and that of Lebesgue measure. Indeed, there are sets in $[0, 1]$ of the first category that are of full measure, and hence uncountable and dense. By the same token, there are generic sets of measure zero. (Some examples are discussed in Exercise 1.)

The main result of Baire is that "the continuum is of the second category." The key ingredient used in his argument is the fact that the real line is complete. This is the main reason why his theorem immediately carries over to the case of a complete metric space.

Theorem 1.1 *Every complete metric space X is of the second category in itself, that is, X cannot be written as the countable union of nowhere dense sets.*

Corollary 1.2 *In a complete metric space, a generic set is dense.*

Proof of the theorem. We argue by contradiction, and assume that X is a countable union of nowhere dense sets F_n,

$$X = \bigcup_{n=1}^{\infty} F_n. \tag{1}$$

By replacing each F_n by its closure, we may assume that each F_n is closed. It now suffices to find a point $x \in X$ with $x \notin \bigcup F_n$.

Since F_1 is closed and nowhere dense, hence not all of X, there exists an open ball B_1 of some radius $r_1 > 0$ whose closure $\overline{B_1}$ is entirely contained in F_1^c.

Since F_2 is closed and nowhere dense, the ball B_1 cannot be entirely contained in F_2, otherwise F_2 would have a non-empty interior. Since F_2 is also closed, there exists a ball B_2 of some radius $r_2 > 0$ whose closure $\overline{B_2}$ is contained in B_1 and also in F_2^c. Clearly, we may choose r_2 so that $r_2 < r_1/2$.

Continuing in this fashion, we obtain a sequence of balls $\{B_n\}$ with the following properties:

(i) The radius of B_n tends to 0 as $n \to \infty$.

(ii) $B_{n+1} \subset B_n$.

(iii) $F_n \cap \overline{B_n}$ is empty.

Choose any point x_n in B_n. Then, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence because of properties (i) and (ii) above. Since X is complete, this sequence converges to a limit which we denote by x. By (ii) we see that $x \in \overline{B_n}$ for each n, and hence $x \notin F_n$ for all n by (iii). This contradicts (1), and the proof of the Baire category theorem is complete.

To prove the corollary, we argue by contradiction and assume that $E \subset X$ is generic but not dense. Then there exists a closed ball $\overline{B}$ entirely contained in E^c. Since E is generic we can write $E^c = \bigcup_{n=1}^{\infty} F_n$ where each F_n is nowhere dense, hence

$$\overline{B} = \bigcup_{n=1}^{\infty} (F_n \cap \overline{B}).$$

It is clear that $F_n \cap \overline{B}$ is nowhere dense, hence the above contradicts Theorem 1.1 applied to the complete metric space $\overline{B}$, and the corollary is proved.

The theorem actually extends to certain cases of metric spaces that are not complete, in particular to open subsets of a complete metric space. To be precise, suppose we are given a subset X_0 of a complete metric space X. Then X_0 is itself a metric space, inheriting its metric from X by restricting the metric on X to X_0. The fact is that if X_0 is an open subset of X, then the conclusion of the theorem holds for it; that is, X_0 cannot be written as a countable union of sets that are nowhere dense (in X_0). See Exercise 3. A simple example is given by the open interval $(0, 1)$ with the usual metric.

1.1 Continuity of the limit of a sequence of continuous functions

Suppose X is a complete metric space, $\{f_n\}$ is a sequence of continuous complex-valued functions on X, and that the limit

$$\lim_{n \to \infty} f_n(x) = f(x)$$

exists for each $x \in X$. It is well known that if the limit is uniform in x, then the limiting function f is also continuous. In general, when the limit is just pointwise, we may ask: must f have at least one point of continuity? We answer this question affirmatively with a simple application of the category theorem.

Theorem 1.3 *Suppose that $\{f_n\}$ is a sequence of continuous complex-valued functions on a complete metric space X, and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for every $x \in X$. Then, the set of points where f is continuous is a generic set in X. In other words, the set of points where f is discontinuous is of the first category.

Therefore f is in fact continuous at "most" points of X.

To show that the set $\mathcal{D}$ of discontinuities of f is of the first category, we use a characterization of points of continuity of f in terms of its oscillations. More precisely, we define the **oscillation** of the function f at a point x by

$$\operatorname{osc}(f)(x) = \lim_{r\to 0} \omega(f)(r,x), \text{ where } \omega(f)(r,x) = \sup\nolimits_{y,z\in B_r(x)} |f(y) - f(z)|.$$

The limit exists since the quantity $\omega(f)(r,x)$ decreases with r. In particular, we see that $\operatorname{osc}(f)(x) < \epsilon$ if there exists a ball B centered at x so that $|f(y) - f(z)| < \epsilon$ whenever $y, z \in B$. Two more observations are in order:

(i) $\operatorname{osc}(f)(x) = 0$ if and only if f is continuous at x.

(ii) The set $E_\epsilon = \{x \in X : \operatorname{osc}(f)(x) < \epsilon\}$ is open.

Property (i) follows immediately from the definition of continuity. For (ii), we note that if $x \in E_\epsilon$, there is an $r > 0$ so that $\sup_{y,z\in B_r(x)} |f(y) - f(z)| < \epsilon$. Consequently, if $x^* \in B_{r/2}(x)$, then $x^* \in E_\epsilon$ because

$$\sup_{y,z\in B_{r/2}(x^*)} |f(y) - f(z)| \leq \sup_{y,z\in B_r(x)} |f(y) - f(z)| < \epsilon.$$

Lemma 1.4 *Suppose $\{f_n\}$ is a sequence of continuous functions on a complete metric space X, and $f_n(x) \to f(x)$ for each x as $n \to \infty$. Then, given an open ball $B \subset X$ and $\epsilon > 0$, there exists an open ball $B_0 \subset B$ and an integer $m \geq 1$ so that $|f_m(x) - f(x)| \leq \epsilon$ for all $x \in B_0$.*

Proof. Let Y denote a closed ball contained in B. Note that Y is itself a complete metric space. Define

$$E_\ell = \{x \in Y : \sup_{j,k\geq \ell} |f_j(x) - f_k(x)| \leq \epsilon\}.$$

Then, since $f_n(x)$ converges for every $x \in X$, we must have

$$(2) \qquad Y = \bigcup_{\ell=1}^{\infty} E_\ell.$$

Moreover, each E_ℓ is closed since it is the intersection of sets of the type $\{x \in Y : |f_j(x) - f_k(x)| \leq \epsilon\}$ which are closed by the continuity of f_j and f_k. Therefore, by Theorem 1.1 applied to the complete metric space Y, some set in the union (2), say E_m, must contain an open ball B_0. By construction,

$$\sup_{j,k \geq m} |f_j(x) - f_k(x)| \leq \epsilon \quad \text{whenever } x \in B_0,$$

and letting k tend to infinity we find that $|f_m(x) - f(x)| \leq \epsilon$ for all $x \in B_0$. This proves the lemma.

To finish the proof of Theorem 1.3, we define

$$F_n = \{x \in X : \ \mathrm{osc}(f)(x) \geq 1/n\},$$

in other words, $F_n = E_\epsilon^c$ with $\epsilon = 1/n$ in the notation of (ii) above.

Then, by our observation (i), we have

$$\mathcal{D} = \bigcup_{n=1}^{\infty} F_n,$$

where we recall that $\mathcal{D}$ is the set of discontinuities of f. The theorem will be proved if we can show that each F_n is nowhere dense.

Fix $n \geq 1$. Since F_n is closed, we must show that it has empty interior. Assume on the contrary, that B is an open ball with $B \subset F_n$. Then, if we set $\epsilon = 1/4n$ in the lemma, we find that there is an open ball $B_0 \subset B$, and an integer $m \geq 1$ so that

$$(3) \qquad |f_m(x) - f(x)| \leq 1/4n, \quad \text{for all } x \in B_0.$$

By the continuity of f_m, we may find a ball $B' \subset B_0$ so that

$$(4) \qquad |f_m(y) - f_m(z)| \leq 1/4n, \quad \text{for all } y, z \in B'.$$

Then, the triangle inequality implies

$$|f(y) - f(z)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(z)| + |f_m(z) - f(z)|.$$

If $y, z \in B'$, the first and third terms are bounded by $1/4n$ because of condition (3). The middle term is also bounded by $1/4n$ due to (4). Therefore

$$|f(y) - f(z)| \leq \frac{3}{4n} < \frac{1}{n} \quad \text{whenever } y, z \in B'.$$

Consequently, if x' denotes the center of B', we have $\text{osc}(f)(x') < 1/n$ which contradicts the fact that $x' \in F_n$. This concludes the proof of the theorem.

1.2 Continuous functions that are nowhere differentiable

Our next application of the category theorem is to the problem of the existence of a continuous function that is nowhere differentiable.

Our first answer to this question appeared in Chapter 4 of Book I where we showed that the complex-valued function f given by the following lacunary Fourier series

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x} \quad \text{with } 0 < \alpha \leq 1$$

is continuous but nowhere differentiable. Moreover, a slight change in the proof shows that both the real and imaginary parts of f are also nowhere differentiable. Other examples arose in Chapter 7 of Book III, in the context of the von Koch and space-filling curves.

Here, we prove the existence of such functions by showing that they are generic in an appropriate complete metric space. The space we have in mind consists of all real-valued continuous functions on $[0, 1]$, which we denote by

$$X = C([0, 1]).$$

This vector space is equipped with the sup-norm

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Together with this norm, $C([0, 1])$ is a complete normed vector space (a Banach space). The completeness follows because the uniform limit of a sequence of continuous functions is necessarily continuous. Finally, the metric d on X is chosen to be $d(f, g) = \|f - g\|$, and hence (X, d) is a complete metric space.

Theorem 1.5 *The set of functions in $C([0, 1])$ that are nowhere differentiable is generic.*

We must show that the set $\mathcal{D}$, of continuous functions in $[0,1]$ that are differentiable at least at one point, is of the first category. To this end, we let E_N denote the set of all continuous functions so that there exists $0 \leq x^* \leq 1$ with

$$|f(x) - f(x^*)| \leq N|x - x^*|, \quad \text{for all } x \in [0,1]. \tag{5}$$

These sets are related to $\mathcal{D}$ by the inclusion

$$\mathcal{D} \subset \bigcup_{N=1}^{\infty} E_N.$$

To prove the theorem it suffices to show that for each N, the set E_N is nowhere dense. This will be achieved by showing successively:

(i) E_N is a closed set.

(ii) the interior of E_N is empty.

Thus $\bigcup E_N$ is of the first category, hence so is the set $\mathcal{D}$.

Proof of property (i)

Suppose that $\{f_n\}$ is a sequence of functions in E_N so that $\|f_n - f\| \to 0$. We must show that $f \in E_N$. Let x_n^* be a point in $[0,1]$ for which (5) holds with f replaced by f_n. We may choose a subsequence $\{x_{n_k}^*\}$ that converges to a limit in $[0,1]$, which we denote by x^*. Then,

$$|f(x) - f(x^*)| \leq |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k}(x^*)| + |f_{n_k}(x^*) - f(x^*)|.$$

On the one hand, since $\|f_n - f\| \to 0$, we see that given $\epsilon > 0$, there exists $K > 0$ so that whenever $k > K$ the first and third terms together are $< \epsilon$. On the other hand, we may estimate the middle term by

$$|f_{n_k}(x) - f_{n_k}(x^*)| \leq |f_{n_k}(x) - f_{n_k}(x_{n_k}^*)| + |f_{n_k}(x_{n_k}^*) - f_{n_k}(x^*)|.$$

Therefore, applying the fact that $f_{n_k} \in E_N$ twice yields

$$|f_{n_k}(x) - f_{n_k}(x^*)| \leq N|x - x_{n_k}^*| + N|x_{n_k}^* - x^*|.$$

Putting all these estimates together, we obtain

$$|f(x) - f(x^*)| \leq \epsilon + N|x - x_{n_k}^*| + N|x_{n_k}^* - x^*|$$

for all $k > K$. Letting k tend to infinity, and recalling that $x_{n_k}^* \to x^*$ we get

$$|f(x) - f(x^*)| \leq \epsilon + N|x - x^*|.$$

Since ϵ is arbitrary, we conclude that $f \in E_N$, and (i) is proved.

Proof of property (ii)

To show that E_N has no interior, let $\mathcal{P}$ denote the subspace of $C([0, 1])$ that consists of all continuous piecewise-linear functions. Also, for each $M > 0$, let $\mathcal{P}_M \subset \mathcal{P}$ denote the set of all continuous piecewise-linear functions, each of whose line segments have slopes either $\geq M$ or $\leq -M$. Functions in $\mathcal{P}_M$ are naturally called "zig-zag" functions. Note the key fact that $\mathcal{P}_M$ is disjoint from E_N if $M > N$.

Lemma 1.6 *For every $M > 0$, the set $\mathcal{P}_M$ of zig-zag functions is dense in $C([0, 1])$.*

Proof. It is plain that given $\epsilon > 0$ and a continuous function f, there exists a function $g \in \mathcal{P}$ so that $\|f - g\| \leq \epsilon$. Indeed, since f is continuous on the compact set $[0, 1]$ it must be uniformly continuous, and there exists $\delta > 0$ so that $|f(x) - f(y)| \leq \epsilon$ whenever $|x - y| < \delta$. If we choose n so large that $1/n < \delta$, and define g as a linear function on each interval $[k/n, (k + 1)/n]$ for $k = 0, \ldots, n - 1$ with $g(k/n) = f(k/n)$, $g((k + 1)/n) = f((k + 1)/n)$, we see at once that $\|f - g\| \leq \epsilon$.

It now suffices to see how to approximate g on $[0, 1]$ by zig-zag functions in $\mathcal{P}_M$. Indeed, if g is given by $g(x) = ax + b$ for $0 \leq x \leq 1/n$, consider the two segments

$$\varphi_\epsilon(x) = g(x) + \epsilon \quad \text{and} \quad \psi_\epsilon(x) = g(x) - \epsilon.$$

Then, beginning at $g(0)$, we travel on a line segment of slope $+M$ until we intersect φ_ϵ. Then, we reverse direction and travel on a line segment of slope $-M$ until we intersect ψ_ϵ (see Figure 1).

We obtain $h \in \mathcal{P}_M$ so that

$$\psi_\epsilon(x) \leq h(x) \leq \varphi_\epsilon(x), \qquad \text{for all } 0 \leq x \leq 1/n,$$

and therefore $|h(x) - g(x)| \leq \epsilon$ in $[0, 1/n]$.

Then, we begin at $h(1/n)$ and repeat this argument on the interval $[1/n, 2/n]$. Continuing in this fashion, we obtain a function $h \in \mathcal{P}_M$ with $\|h - g\| \leq \epsilon$. Hence $\|f - h\| \leq 2\epsilon$, and the lemma is proved.

We deduce at once from this lemma that E_N has no interior points. Indeed, given any $f \in E_N$ and $\epsilon > 0$, we first choose a fixed $M > N$. Then, there exists $h \in \mathcal{P}_M$ so that $\|f - h\| < \epsilon$, and moreover $h \notin E_N$ since $M > N$. Therefore, no open ball around f is entirely contained in E_N, which is the desired conclusion. Theorem 1.5 is proved.